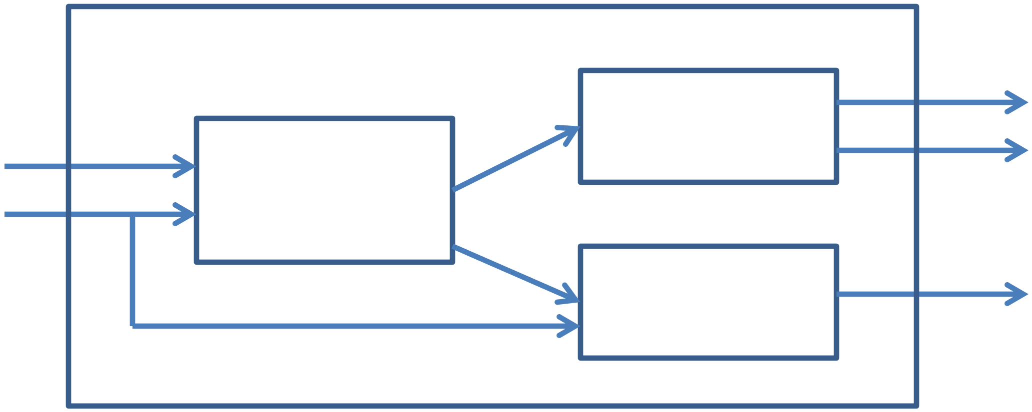
CS:4980
Foundations of Embedded Systems

Dynamical Systems
Part I

Dynamical Systems

- Controller interacting with the physical world via sensors and actuators
  - Thermostat controlling temperature
  - Cruise controller regulating speed of a car
- System variables: (measures of) physical quantities evolving continuously over time
  - Temperature, pressure, velocity …
- Continuous-time models using differential equations

