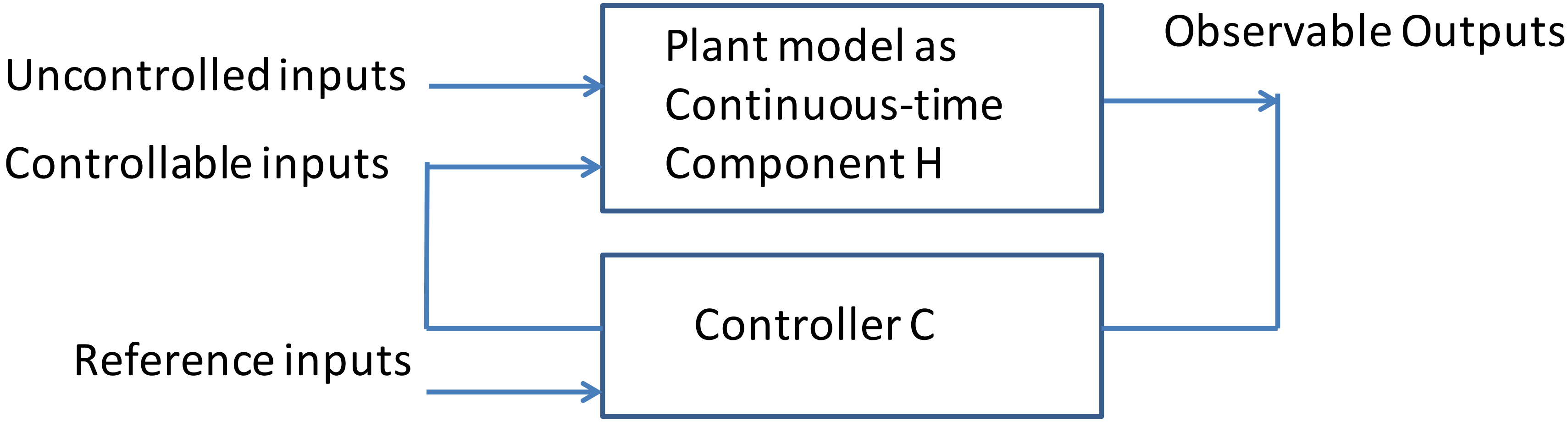
CS:4980
Foundations of Embedded Systems

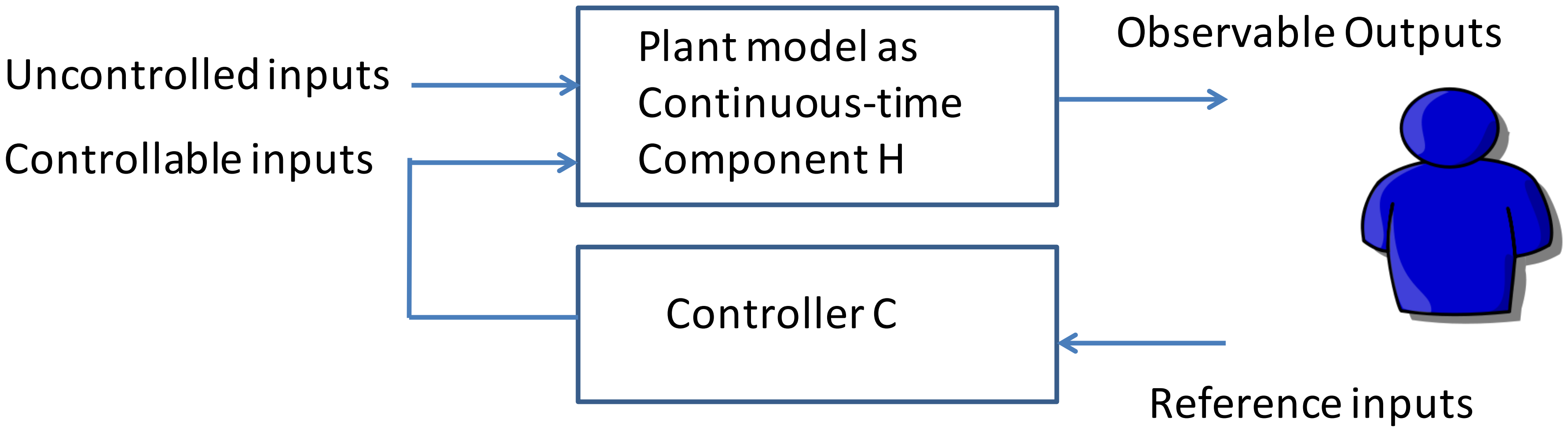
Dynamical Systems
Part IV

Control Design Problem



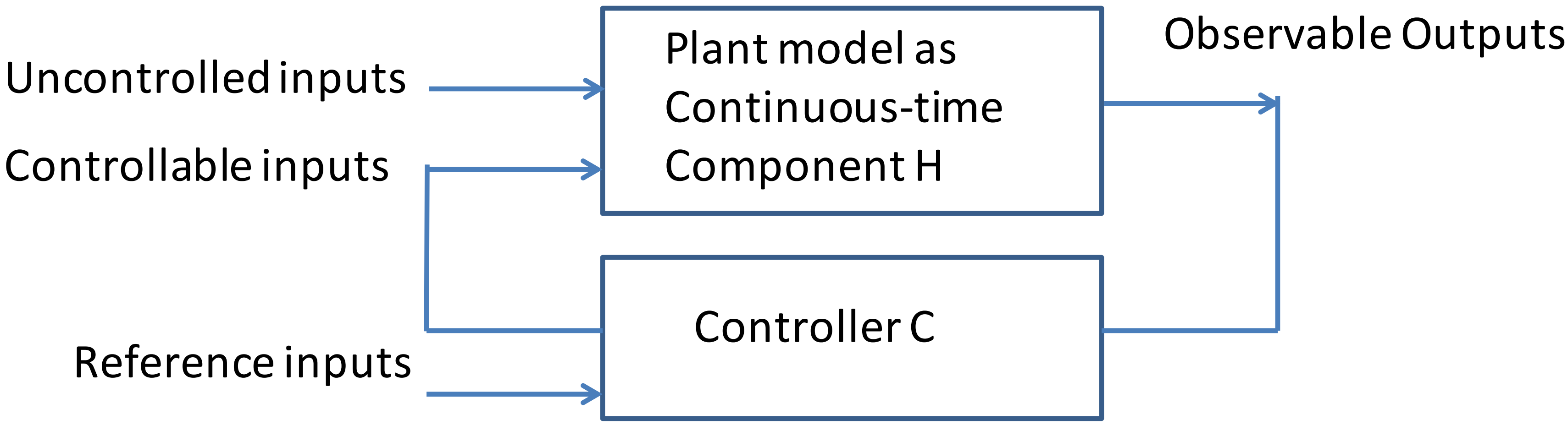
- Design a controller C so that the composed system C || H is stable
- Reference inputs are high-level commands supplied by humans (e.g. desired speed of the car, temperature in the room)
- Controller should satisfy additional safety/liveness requirements corresponding to reference inputs (e.g. speed of car eventually becomes close to desired cruising speed)

Open Loop Controller



- ❑ Plant outputs not fed to the controller
