CS:4420 Artificial Intelligence

Spring 2019

Probabilistic Reasoning

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Readings

- Chap. 14 of [Russell and Norvig, 3rd Edition]

Making Probabilistic Reasoning Feasible

Recall:

- A joint probability distribution (JPD) contains all the relevant information to reason about the various kinds of probabilities of a set $\{X_1, \ldots, X_n\}$ of random variables.
- Unfortunately, JPD tables are difficult to create and also very expensive to store.
- One possibility is to work with conditional probabilities and exploit the fact that many random variables are conditionally independent.
- Belief Networks are a successful example of probabilistic systems that exploit conditional independence to reason *effectively* under uncertainty.

Review of Basic Concepts

The JPD is a collection of probabilities:

$$\mathbf{P}(X_1, \ldots, X_n) = \{P(X_1 = x_1 \wedge \cdots \wedge X_n = x_n) \mid x_i \in Domain(X_i)\}$$

Conditional Probability:

$$P(X_1 = x_1 \mid X_2 = x_2) \quad = \quad \frac{P(X_1 = x_1 \wedge X_2 = x_2)}{P(X_2 = x_2)}$$

or

$$P(X_1 = x_1 \wedge X_2 = x_2) \quad = \quad P(X_1 = x_1 \mid X_2 = x_2) P(X_2 = x_2)$$

Review of Basic Concepts (2)

Chain rule:

$$P(X_1 = x_1 \mid X_2 = x_2, ..., X_n = x_n) \quad = \quad \frac{P(X_1{=}x_1 \wedge X_2{=}x_2 \wedge \cdots \wedge X_n{=}x_n)}{P(X_2{=}x_2 \wedge \cdots \wedge X_n{=}x_n)}$$

or

$$\begin{aligned} & P(X_1 = x_1 \wedge X_2 = x_2 \wedge \cdots \wedge X_n = x_n) \\ & \quad = \; P(X_1 = x_1 \mid X_2 = x_2, ..., X_n = x_n) P(X_2 = x_2 \wedge \cdots \wedge X_n = x_n) \\ & \quad = \; \prod_{i=1}^{n} P(X_i = x_i \mid X_{i+1} = x_{i+1}, ..., X_n = x_n) \end{aligned}$$

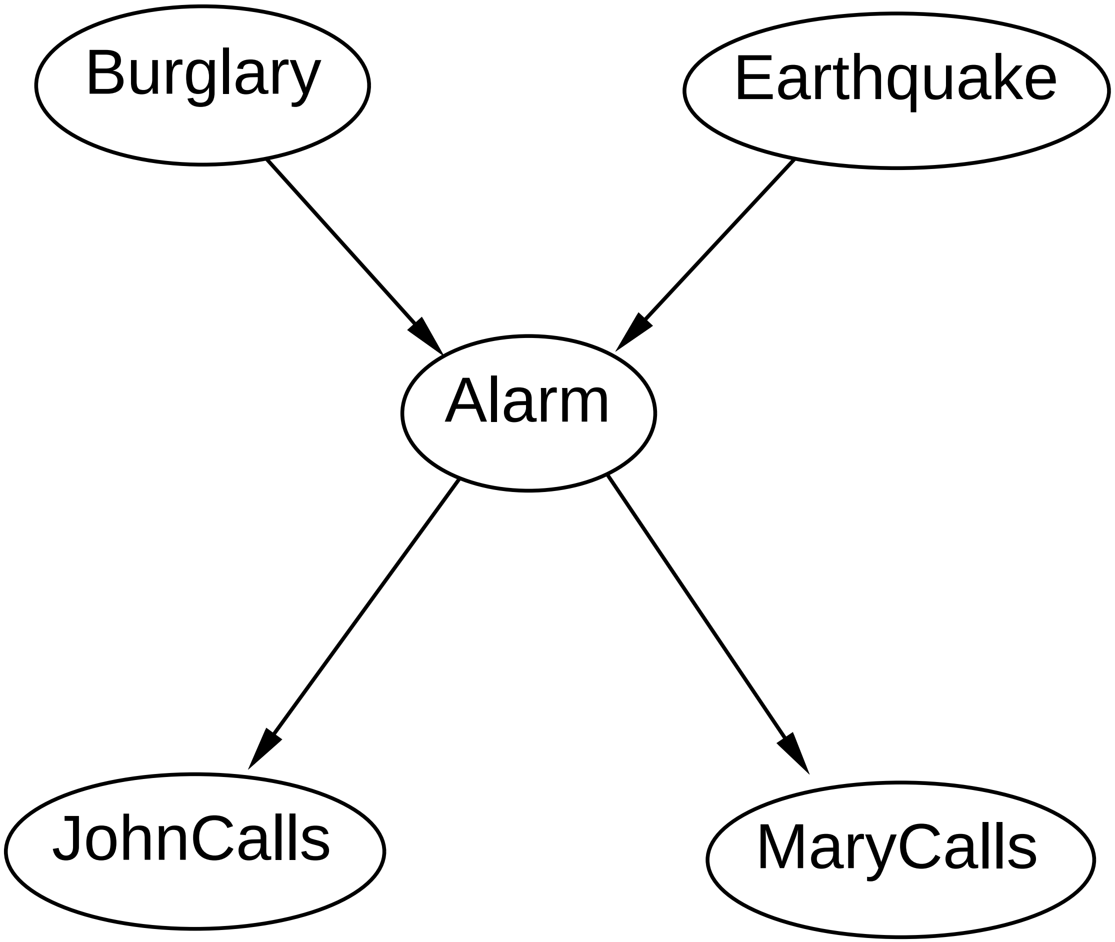
Conditional Independence:

If

$$\begin{aligned} P(X_1 = x_1 \mid X_2 = x_2, ..., X_n = x_n) \quad = \\ P(X_1 = x_1 \mid X_2 = x_2, ..., X_{n-1} = x_{n-1}) \end{aligned}$$

then $X_1 = x_1$ is conditionally independent from $X_n = x_n$ given the evidence $X_2 = x_2, ..., X_{n-1} = x_{n-1}$

A Belief Network



Belief Networks

Let $X_1, \ldots, X_n$ be discrete random variables.

A *belief network (or Bayesian network)* for $X_1, \ldots, X_n$ is a graph with m nodes such that

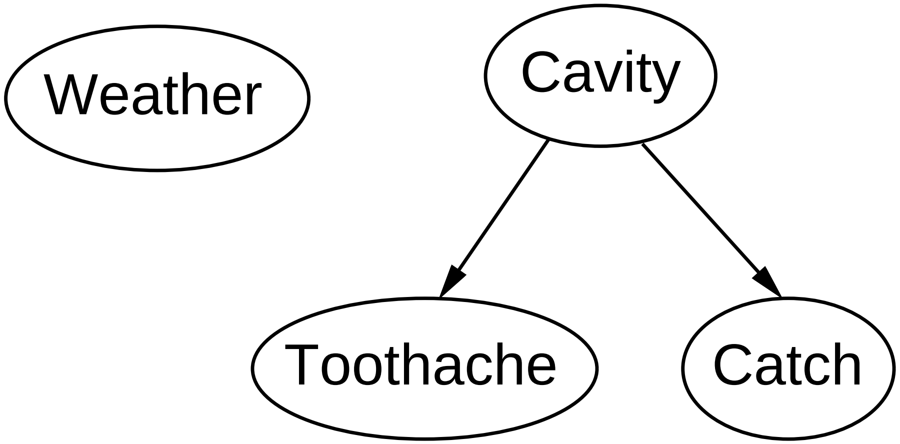
- there is a node for each X_i
- all the edges between two nodes are directed
- there are no cycles
- each node has a conditional probability table (CPT), given in terms of its parents

The intuitive meaning of an edge from a node X_i to a node X_j is that X_i has a direct influence on X_j

Network Semantics

The topology of the network encodes conditional independence assertions

Example:



- $Weather$ is independent from the other variables
- $Toothache$ and $Catch$ are conditionally independent given $Cavity$

Conditional Probability Tables

Each node X_i in a belief network has an associated CPT expressing the probability of X_i, given its parents as evidence

Example:

CPT for $Alarm$:

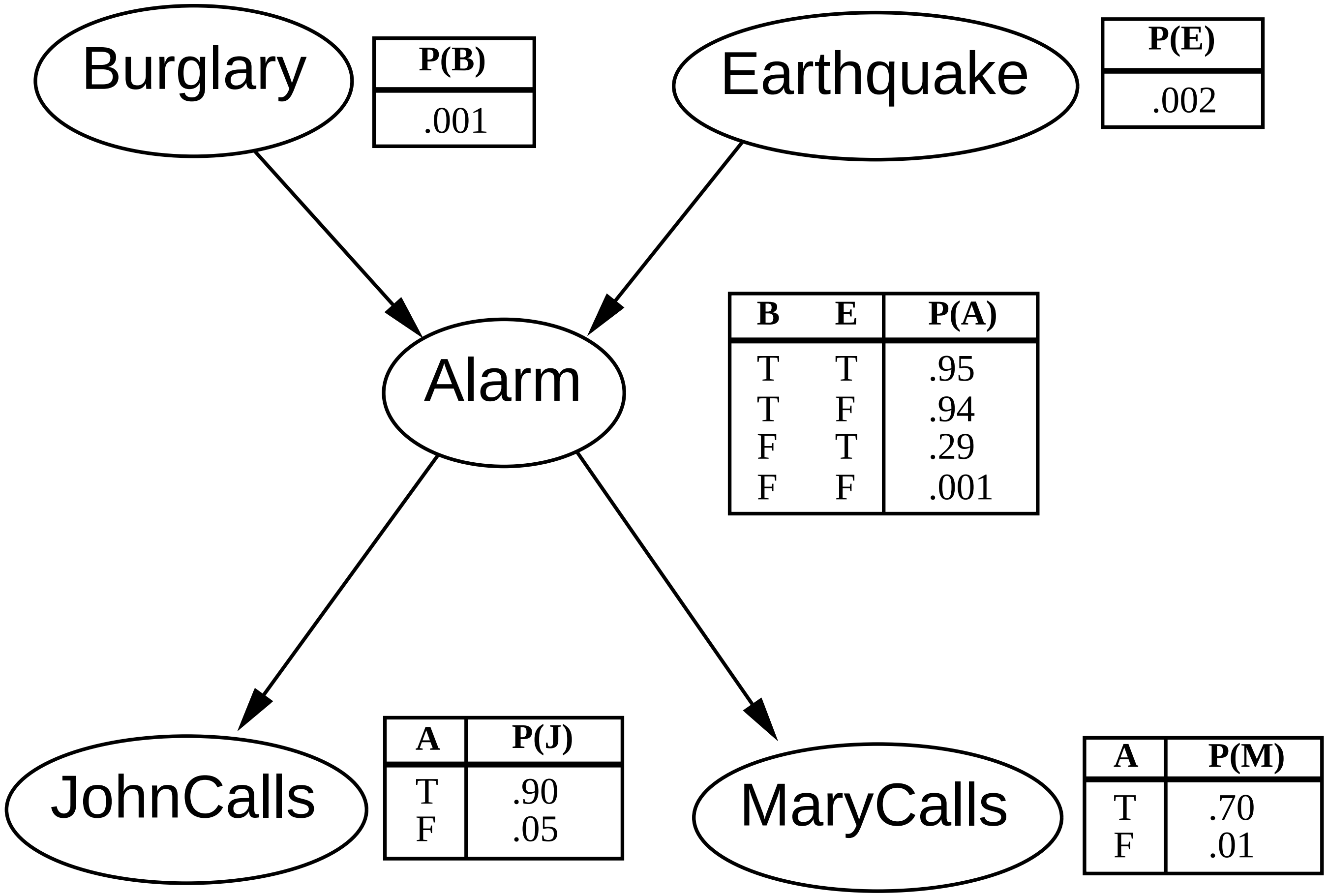
$Burglary$	$Earthquake$	$Alarm$ T	$Alarm$ F
T	T	0.950	0.050
T	F	0.940	0.060
F	T	0.290	0.710
F	F	0.001	0.999

$$P(alarm \mid burglary \wedge earthquake) = 0.950$$

$$P(\neg alarm \mid \neg burglary \wedge earthquake) = 0.710$$

...