Model-Based Design



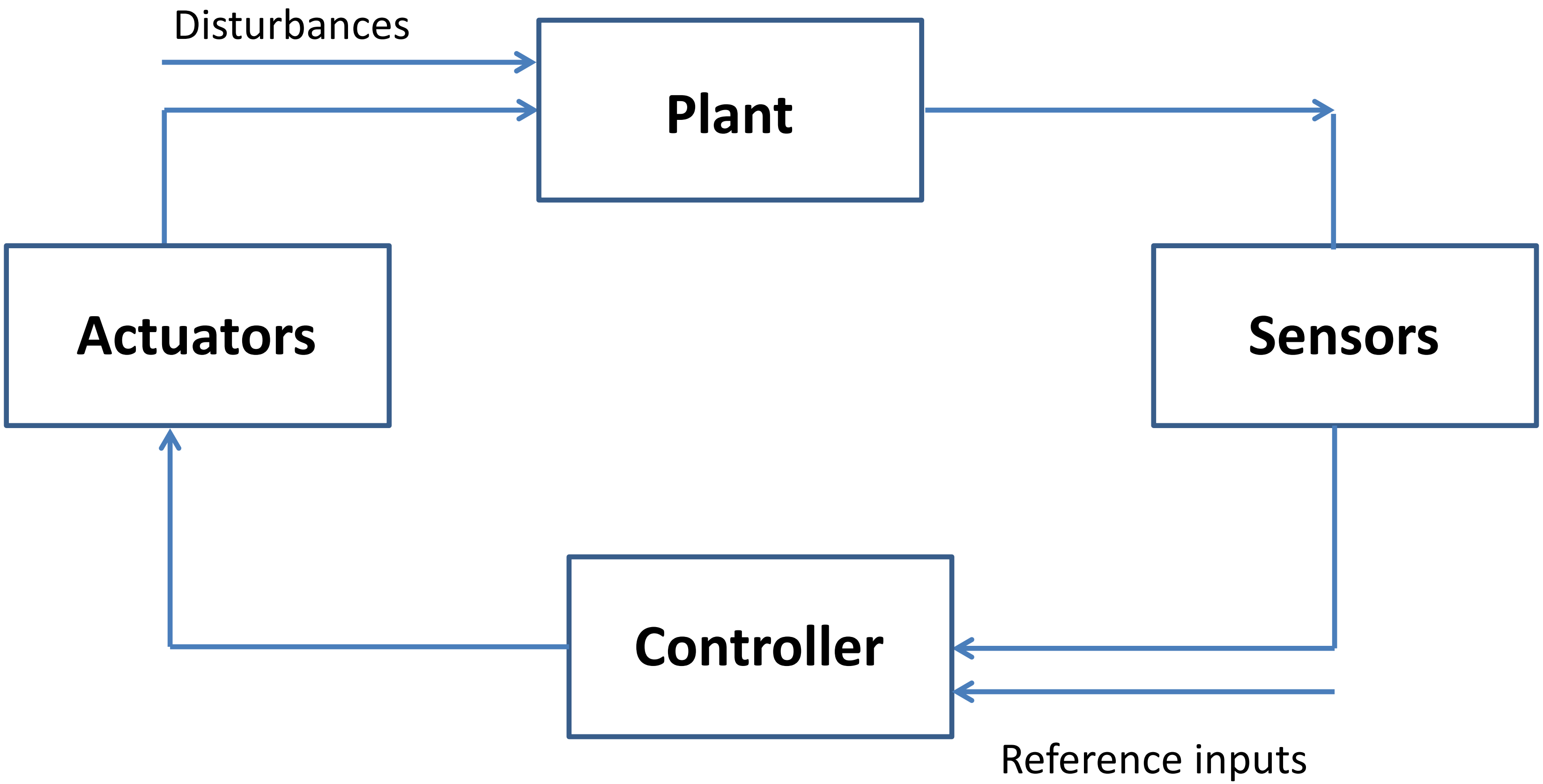
Block Diagrams

- Widely used in industrial design
- Tools: Simulink, Modelica, RationalRose…

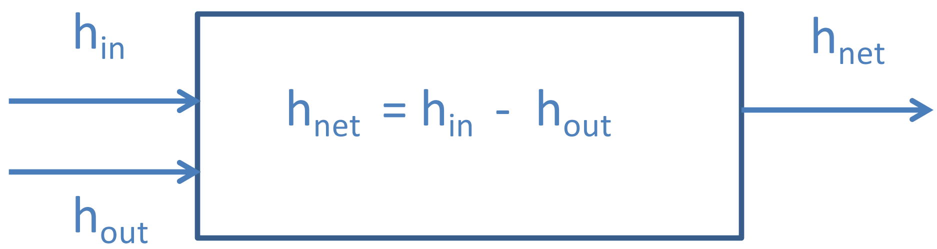
Key question: what is the execution semantics?

- Similar to synchronous model but continuous-time instead of discrete-time

Traditional Feedback Control Loop



Example: Heat Flow



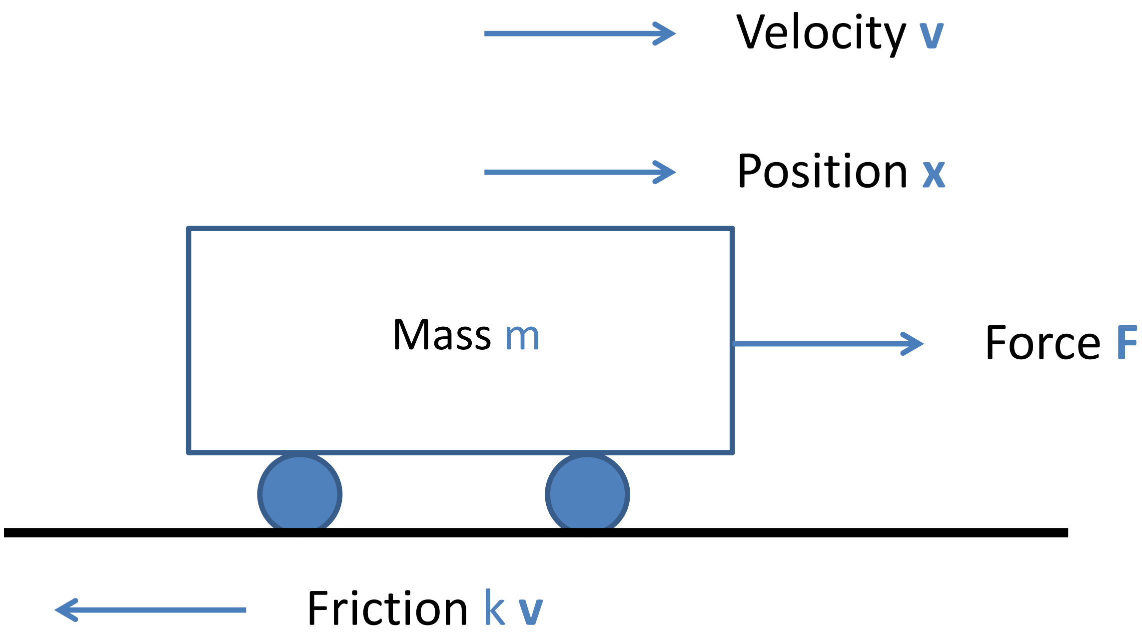
Input variables: h_{in} and h_{out} of type real