  - Benefit: Sensors not needed (less expensive)
- ❑ Controller simply maps reference inputs to controllable inputs
  - Knowledge of plant dynamics hard-coded in this algorithm
- ❑ Human intervention typically necessary to maintain acceptable performance

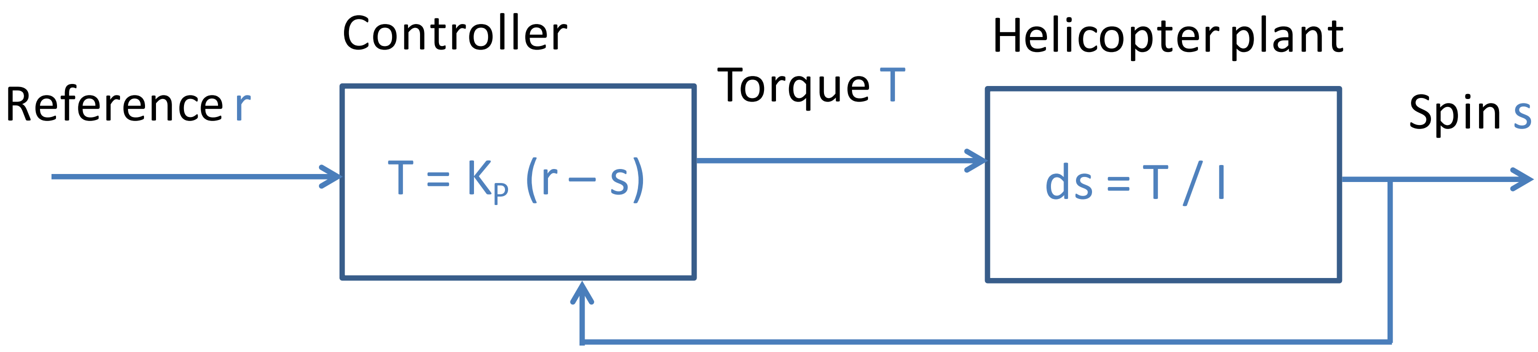
Feedback Controller



- Controller adjusts controllable inputs in response to outputs
  - Can respond better to variations in disturbances
  - Performance depends on how well outputs can be measured

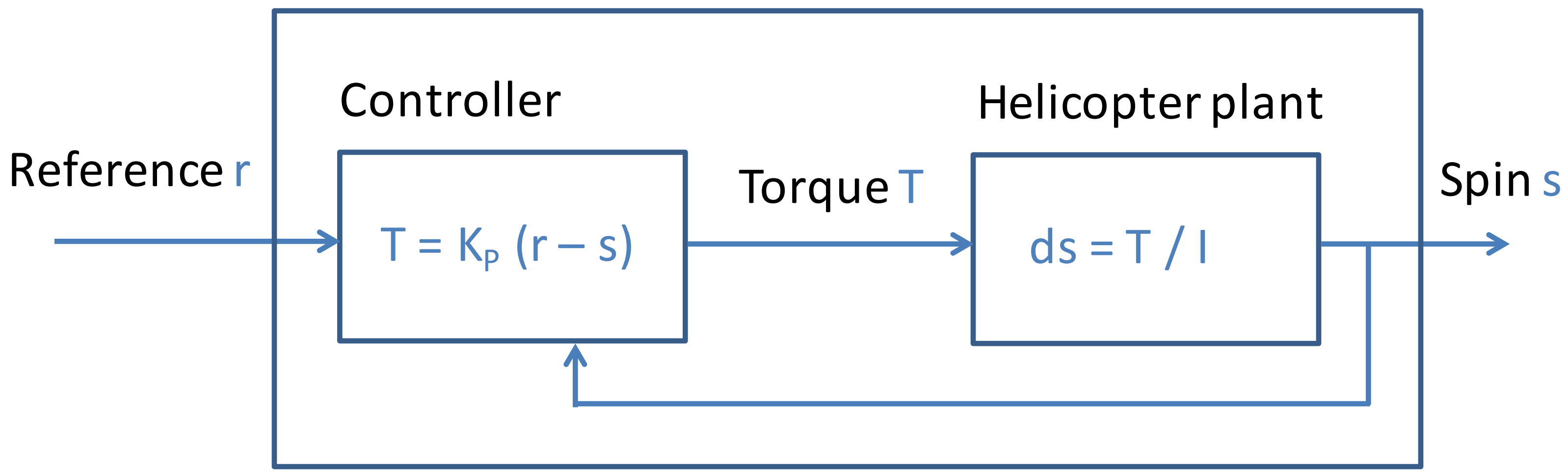
- Two control design techniques:
  1. Mathematical, based on theory of linear systems
  2. PID controllers, widely used in practice