A Belief Network with CPTs



P(B)
.001

P(E)
.002

B	E	P(A)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J)
T	.90
F	.05

A	P(M)
T	.70
F	.01

Note: The tables only show $P(X = true)$ here because $P(X = false) = 1 - P(X = true)$

The Semantics of Belief Networks

There are two equivalent ways to interpret a belief network for the variables $X_1, \ldots, X_n$:

1. The network is a representation of the JPD $\mathbf{P}(X_1, \ldots, X_n)$
2. The network is a collection of conditional independence statements about $X_1, \ldots, X_n$

Interpretation 1 is helpful when constructing belief networks

Interpretation 2 is helpful in designing inference procedures based on them

Belief Network as JPDs

The whole JPD $\mathbf{P}(X_1, \ldots, X_n)$ can be computed from a belief network for $X_1, \ldots, X_n$ and its CPTs

For each tuple $\langle x_1, \ldots, x_n \rangle$ of possible values for $\langle X_1, \ldots, X_n \rangle$,

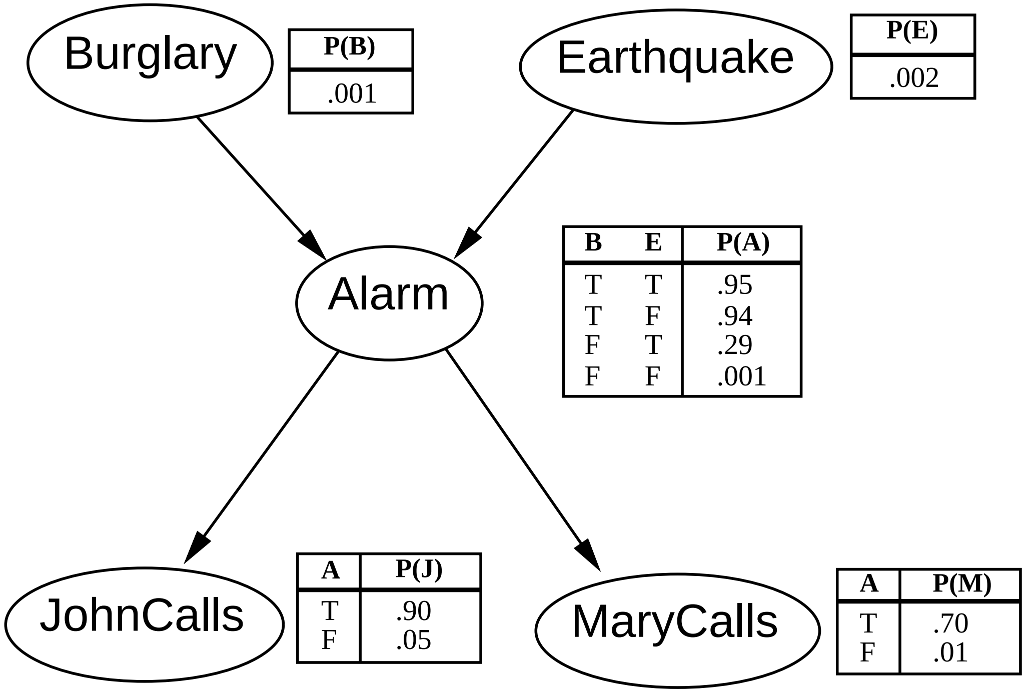
$$P(X_1 = x_1 \wedge \cdots \wedge X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i \mid Parents(X_i))$$

where

$$Parents(X_i) = \{X_j = x_j \mid 1 \leq j \leq n \text{ and } X_j \text{ is a parent of } X_i\}$$

