# CS:4980
# Foundations of Embedded Systems

## Hybrid Systems
## Part I

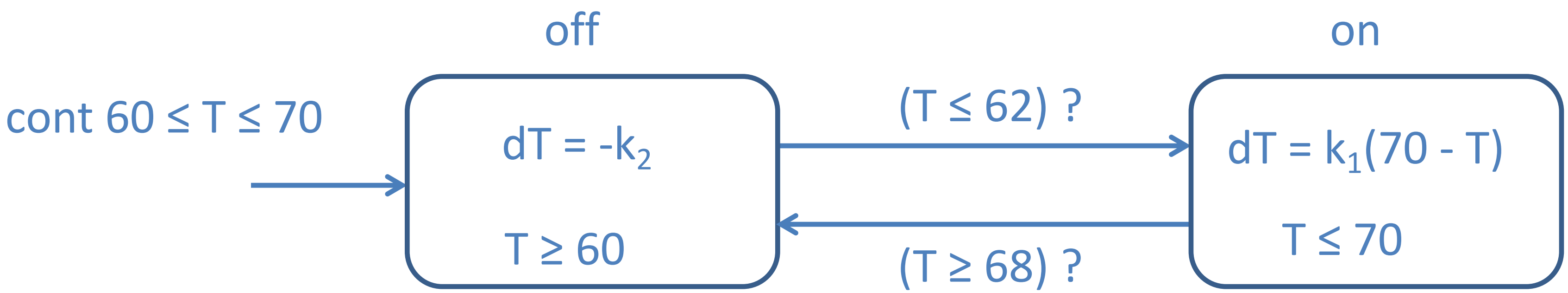
# Models of Reactive Computation

- **Continuous-time model** for dynamical system
  - Synchronous, where time evolves continuously
  - Execution of system: Solution to algebraic / differential equations

- **Timed model**
  - Like asynchronous for communication of information
  - Clocks evolve continuously, and constraints on delays allow synchronous/global coordination

- **Hybrid systems**
  - Generalization of timed processes
  - During timed transitions, evolution of state/output variables specified using differential equations as in dynamical systems

# Self-Regulating Switching Thermostat




State machine with two modes: on and off

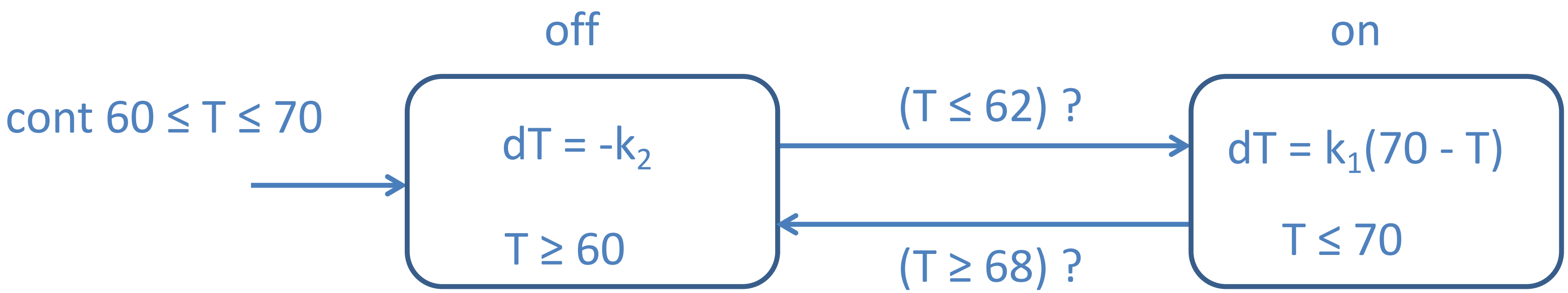
State variable T of type cont (continuous), to model temperature

T can be tested and updated during mode-switches

T changes continuously during timed transitions given by differential equations

Invariants (as in timed model) constrain how long can a timed transition be

# Executions of Thermostat




Initial state = $(\text{off}, T_0)$ with $T_0$ in the interval $[60,70]$

While in off mode, T decreases continuously: $\mathbf{T}(t) = T_0 - k_2\, t$

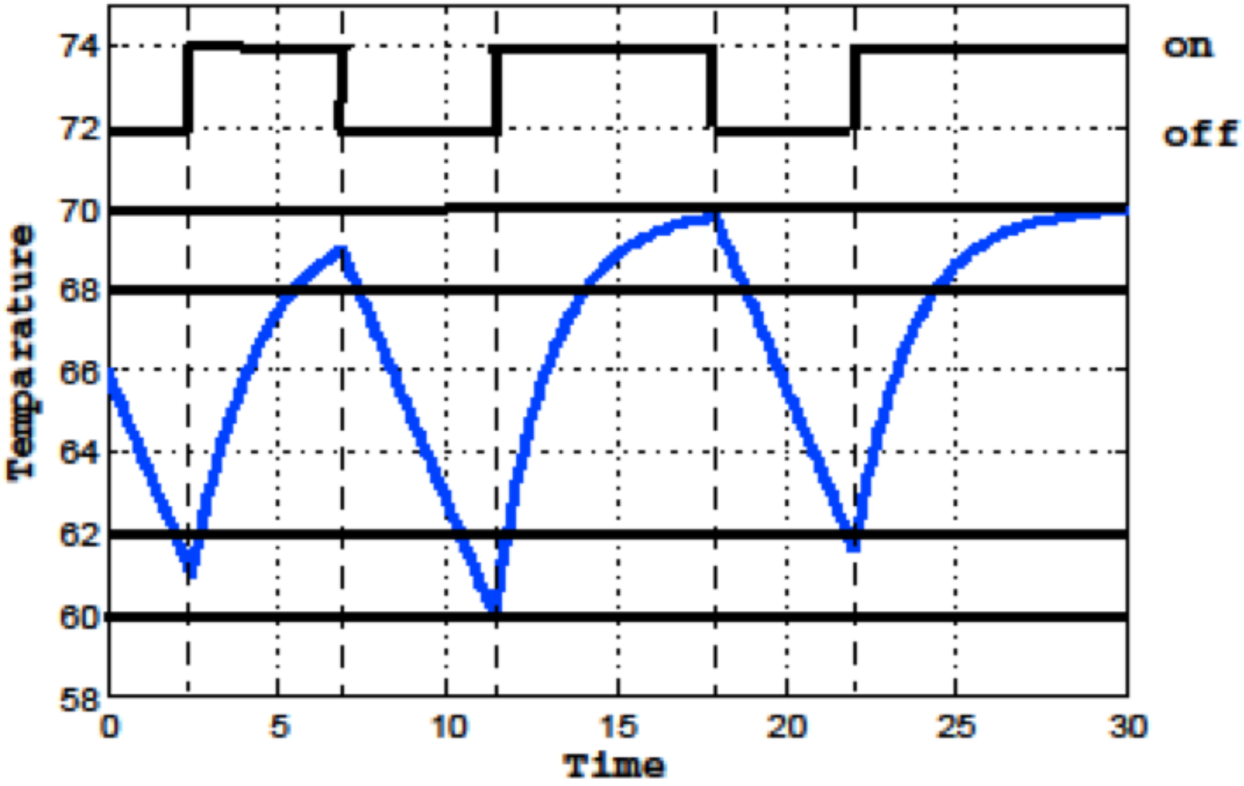
Mode-switch to on enabled when $T \le 62$, and must happen before T passes 60