Feedback Controller for Helicopter Model



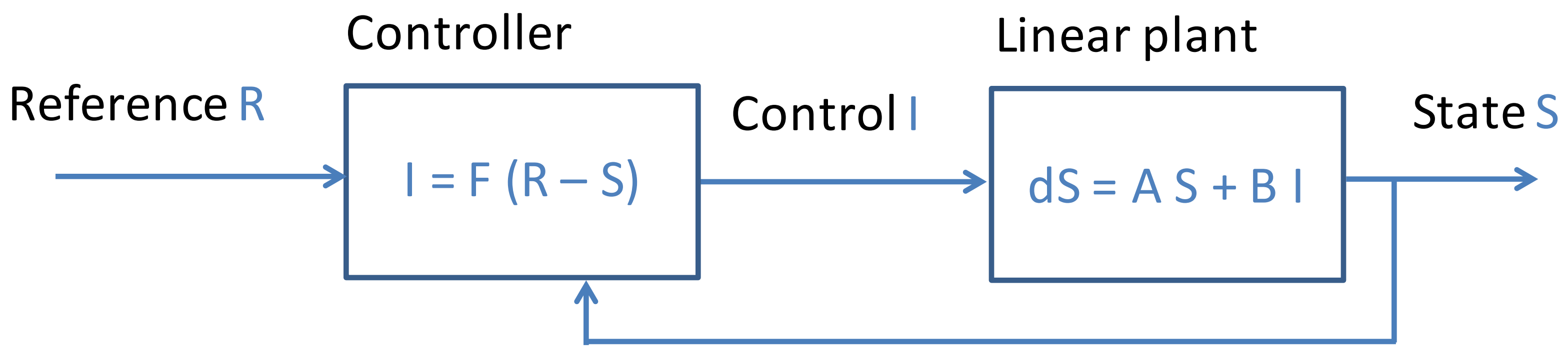
- Design controller so that composed system is stable
- Error $e = (r - s)$: difference in desired value and observed output
- Proportional controller: output T is proportional to error e
- Constant K_P: *proportional gain*
- Note: the direction of torque changes with sign of the error

Stabilizing Controller for Helicopter Model



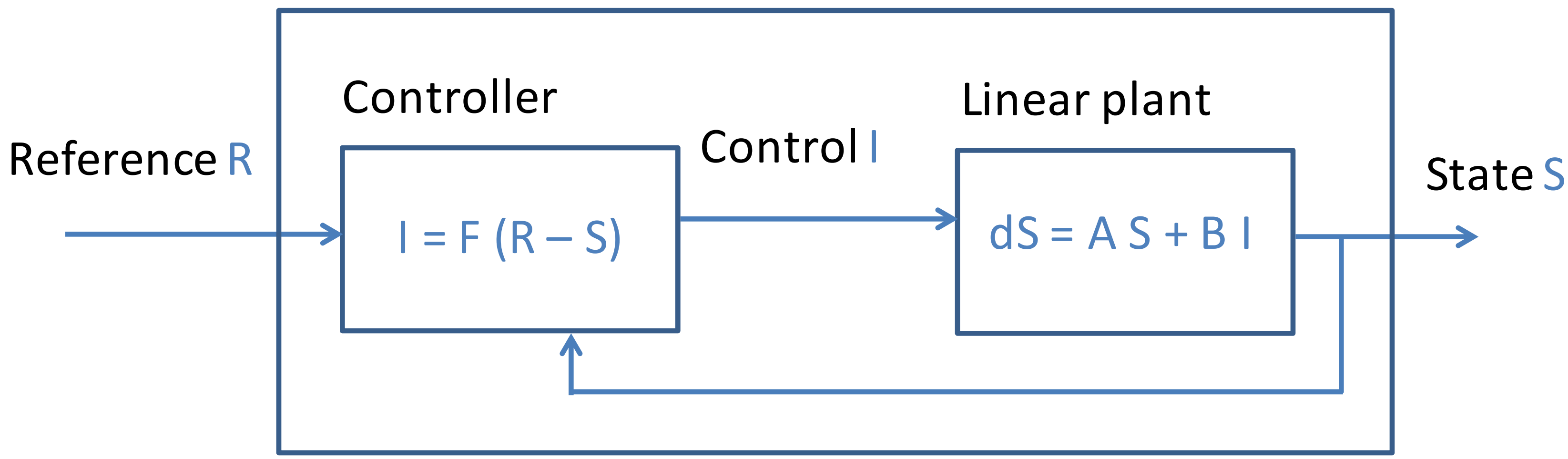
- Dynamics of the composed system: $ds/dt = K_P (r - s) / I$
- When is this system asymptotically stable? BIBO stable?
  - When the coefficient $-K_P / I$ is negative
- Control design: choose a positive *gain constant* K_P
  - Rate of convergence depends on magnitude of K_P

Feedback Controller for Linear Systems



- Assume the controller observes the complete state vector S
- Reference signal R has same dimension as state vector S
- State feedback controller: linear transformation
- Matrix F: *gain matrix* of dimension $m \times n$, with $m = |I|$, $n = |S|$