Output variable: h_{net} of type real

State variables: none

- *Signal*: assignment of values to variables as function of time t
- At each time t, value of output signal $h_{net}(t)$ equals $h_{in}(t) – h_{out}(t)$
- Output as a function of inputs/state, specified using algebraic equations (as opposed to assignments)

Car Model



- **v**, **x**, **F** and **v** are all functions of time t; k is a friction constant
- Newton's law of motion gives:

$$\mathbf{F} - k\,\mathbf{v} = m\, d^2\mathbf{x}/dt$$

Notation

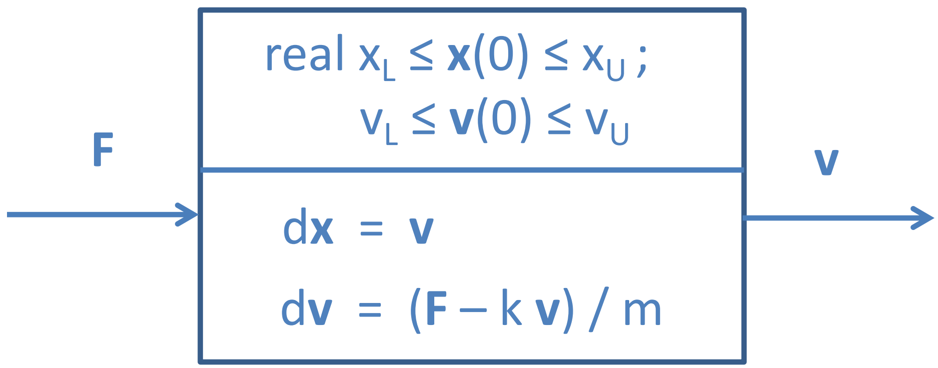
First derivative of function $f(t)$ with respect to t :

- $df(t)/dt$ (full notation)
- df/dt (with the understanding that f is a function of t)
- df (dimension t is implicit)
- f' (same as df)
- $\dot{f}$ (same as df)

Second derivative of function $f(t)$ with respect to t :

- $d^2f(t)/dt^2$ (full notation)
- d^2f/dt^2 (with the understanding that f is a function of t)
- d^2f (dimension t is implicit)
- f'' (same as d^2f)
- $\ddot{f}$ (same as d^2f)

Continuous-time Component Car



- The value of output variables is defined in terms of input and state variables
- For each state variable **s**, its *rate of change* d**s**/dt is defined in terms of input and state variables

Executions of Car

Input signal: function **F** : $real_{>=0} \rightarrow real$ that gives value of force over time

- should be continuous or piecewise-continuous

Given an initial state (x_0, v_0) and input signal **F**, the execution of the system is defined by state-signals (ie, functions)

$$\mathbf{x}: real_{>=0} \rightarrow real \quad \text{and} \quad \mathbf{v}: real_{>=0} \rightarrow real$$

that satisfy the initial-value problem:

1. $\mathbf{x}(0) = x_0$
2. $\mathbf{v}(0) = v_0$
3. $d\mathbf{x}(t)/dt = \mathbf{v}(t)$
4. $d\mathbf{v}(t)/dt = d^2\mathbf{x}(t)/dt^2 = (\mathbf{F}(t) - k\,\mathbf{v}(t)) / m$

Executions of Car: Example 1

Suppose force $\mathbf{F}(t)$ is always 0, and initial position is 0.