While in on mode, T increases according to $\mathbf{T}(t) = 70 - (70 - T^*)\, e^{-k1(t-t^*)}$

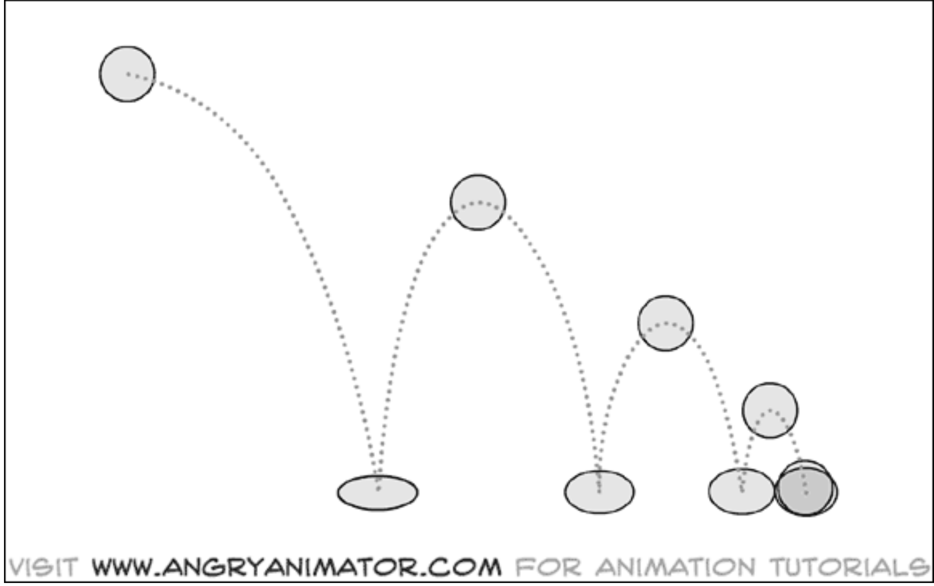
$t^*$, $T^*$ : time and temperature upon entry to mode on