Stabilization by Linear State Feedback



- Dynamics of the composed system:

$$dS/dt = (A - B\,F)\,S + B\,F\,R$$

- Goal of control design: define the gain matrix F so that the composed system is asymptotically, and so BIBO, stable
  - Given matrices A and B, find F such that each eigenvalue of $A - B\,F$ has negative real part

Design of Gain Matrix

- System dynamics: $dS/dt = A\,S + B\,I$ with n state and m input vars
- Design goal: given matrices A and B, find F such that each eigenvalue of $A - B\,F$ has negative real part
- When is this possible ?
- Suppose we choose desired eigenvalues $\lambda_1, \ldots, \lambda_n$ and solve the system of equations

$$\det(A - B\,F - \lambda I) = (\lambda - \lambda_1)\,(\lambda - \lambda_2)\,\ldots\,(\lambda - \lambda_n)$$

  where the $m \times n$ entries of matrix F are the unknowns
- When is this system guaranteed to be solvable?
- Does the existence of a solution depend on the choice of eigenvalues?

Controllability

- Given an $n \times n$ matrix A and $n \times m$ matrix B, consider the *controllability* $n \times mn$ matrix

$$\mathbf{C}[A,B] = (B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B)$$

  m columns of B followed by m columns of $A\,B$, then of $A\,A\,B$, …

- Recall: the *rank* of a matrix is the maximum number of linearly independent rows

- The matrix pair (A, B) is *controllable* if $\mathbf{C}[A,B]$ has rank n

Theorem: The following are equivalent:

1. The matrix pair (A, B) is controllable
2. For any set $\{\lambda_1, \ldots, \lambda_n\}$ of complex numbers such that $a + bj$ is in the set iff its conjugate $a - bj$ is in the set, there is a $n \times m$ gain matrix F such that the eigenvalues of $A - B\,F$ are $\lambda_1, \ldots, \lambda_n$

Example: Controllability test

Consider 2-dimensional system with one input u, with dynamics given by

$$d\,s_1 = 4\,s_1 + 6\,s_2 + 2\,u$$
$$d\,s_2 = s_1 + 3\,s_2 + u$$

- What are the matrices A, B, $\mathbf{C}[A, B]$?
- What is the rank of $\mathbf{C}[A, B]$?

Advantages of Controllability

- Consider a linear system with dynamics:

$$dS/dt = A\,S + B\,I \text{ ; initial state } s_0$$

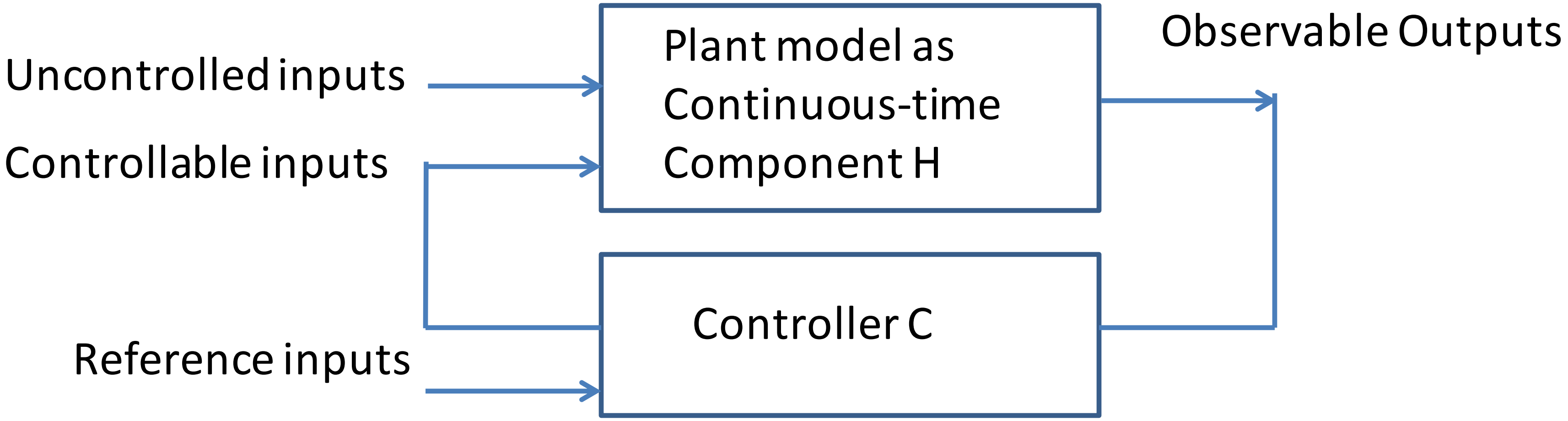
- Suppose (A, B) is controllable

- Then, for every system state s there is an input signal **I** and a time t_g such that