We need to solve:

- $\mathbf{x}(0) = 0$
- $\mathbf{v}(0) = v_0$
- $d\mathbf{x}/dt = \mathbf{v}$
- $d\mathbf{v}/dt = - k\, \mathbf{v} / m$

Solution:

- Velocity decreases exponentially fast, converging to 0

  $$\mathbf{v}(t) = v_0\, e^{-k\, t / m}$$

- Position converges exponentially fast to $m\, v_0 / k$

  $$\mathbf{x}(t) = (m\, v_0 / k)\, (1 - e^{-k\, t / m})$$

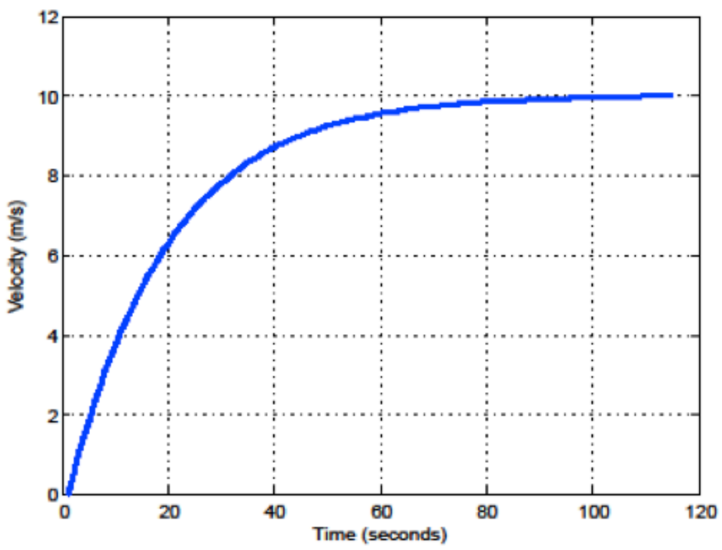
Executions of Car: Example 2

Suppose initial position is 0, initial velocity is 0, and force is constant F_0. Then, to get executions, we need to solve:

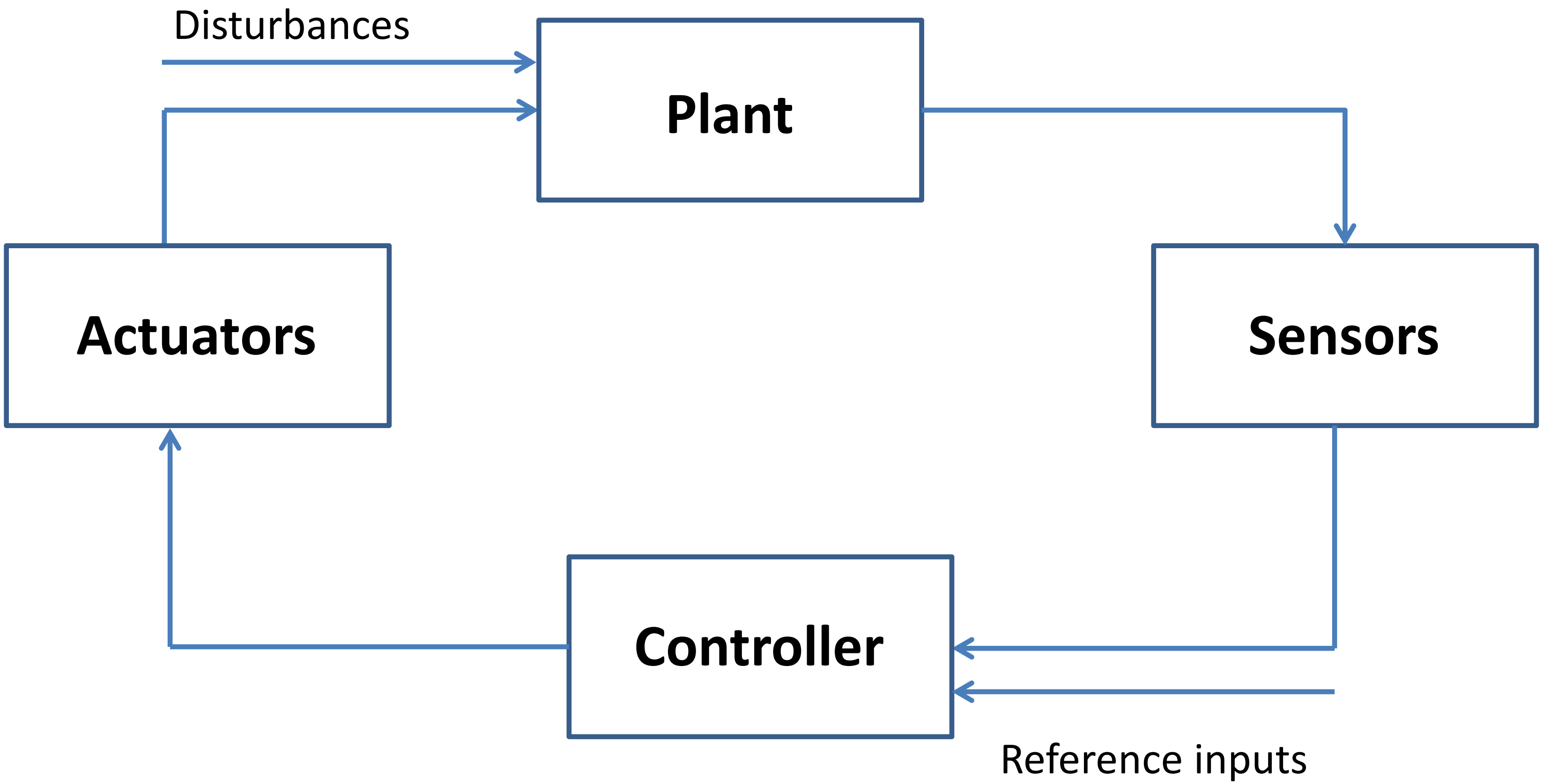
- $\mathbf{x}(0) = 0$
- $\mathbf{v}(0) = 0$
- $d\mathbf{x}/dt = \mathbf{v}$
- $d\mathbf{v}/dt = (F_0 - k\ \mathbf{v}) / m_c$