Mode-switch to off enabled when $T \ge 68$, and must happen before T passes 70

# Simulation Plot of an Execution

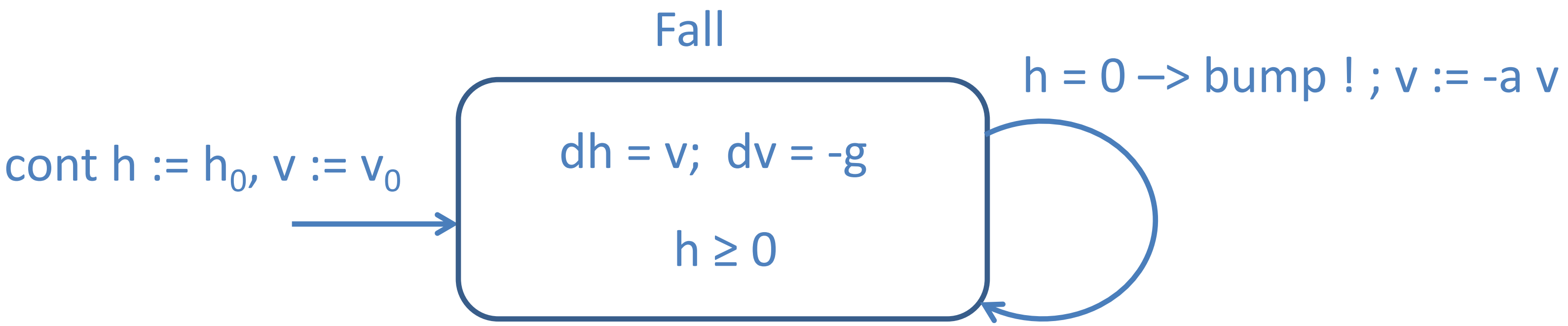


# Modeling a Bouncing Ball



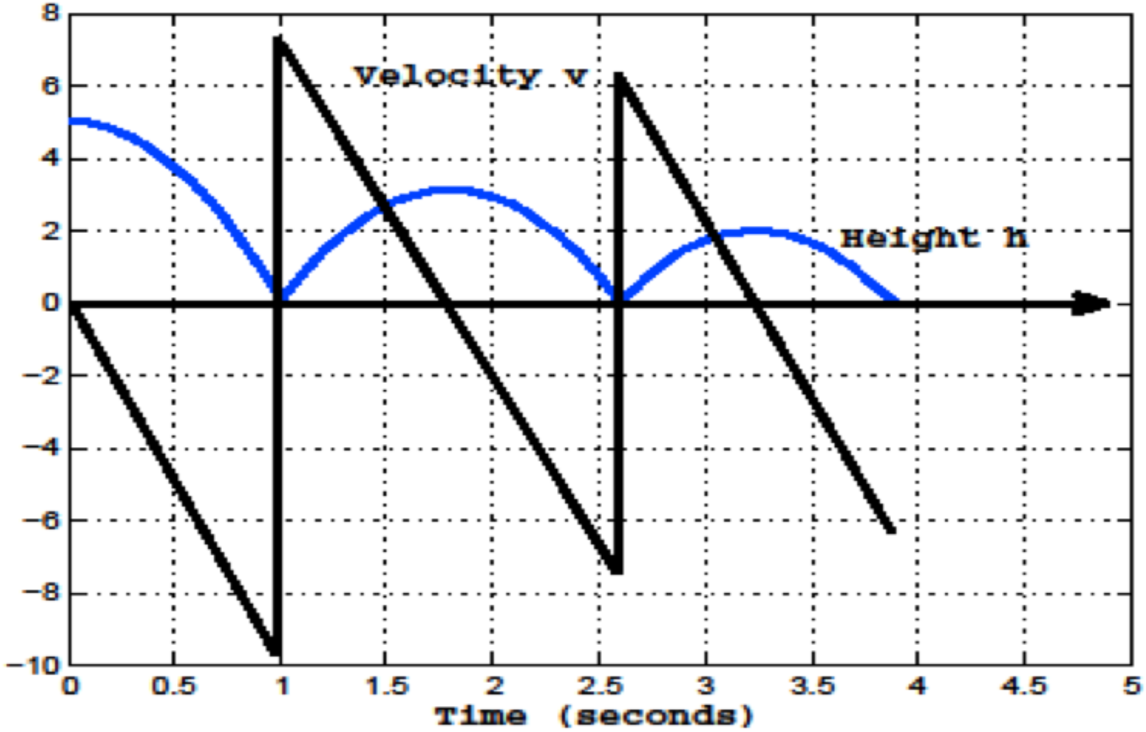

- Ball dropped from an initial height $h_0$ with an initial velocity $v_0$
- Velocity changes according to the differential equation $dv/dt = -g$
- When the ball hits the ground (height $h = 0$), velocity changes discretely: $v := -a\ v$, where $0 < a < 1$ is *dampening* constant
- Modeled as a *hybrid system*: mix of discrete and continuous behaviors

# Hybrid Process for Bouncing Ball



# Execution of the Bouncing Ball Process

$h_0 = 5$
$v_0 = 0$



# Definition of Hybrid Process: Syntax

A *hybrid process* HP consists of

1. An asynchronous process P, with
   - continuous ($I_c$) and discrete ($I_d$) input variables I
   - continuous ($S_c$) and discrete ($S_d$) state variables S
   - continuous ($O_c$) and discrete ($O_d$) output variables O
2. A continuous-time invariant CI, a Boolean expression over $S_c$
3. For every $y \in O_c$, a Lipschitz-continuous real-valued expression $h_y(S_c, I_c)$ defining $y$
4. For every $x \in S_c$, a Lipschitz-continuous real-valued expression $f_x(S_c, I_c)$ defining the rate of change of $x$
5. Input, output, internal and timed actions

# Definition of Hybrid Process: Semantics

- Inputs, outputs, states, initial states, internal actions, input actions, output actions: Defined exactly as in the asynchronous model

- Timed actions: Given
  - a state $s_0$, a real-valued time $\delta > 0$ and
  - a continuous input signal $\mathbf{I}(t)$ giving values for $I_c$ over interval $[0, \delta]$,

  signals $\mathbf{S}(t)$ and $\mathbf{O}(t)$ over $[0, \delta]$ are uniquely defined so that

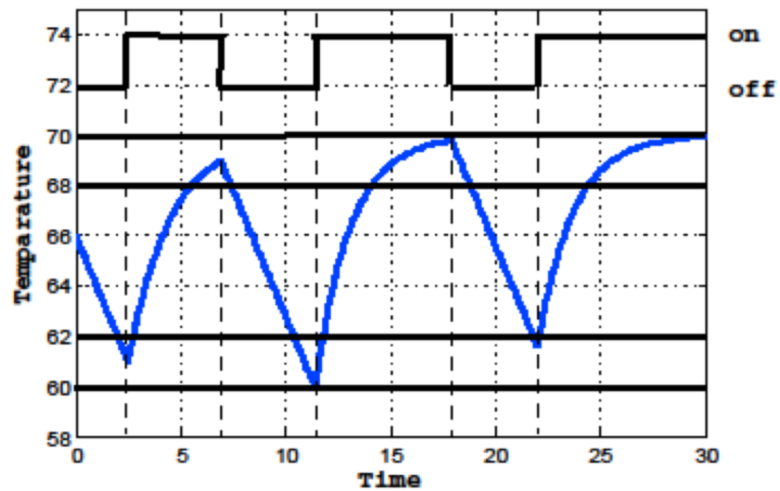
  1. $\mathbf{S}(0) = s_0$
  2. For each $y \in O_c$, $\mathbf{O}_y(t) = h_y(\mathbf{S}(t), \mathbf{I}(t))$
  3. For each $x \in S_c$, $d\mathbf{S}_x(t) = f_x(\mathbf{S}(t), \mathbf{I}(t))$
  4. For all $t \in [0, \delta]$, $\mathbf{S}(t)$ satisfies the invariant $CI$

  **Note:** At all times $t \in [0, \delta]$, discrete state variables stay unchanged

# Executions of Hybrid Processes

Starting from an initial state, execute either

- a timed step of some duration $\delta > 0$ (only continuous variables change) or
- a discrete, instantaneous step: input, output, or internal action

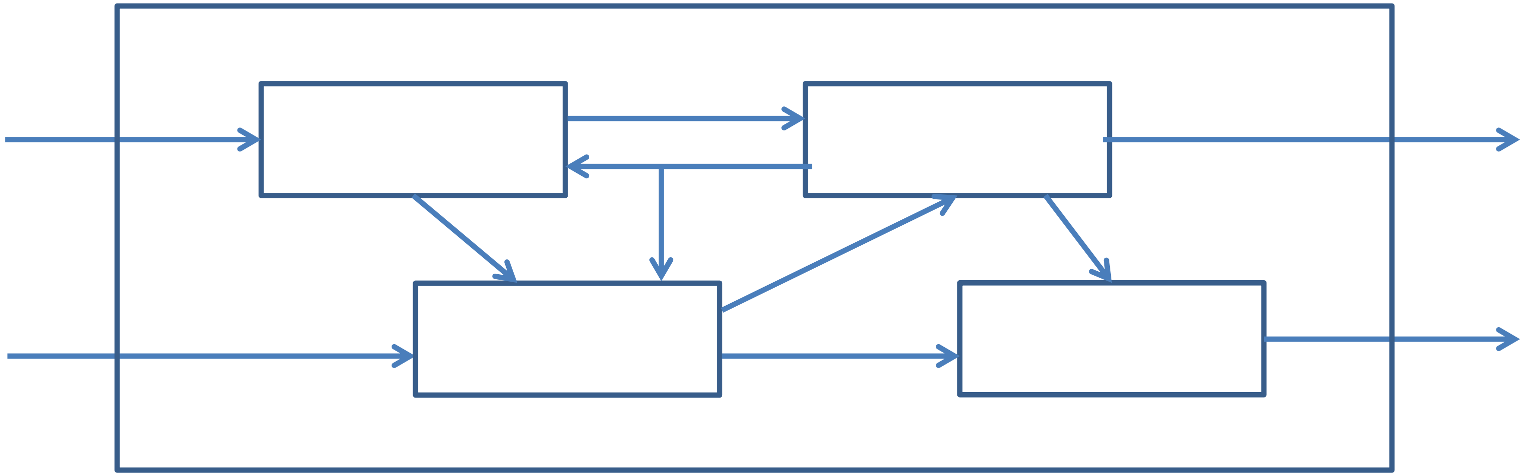



(mode, temp)