$$\mathbf{S}(t_g) = s$$

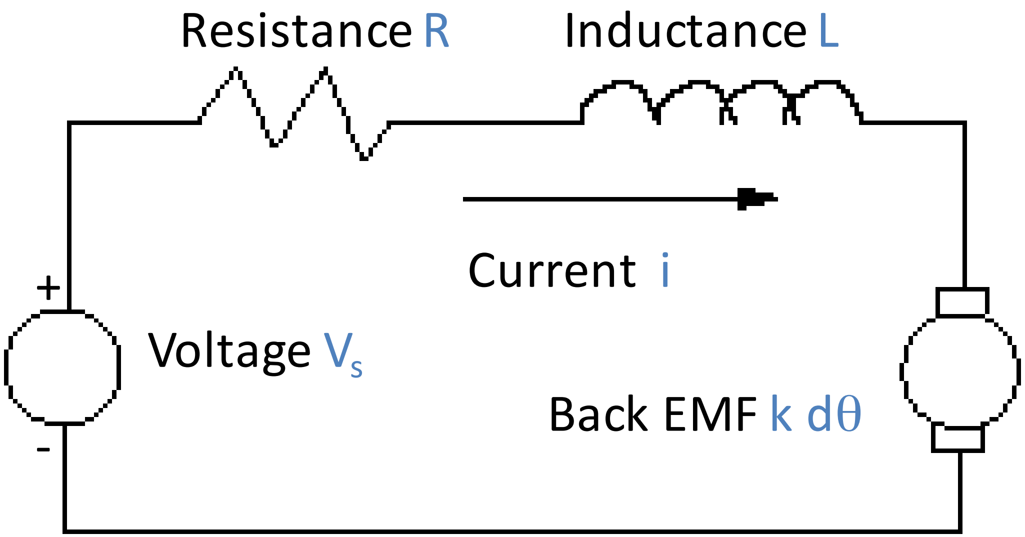
  where **S** is the unique response signal for **I** and s_0

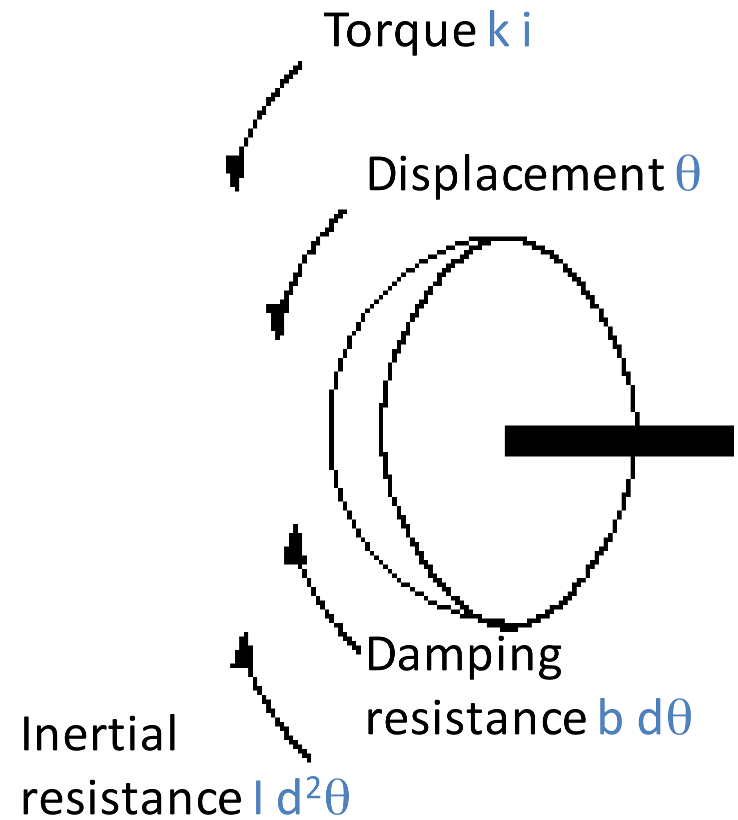
PID Controllers



- Strategy for designing controllers that is widely used in practice
- Error = Reference Inputs – Observable Outputs
- Controller's output is sum of 3 terms:
  - Term proportional to error
  - Integral term to handle cumulative error
  - Derivative term in response to rate of change of error

DC Motor





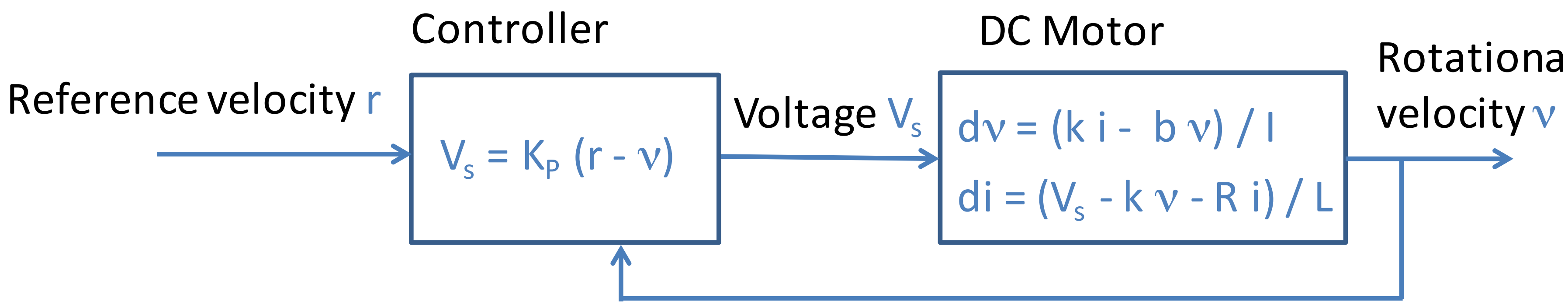
Laws of electrical circuits:

$$L\ di/dt + R\ i + k\ d\theta/dt = V_s$$

Laws of motion for the shaft:

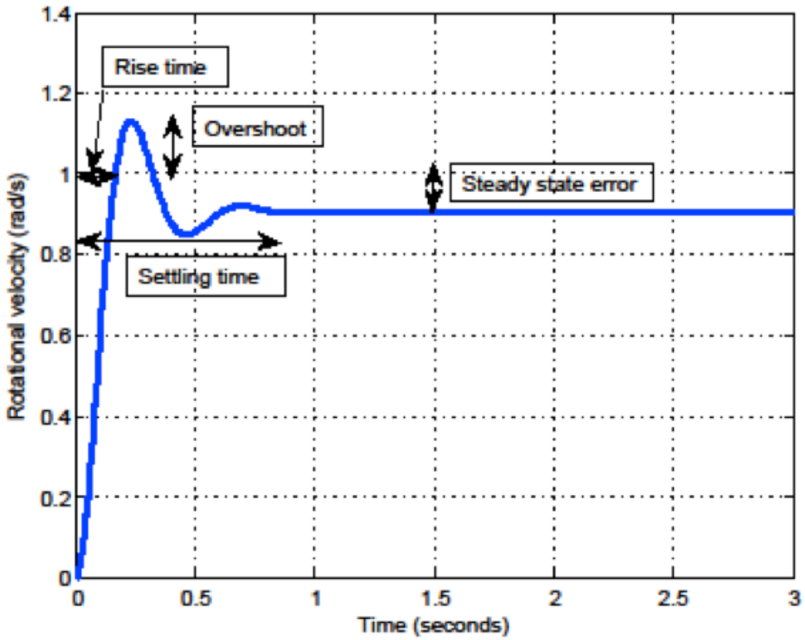
$$I\ d^2\theta/dt^2 + b\ d\theta/dt = k\ i$$

Proportional Controller for DC Motor



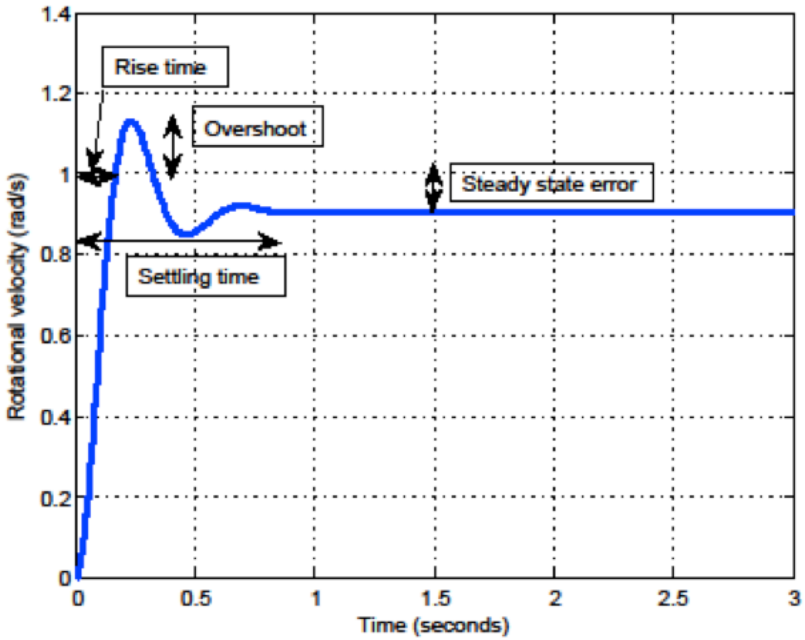
- DC Motor modeled as a linear system with 2 state variables, 1 input variable, and 1 output variable
- Feedback controller observes rotational velocity ν, and adjusts voltage to make ν equal to desired velocity r
- First attempt: proportional controller (*P controller*)

Step Response of P Controller



- Step response: How will system output change if at time 0, with $\nu = 0$, we change reference input r to 1?
- Plotted using MATLAB (see notes for values of various parameters)
- Beyond stability and convergence, what are desired characteristics of the response?

Characteristics of the Step Response



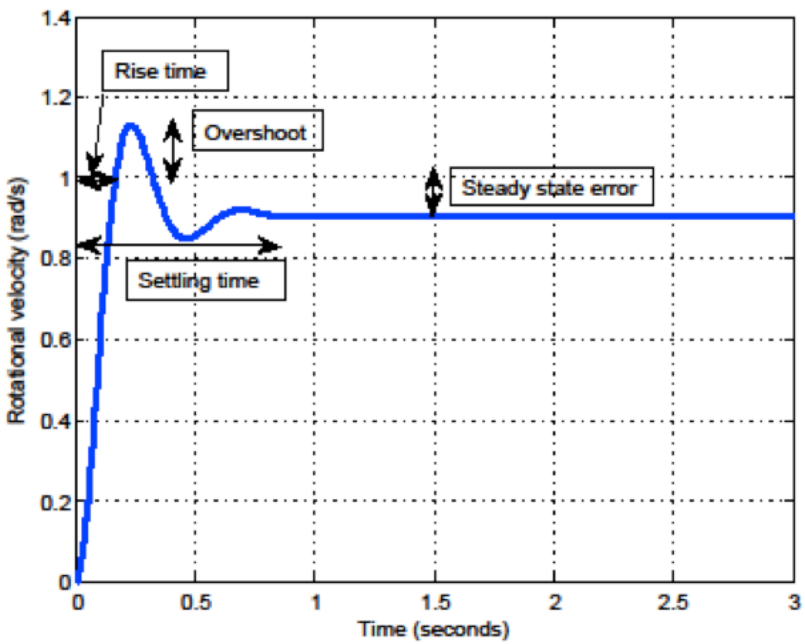
1. *Overshoot*: Difference between maximum output value and reference value (12% in this plot)

2. *Rise Time*: Time at which the output value crosses reference value (0.15sec in this plot)