Compute the solution using MATLAB

- Mass m_c = 1000kg
- Coefficient of friction $k = 50$
- Force F_0 = 500 Newton
- Velocity converges to 10 m/s



Traditional Feedback Control Loop



Continuous-Time Component Definition

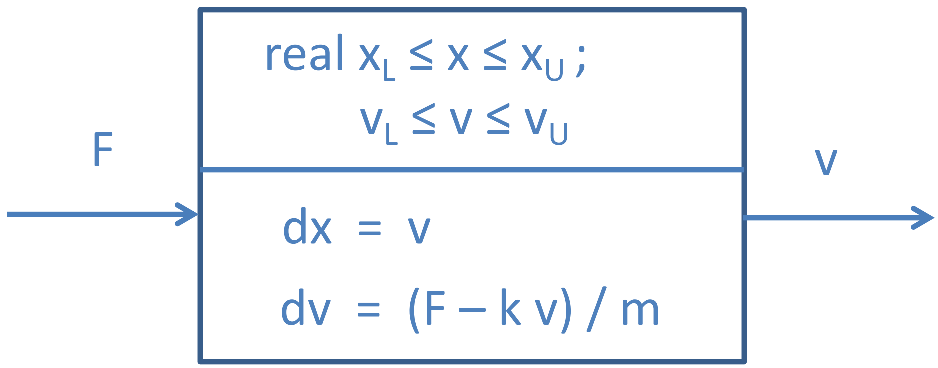
- Set I of real-valued *input variables*; type is either real or interval of real, real[L, U]
- Set O of real-valued *output variables*
- Set S of real-valued *state variables*
- *Initialization* Init specifying set [Init] of initial states
- For each output var y, a real-valued expression h_y over I U S
- For each state variable x, a real-valued expression f_x over I U S

Execution

Given an input signal $\mathbf{I}(t) : real_{>=0} \rightarrow real^{|\mathbf{I}|}$, an *execution* consists of a differentiable state signal $\mathbf{S}(t)$ and output signal $\mathbf{O}(t)$ such that

1. $\mathbf{S}(0)$ is in [Init]
2. For each output var y and time t, $\mathbf{y}(t) = h_y(\mathbf{I}(t), \mathbf{S}(t))$
3. For each state var x, $d\mathbf{x}(t)/dt = f_x(\mathbf{I}(t), \mathbf{S}(t))$

Continuous-time Component Car



- The value of output variables is defined in terms of input and state variables
- For each state variable s, its *rate of change* ds is defined in terms of input and state variables

Existence and Uniqueness

- Given an input signal $\mathbf{I}(t)$, when are we guaranteed that the system has at least/exactly one execution?
- The input signal should be continuous (or at least piecewise continuous), but answer also depends on right-hand-sides of equations defining state and output dynamics
- Related to classical theory of Ordinary Differential Equations (ODEs)
- Consider the initial value problem

  $d\mathbf{x}/dt = F(\mathbf{x})$; $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}(t)$ is n-dimensional vector
- When do we have a unique differentiable function as a solution for $\mathbf{x}$?