(off, 66) –2.5–> (off, 61) –> (on, 61) –3.7–> (on,69.02) –> (off, 69.02)
–4.4–> (off, 60.22) –> (on, 60.22) –7.6–> (on, 69.9) –> (off, 69.9)
–4.1–> (off, 61.7) –> (on, 61.7) –> …

Concepts based on transition systems such as reachable states, safety and liveness requirements, all apply to hybrid systems

# Block Diagrams



- Component processes can now be hybrid processes
  - Need to define instantiation, composition, output hiding

- Channels connecting processes of two types
  1. Sender/receiver communication during discrete steps, as in the asynchronous model
  2. Continuously evolving signals during timed steps, as in the model of continuous-time dynamical systems

# Composition of Hybrid Processes

**Instantiation, variable renaming and output hiding:**

- Defined as usual

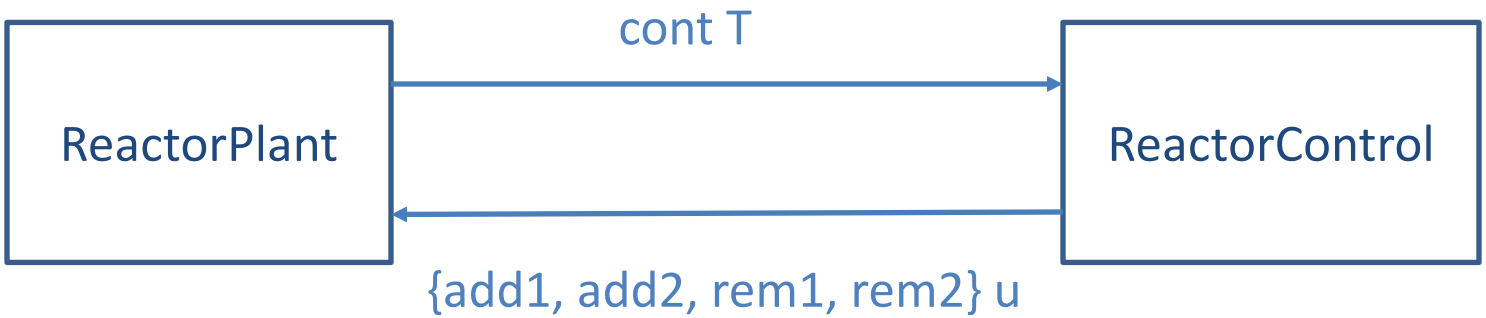
**Composition:**

- Compose discrete parts together as in the asynchronous model
- Compose continuous parts of internal actions together as in dynamical systems
- Generate continuous-time invariants of as conjunction of invariants of component processes

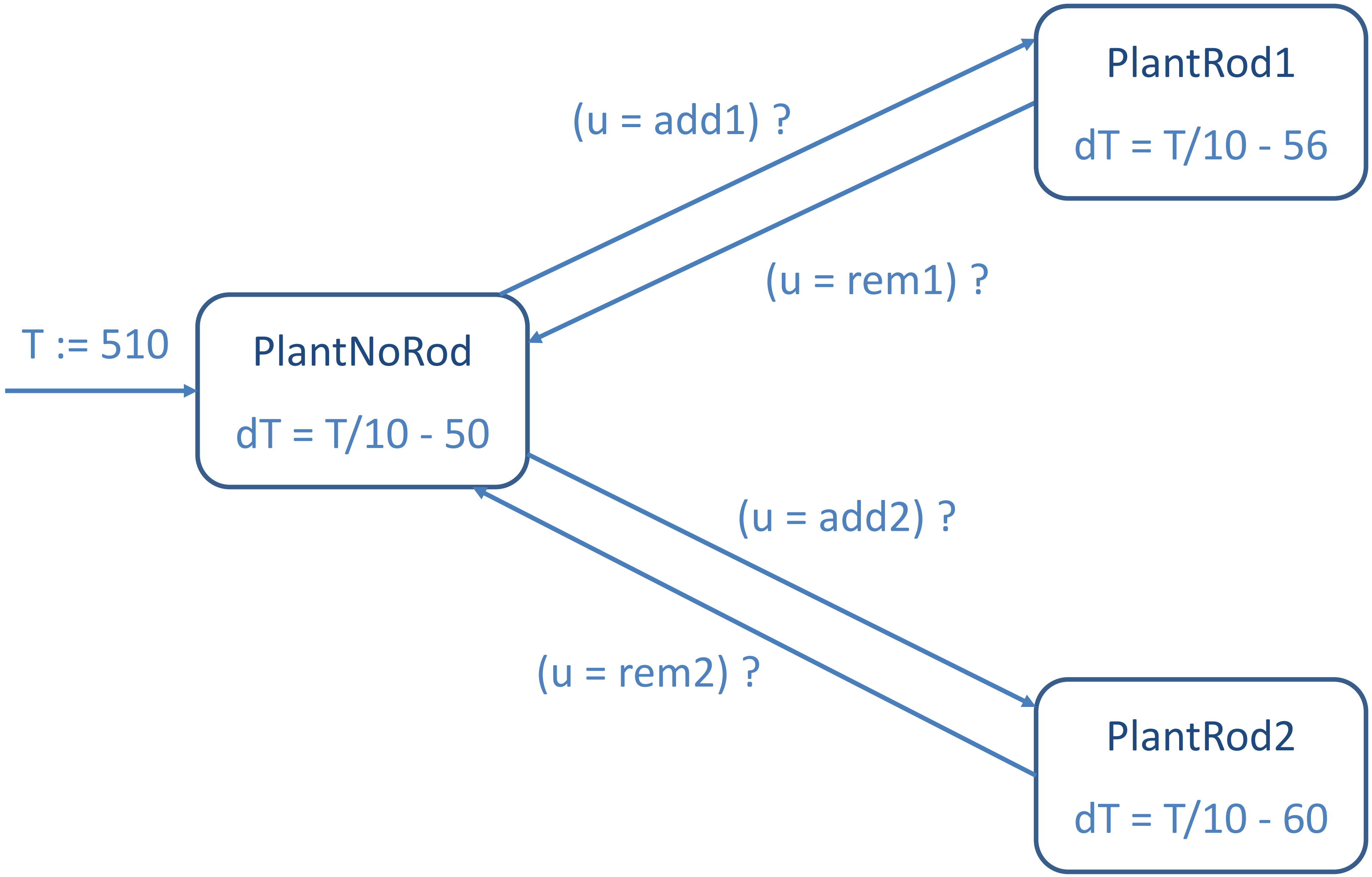
**Compatibility of two hybrid processes:**