3. Settling Time: Time at which output value reaches steady-state value (0.8sec in this plot)

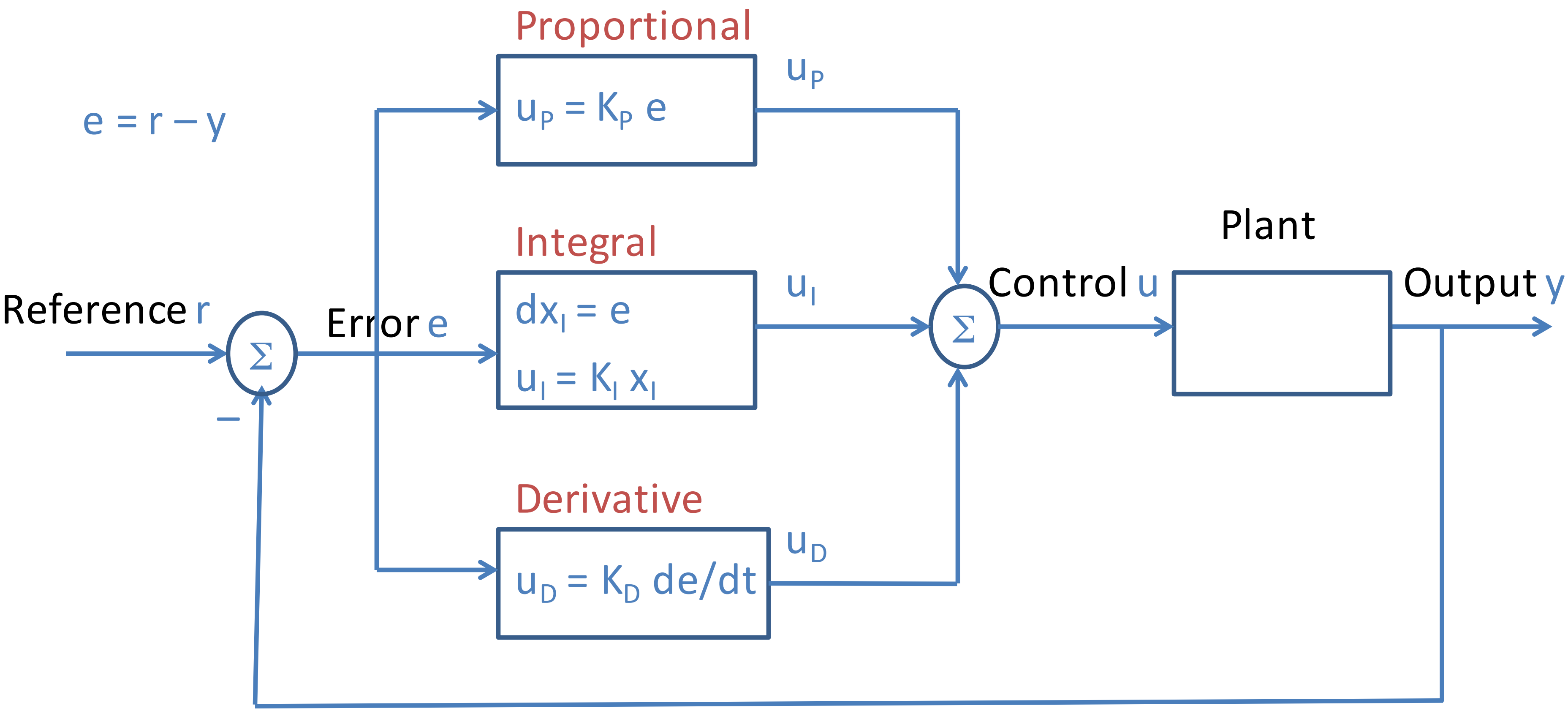
4. *Steady State Error*: Difference between steady-state output value and reference (10% in this plot)

Improving the Step Response



- Performance of the P-controller depends on the value of the proportional gain constant K_P
- What happens if we increase it?
- Rise time decreases, but overshoot increases
- Steady-state error remains!
- Solution: Use integral and derivative gains