Solution Existence

Initial value problem:

$d\mathbf{x}/dt = F(\mathbf{x})$; $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}(t)$ is n-dimensional vector

- ❑ The problem has a solution $\mathbf{x}(t)$ if function F is continuous

- ❑ Example when solution does not exist:

  dx/dt = if (x = 0) then 1 else 0

- ❑ It is natural to require all right-hand-side expressions h_y and f_x in definition of a continuous-time component to be continuous
  - Discontinuous case -> Hybrid Systems (Chap. 9)

Continuous Function

Definition of continuity relies on a given notion of distance ||_|| between points (e.g., Euclidean distance)

A function $f: real^m \rightarrow real^n$ is *(uniformly) continuous* if

for all $\varepsilon > 0$,

there is a $\delta > 0$ such that

for all $u, v \in Real^m$,

if $||u - v|| < \delta$ then $||f(u) - f(v)|| < \varepsilon$

Solution Uniqueness

Initial value problem:

$d\mathbf{x}/dt = G(\mathbf{x})$; $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}(t)$ is n-dimensional vector

Theorem: There exists a unique solution $\mathbf{x}(t)$ if the function G is Lipschitz-continuous

Examples:

- A linear function such as $(F - k\,v) / m$ is Lip-continuous
- Quadratic function x^2 is Lip-continuous if domain of x is bounded

Counterexamples:

- $x^{1/3}$ is not Lip-continuous: $dx/dt = x^{1/3}$; $x(0) = 0$ has multiple solutions:
  1. $\mathbf{x}(t) = 0$
  2. $\mathbf{x}(t) = (2t/3)^{3/2}$

Lipschitz-Continuous Function

Informally, Lipschitz-continuous means that there is a constant upper bound on how much a function's output changes

A function $f: real^m \rightarrow real^n$ is *Lipschitz-continuous* if there exists a constant c such that

for all u, v in $real^m$,

$$||f(u) - f(v)|| \leq c \; ||u - v||$$

Lipschitz-Continuous Component

Definition: A continuous-time component has *Lipschitz-continuous dynamics* if

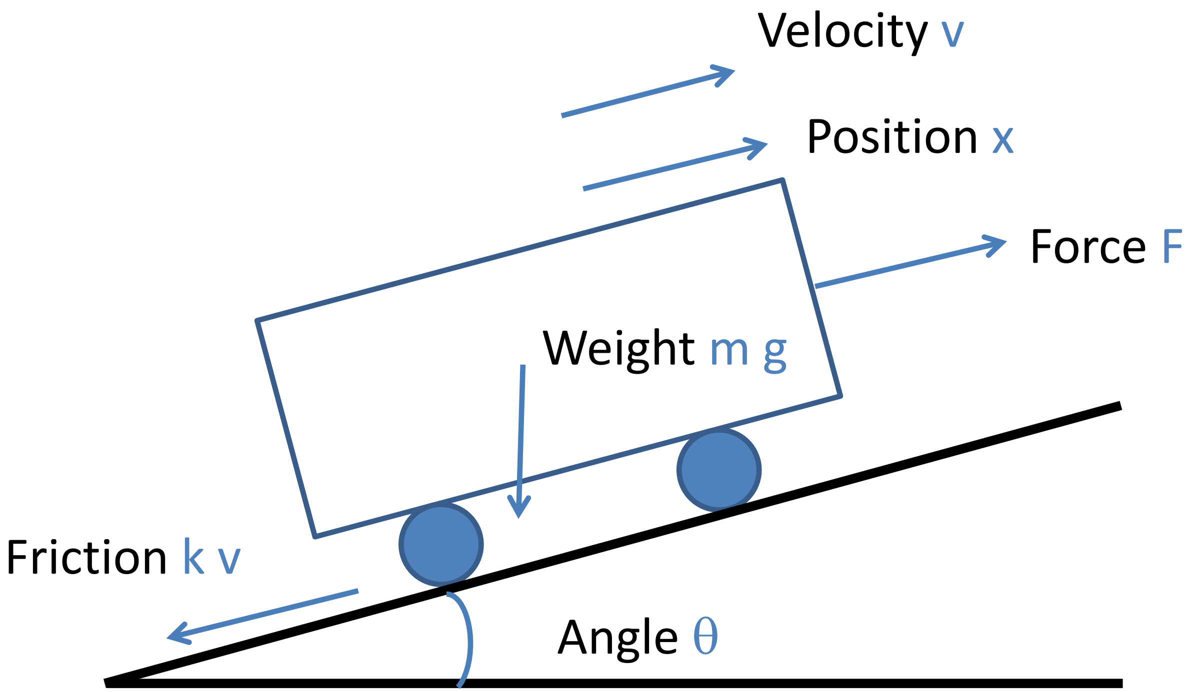
- each expression h_y corresponding to output variable y is a Lipschitz-continuous function of $I \cup S$
- each expression f_x corresponding to state variable x is a Lipschitz-continuous function over $I \cup S$