- State variables are disjoint and output variables are disjoint
- No cyclic await dependencies among shared input/output variables

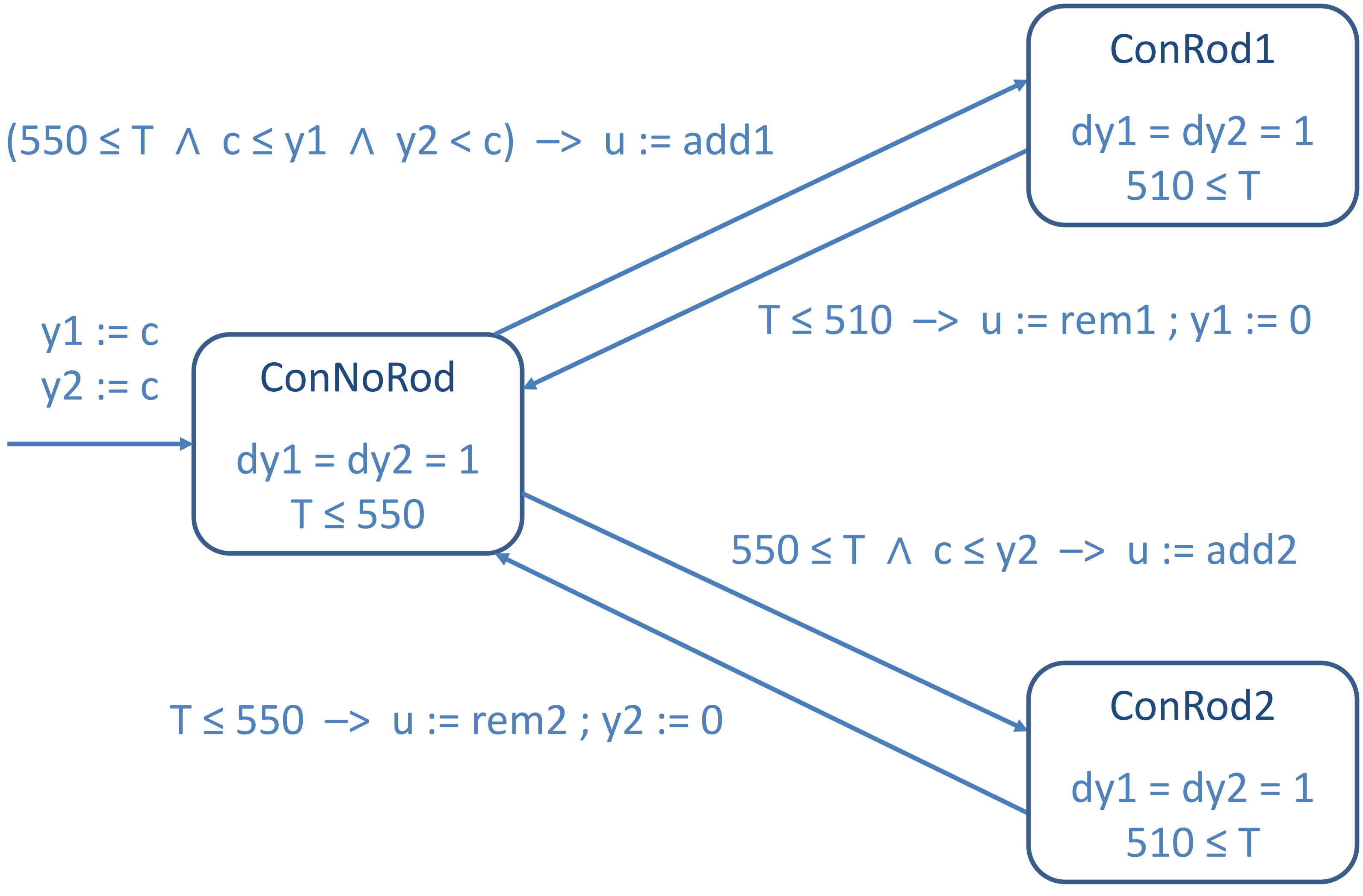
# Nuclear Reactor Example



# Reactor Plant



# Reactor Controller



# Summary of the Model

+ Generalizes timed model

  Variables evolving continuously during a timed action can have complex dynamics, clocks being a very special case

+ Generalizes continuous-time dynamical systems

  Discontinuous changes to system state now can be modeled

+ Generalizes asynchronous model

  Distributed/multi-agent systems can be modeled

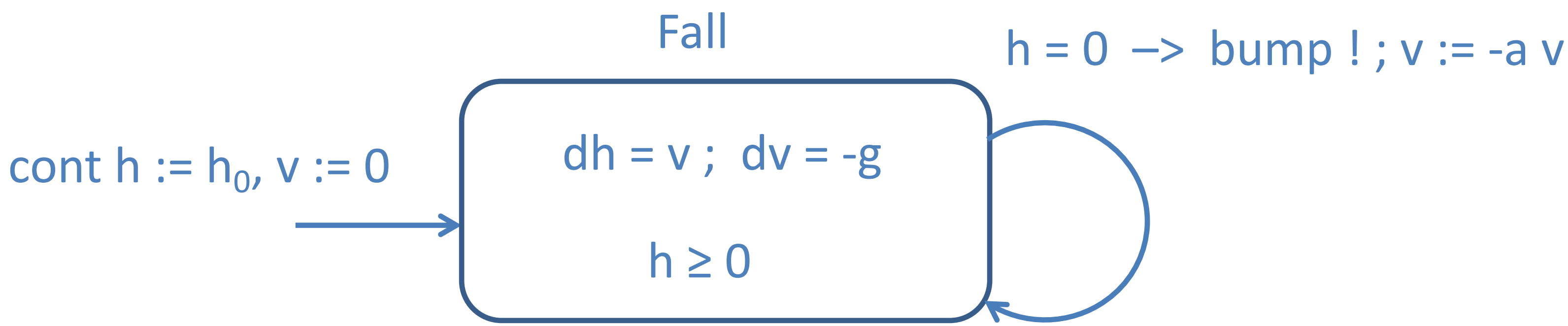
+ Suitable for modeling of cyber-physical systems in full generality