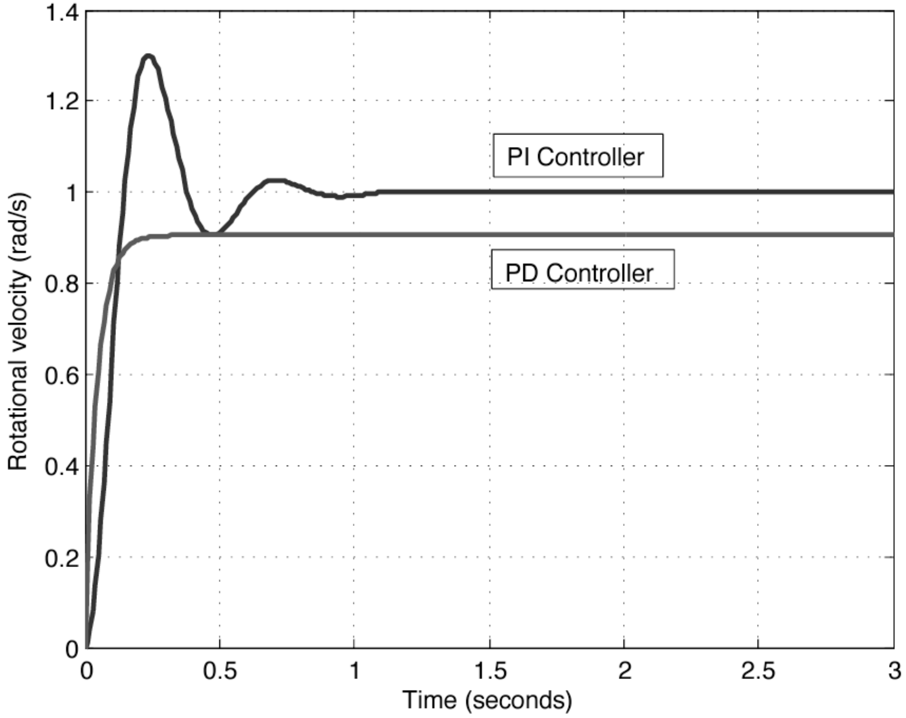
Generic PID Controller



PID Controller

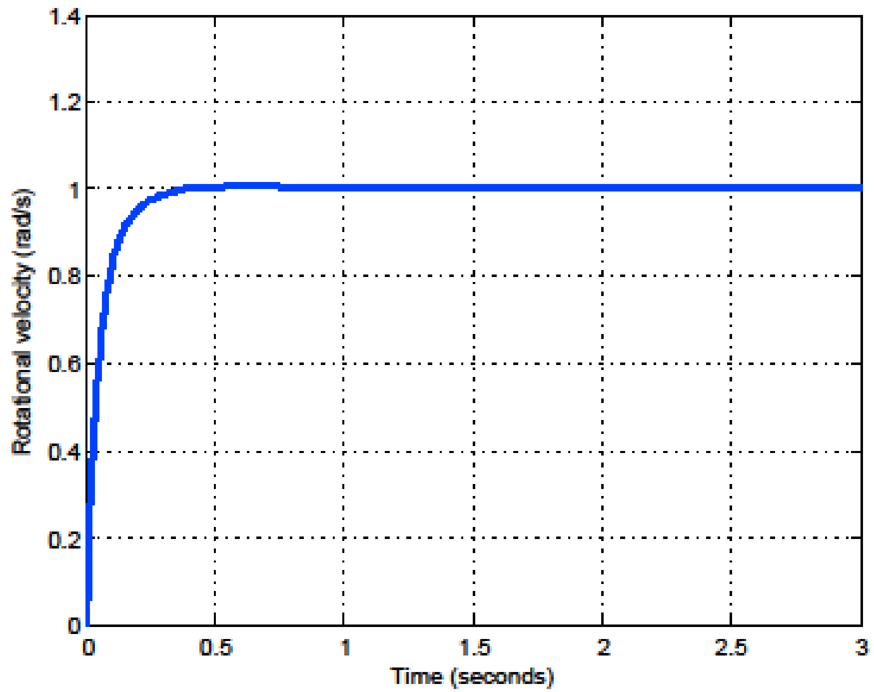
- If **e**(t) is the error signal, then the output **u**(t) of the PID controller is sum of 3 terms:
  - Proportional term: K_P **e**(t), where K_P is the *proportional gain* (response to current error)
  - Integral term: $K_I \int_0^t$ **e**(t) dt, where K_I is the *integral gain* (response to error accumulated so far)
  - Derivative term: K_D (d/dt)**e**(t), where K_D is the *derivative gain* (response to current rate of change of error)
- Special cases of controllers: P, PD, PI

PI and PD Controllers for DC Motor



- PI Controller: adding integral term to proportional controller gets rid of steady state error
  - Overshoot, rise time, setting time increase (why?)
- PD controller: adding derivative term to proportional controller gets rid of overshoot
  - Steady state error remains

PID Controller for DC Motor



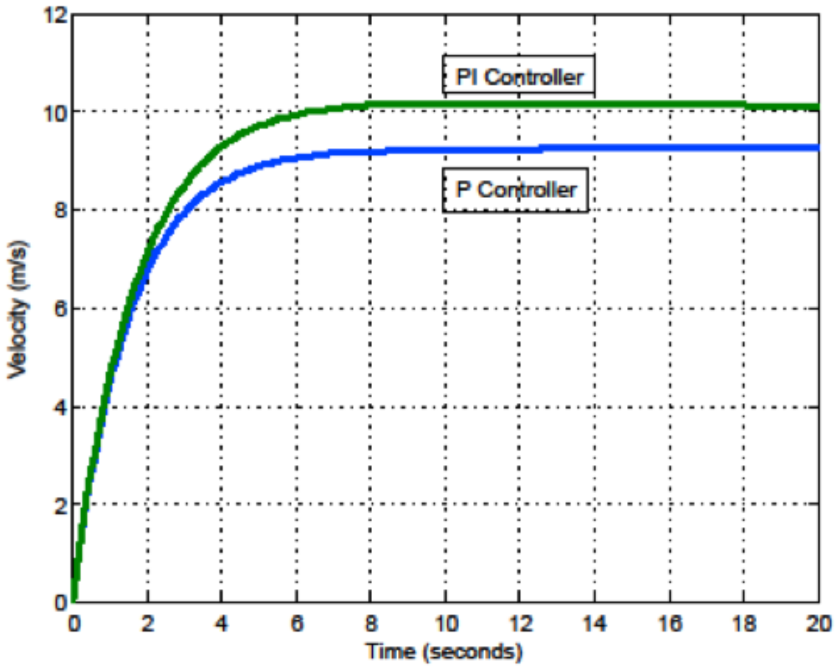
Excellent performance on all metrics: $K_P = 100$, $K_D = 10$, $K_I = 200$

Small rise time, settling time, negligible steady state error, no overshoot

Designing PID Controllers

- What are the effects of changing the gain constants K_P, K_D, K_I ?
- Broad co-relationships well understood
- Control toolboxes allow automatic tuning of parameters
- PID controllers seem to work well even when the actual system differs significantly from the plant model
    - Computation of control output depends only on the measured error, and not on the model!

PI Cruise Controller



- Desired change in velocity: 10 m/s
- PI controller: $K_P = 600$, $K_I = 40$
- Settling time: 7s, with negligible overshoot and steady-state error
- Works in a real car!

Credits

Notes based on Chapter 6 of

Principles of Cyber-Physical Systems
by Rajeev Alur
MIT Press, 2015