Theorem: Given a continuous input signal $\mathbf{I}(t)$, a component with Lipschitz-continuous dynamics has unique and continuous response signals $\mathbf{S}(t)$ and $\mathbf{O}(t)$

Note: Continuity of output signals means they can be fed to other components in a block diagram

Henceforth, we will consider only Lipschitz-continuous components

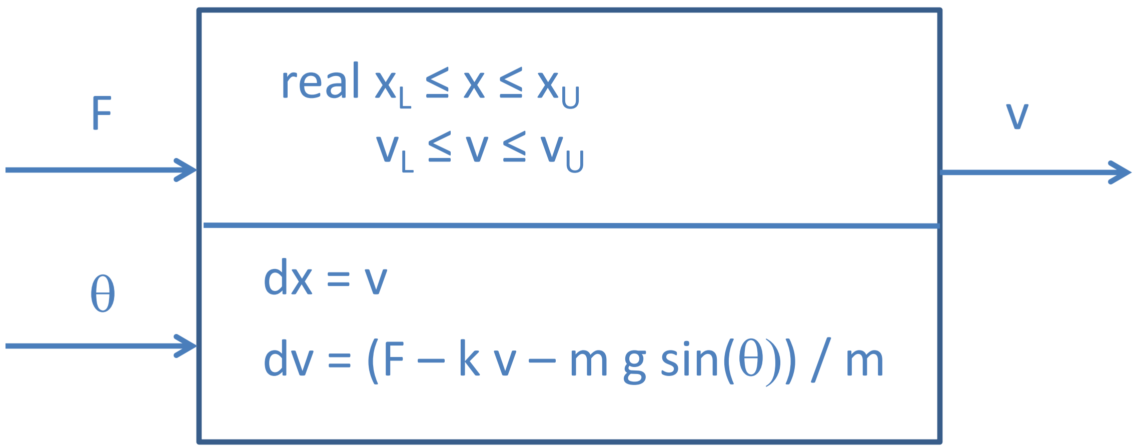
Car on a non-level road



Newton's law of motion gives

$$F - k\,v - m\,g\,\sin(\theta) \;=\; m\,d^2x/dt$$

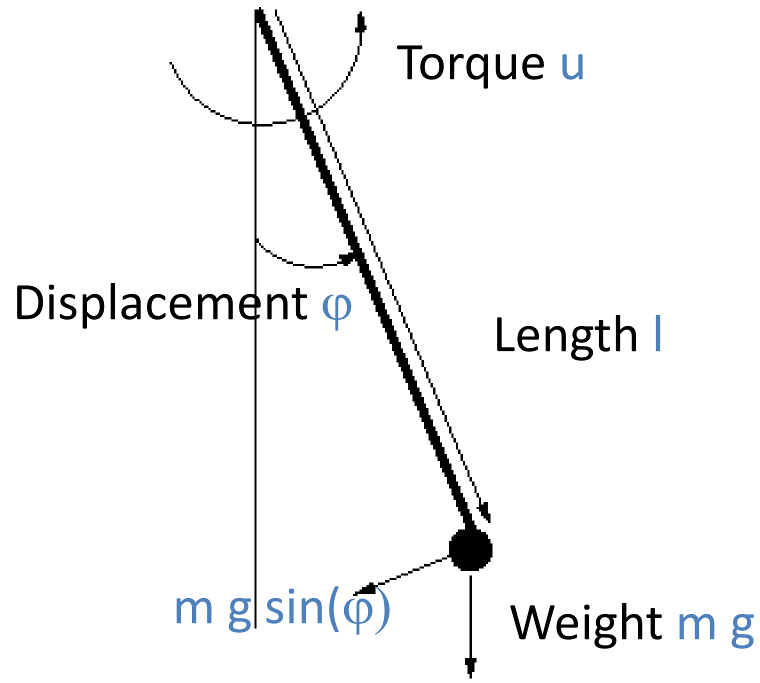
Continuous-time Component Car 2



The road's slope, denoted by θ, models disturbance, or an uncontrolled input