+ Commercial tool support: Modelica, Stateflow/Simulink

– Analysis is challenging

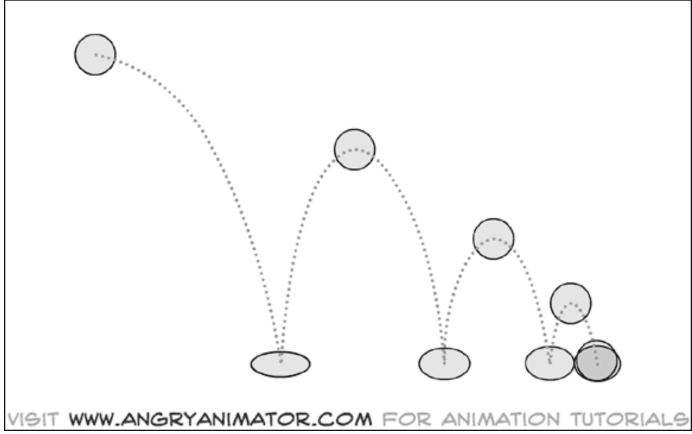
  Even if dynamics in individual modes is linear, due to discrete changes it is not possible to obtain closed-form solutions, or general theorems about stability

# Analysis of Bouncing Ball Model



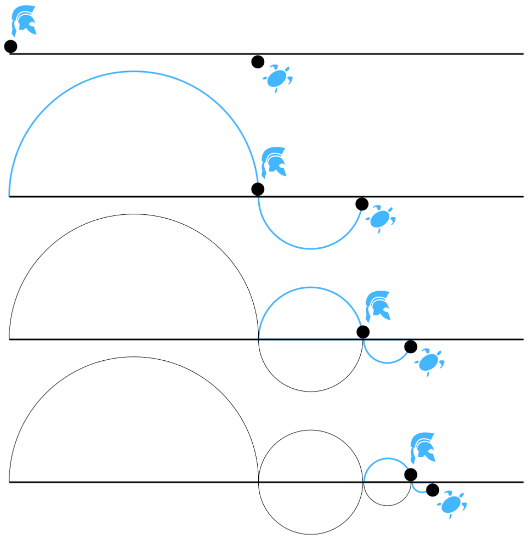

| | |
|---|---|
| Evolution in height during first bounce: | $\mathbf{h}(t) = h_0 - g\, t^2 / 2$ |
| Time at which first bump occurs: | $t_1 = \text{Sqrtr}\ (2\, h_0 / g)$ |
| Speed just before first bump occurs: | $v_1 = \text{Sqrt}\ (2\, g\, h_0)$ |
| Speed just after first bump : | $v_2 = a\, v_1$ |
| Evolution of height during second bounce: | $\mathbf{h}(t) = v_2\, t - g\, t^2 / 2$ |
| Time between first and second bump: | $t_2 = 2\, v_2 / g$ |
| Speed just before second bump occurs: | $v_2$ and after 2nd bump $v_3 = a\, v_2$ |

# Modeling a Bouncing Ball




- Speed after k bumps: $a^k v_1$
- Duration between $k^{th}$ and following bump: $a^k v_1 / g$
- Sum of durations between successive bumps converges to $v_1 (1 + a) / (1 - a)$
- Infinitely many discrete actions in finite time: Zeno behavior!

# Zeno's Paradox



- Described by Greek philosopher Zeno in context of a race between Achilles and a tortoise
- Tortoise has a head start over Achilles, but is much slower
- In each discrete round, suppose Achilles is $d$ meters behind at the beginning of the round
- During the round, Achilles runs $d$ meters, but by then, tortoise has moved a little bit further
- At the beginning of the next round, Achilles is still behind, by a distance of $a\,d$ meters, with $0 < a < 1$
- By induction, if we repeat this for infinitely many rounds, Achilles will never catch up!
- If the sum of durations between successive discrete actions converges to a constant $K$, then an execution with infinitely many discrete actions describes behavior only up to time $K$ (and does not tell us the state of the system at time $K$ and beyond)

# Formalization

- An infinite execution of a hybrid process HP is of the form $s_0 -t_1\!\!-> s_1 -t_2\!\!-> s_2 -t_3\!\!-> s_3 \ldots$, where $t_i$ is the duration of $i^{th}$ step
  - Input/output/internal actions are instantaneous (duration 0)

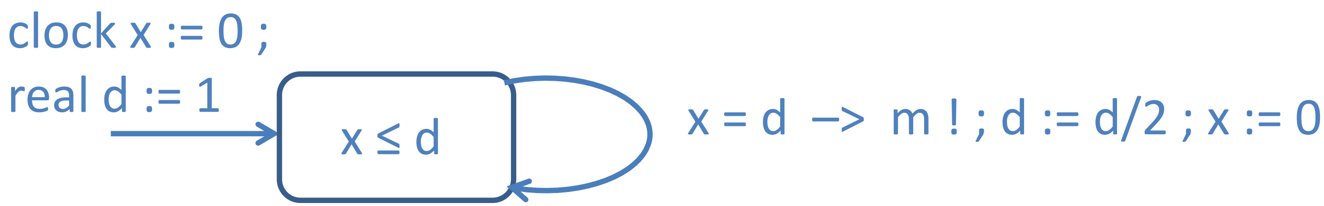
- An infinite execution is called
  - *Zeno* if the infinite sum of all the durations is bounded above by a constant, (e.g., $\sum_{n>0} 1/(n^2+n) = 1$) and
  - *non-Zeno* if the sum diverges (e.g., $\sum_{n>0} 1/n$)

- A state s of the process HP is called
  - *Zeno* if every execution starting in state s is Zeno
  - *Non-Zeno* if there is some non-Zeno execution starting in s

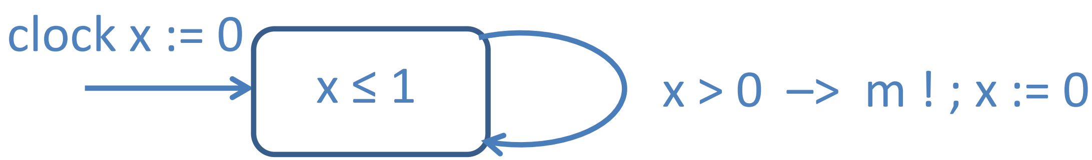
# Formalization

- A hybrid process HP is called *non-Zeno* if every reachable state of HP is non-Zeno
  - At every point during an execution it is possible for time to diverge
- A Zeno system could end up in a state from which duration between successive steps must get smaller and smaller
- Examples
  - Thermostat: non-Zeno
  - Bouncing ball: Zeno

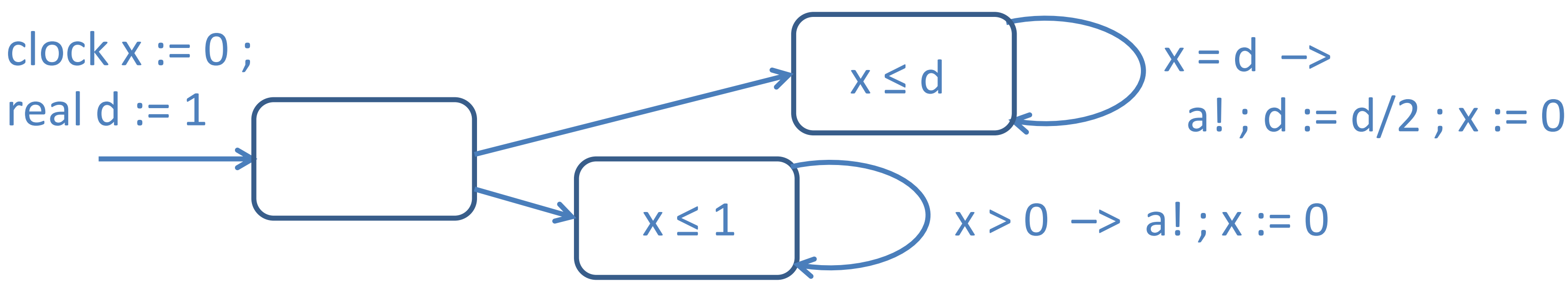
# Zeno vs Non-Zeno




Zeno! Every possible execution is Zeno




Non-Zeno! Some executions are Zeno and some are non-Zeno



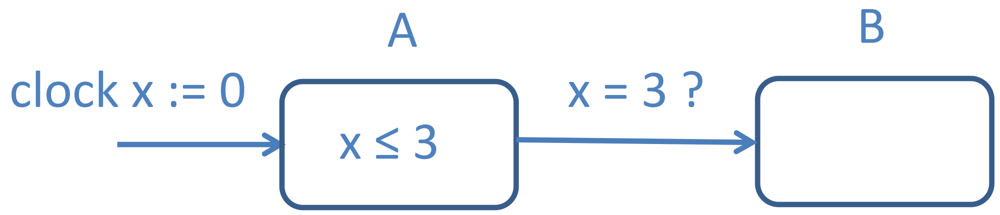

Zeno! System may end up in a state from which only Zeno executions are possible

# Zeno Processes and Reachability

How does existence of Zeno processes influence analysis?

**Recall:**

- A state s of a system H is *reachable* if there exists a finite execution starting in an initial state and ending in state s
- A property P is *invariant* for H all reachable states satisfy P

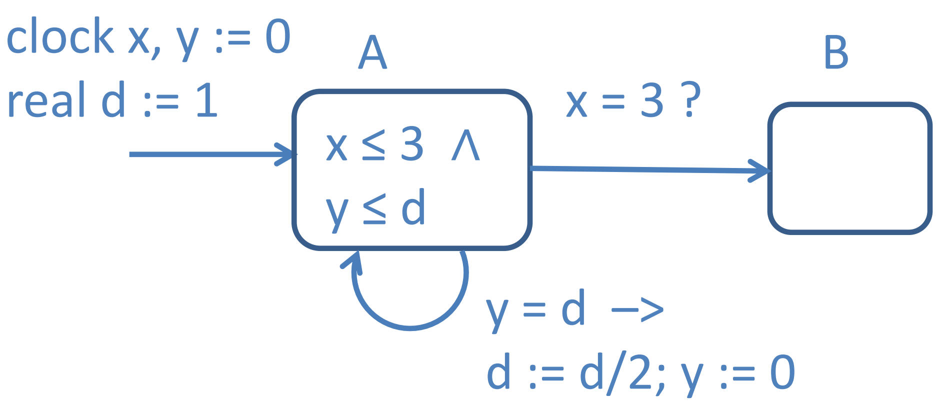



Is mode B reachable ?

# Zeno Processes and Reachability

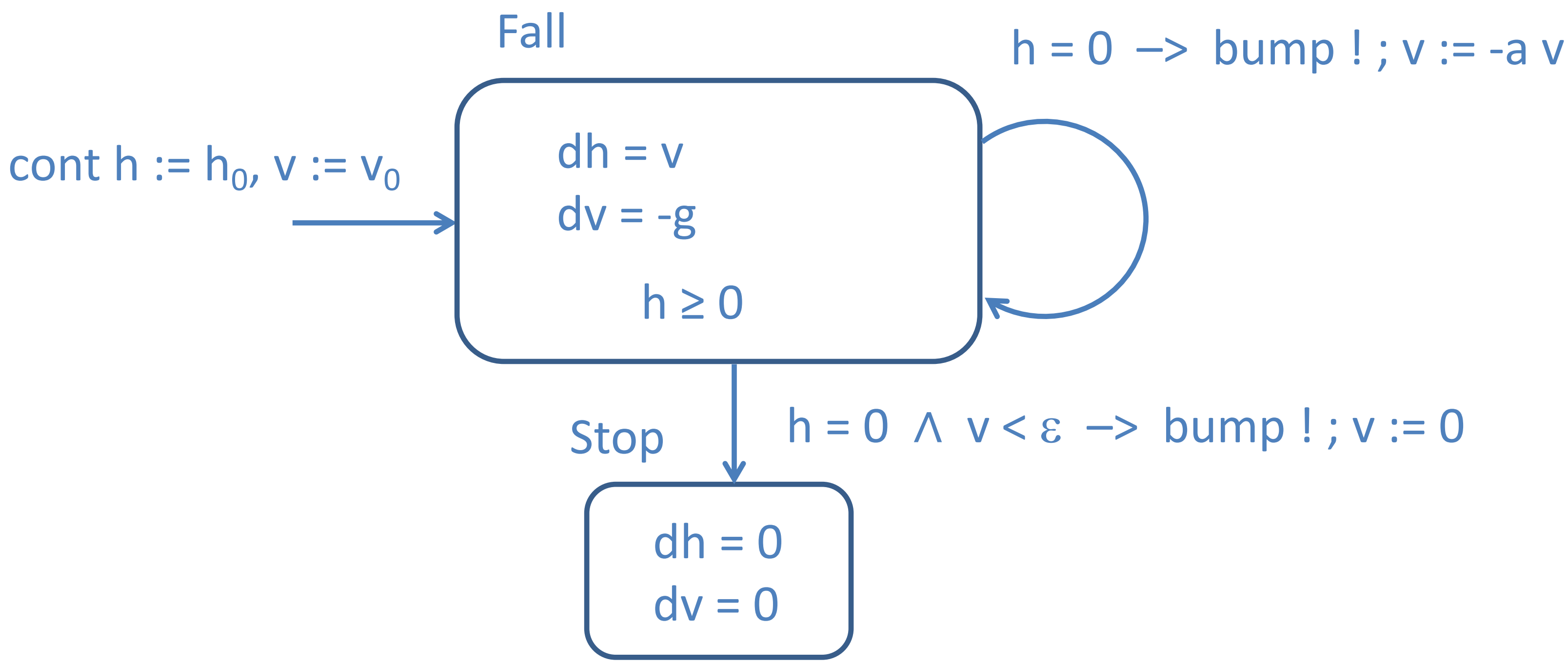

composition

Is mode B reachable ?