Design problem: Find a controller with v as input and F as output such that the composed system works correctly for all continuous input signals $\mathbf{q}(t)$ for θ, with $\mathbf{q}(t)$ always in $[-\pi/6, \pi/6)$

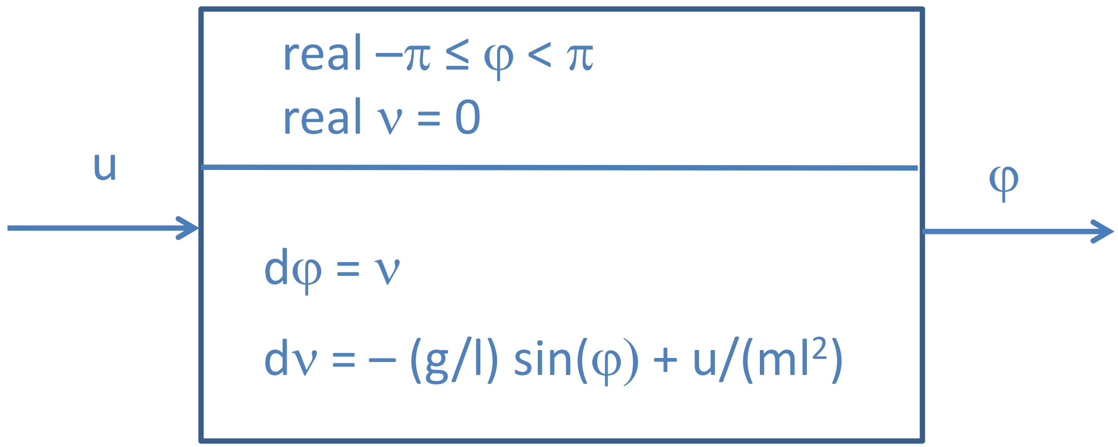
Simple Pendulum



- External torque applied by the motor at the pivot: u
- Dynamics captured by second-order non-linear differential equation:

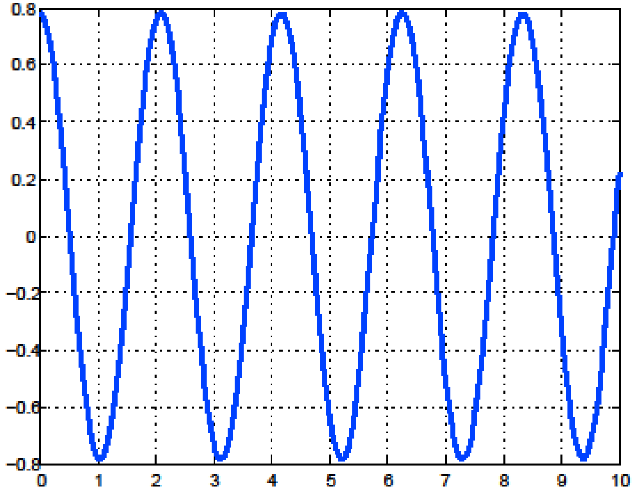
$$m\, l^2\, (d^2\varphi/dt^2) \;=\; u - m\, g\, l \sin(\varphi)$$

Pendulum Model



Angular Displacement

- External torque = 0; Initial displacement = $\pi/4$
- Oscillatory motion plotted by MATLAB
- What are the equilibria of this pendulum ?



Credits

Notes based on Chapter 6 of

Principles of Cyber-Physical Systems
by Rajeev Alur
MIT Press, 2015