Presence of a Zeno process in the system can stop time from advancing and make states of other processes unreachable!

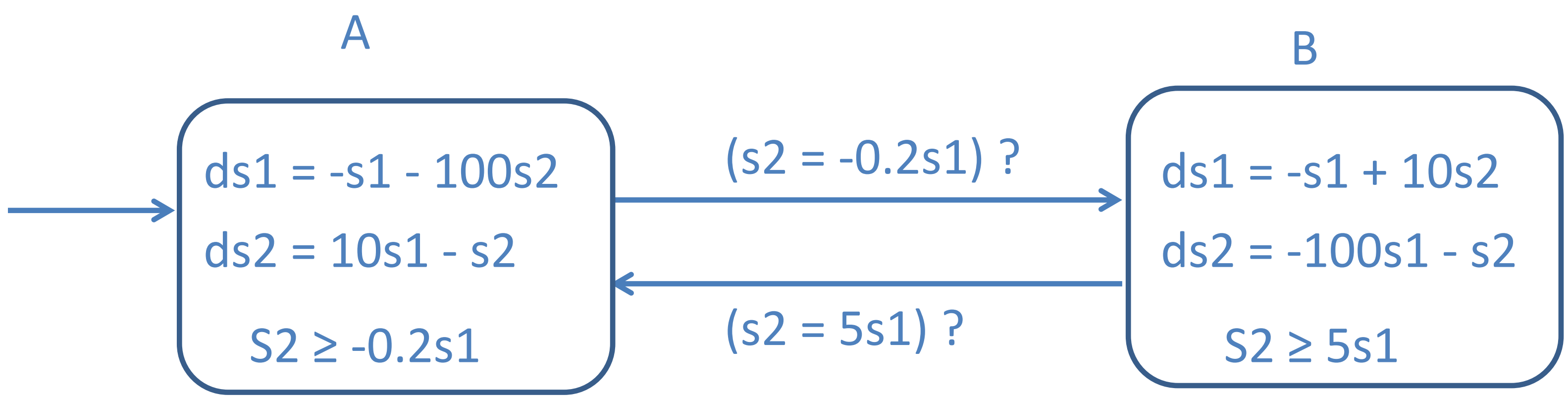
# Making Bouncing Ball Non-Zeno




If velocity is too small, stop modeling dynamics precisely

In this model, there is a lower bound on duration between successive bumps

# Stability of Hybrid Systems

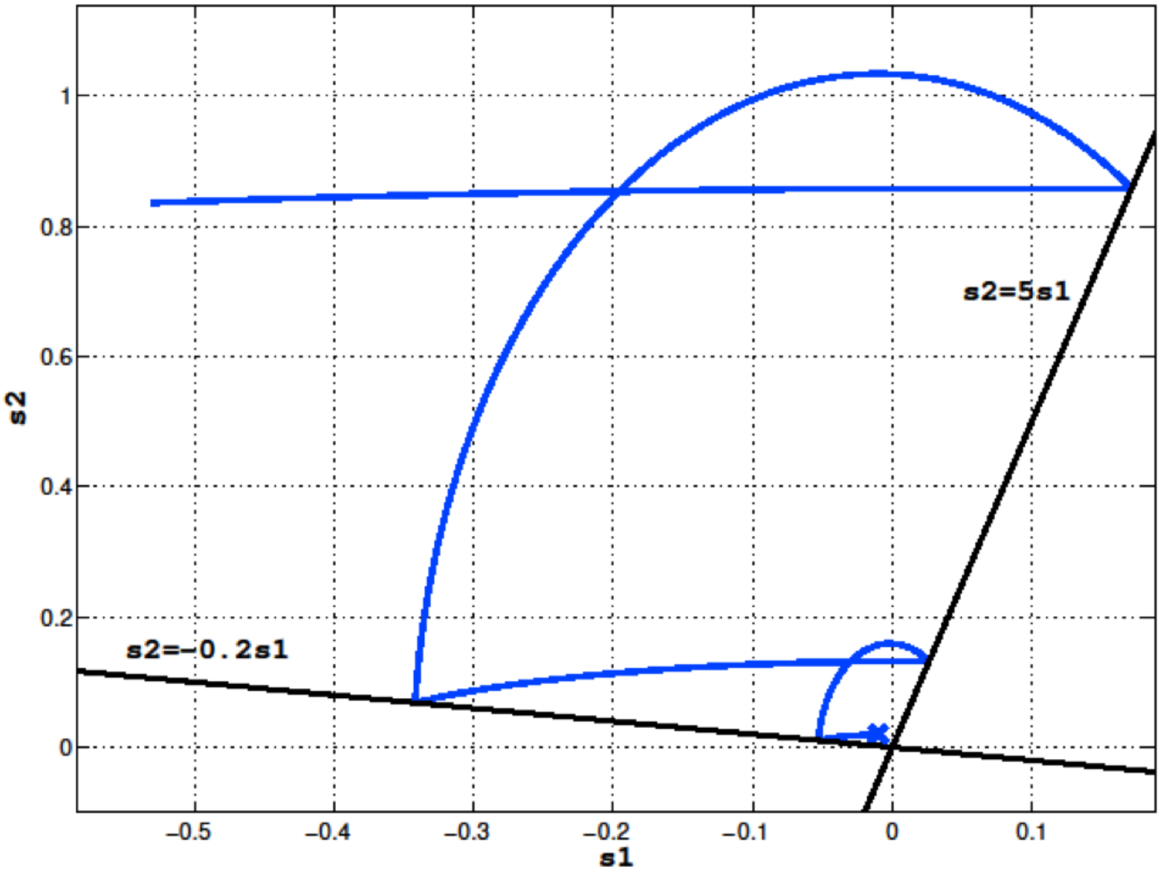



Is the dynamics in mode A stable?

Is the dynamics in mode B stable?

Each mode has stable dynamics but switching causes instability!

# Stability of Hybrid Systems



# Credits

Notes based on Chapter 9 of

**Principles of Cyber-Physical Systems**
by Rajeev Alur
MIT Press, 2015