Belief Network as JPDs

$$P(X_1 = x_1 \wedge \cdots \wedge X_n = x_n) \; = \; \prod_{i=1}^{n} P(X_i = x_i \mid Parents(X_i))$$



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A	P(J)
T	.90
F	.05

A	P(M)
T	.70
F	.01

$$\begin{aligned} & P(j \wedge m \wedge a \wedge \neg b \wedge \neg e) \\ & \quad = \; P(j \mid a)\, P(m \mid a)\, P(a \mid \neg b \wedge \neg e)\, P(\neg b)\, P(\neg e) \\ & \quad = \; 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 = 0.00062 \end{aligned}$$

Belief Networks and Cond. Independence

Let $\{X_1, X_2, \ldots, X_n\}$ be any set of nodes in the network such that

- all the parents of X_1 are in $\{X_2, \ldots, X_n\}$
- no node in $\{X_2, \ldots, X_n\}$ is a descendant of X_1

Let $\langle x_1, \ldots, x_n \rangle$ be a value assignment for $\langle X_1, \ldots, X_n \rangle$

From the equation

$$P(X_1 = x_1 \wedge \cdots \wedge X_n = x_n) \ = \ \prod_{i=1}^{n} P(X_i = x_i \mid Parents(X_i))$$

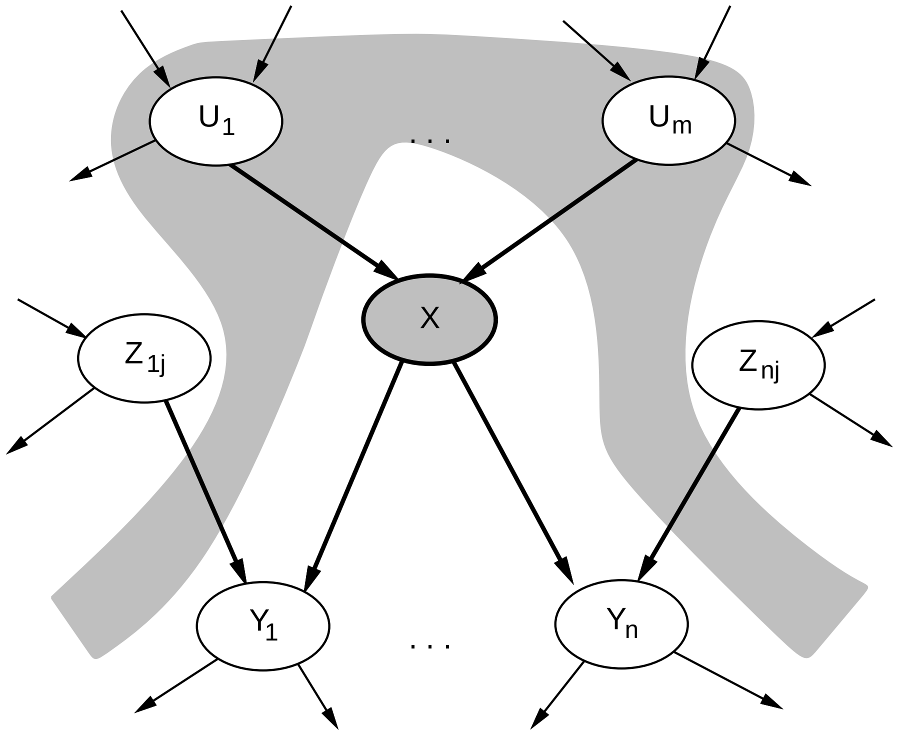
we can show that

$$P(X_1 = x_1 \mid X_2 = x_2 \wedge \cdots \wedge X_n = x_n) \ = \ P(X_1 = x_1 \mid Parents(X_1))$$

Belief Networks and Cond. Independence

A consequence of the last equation is:

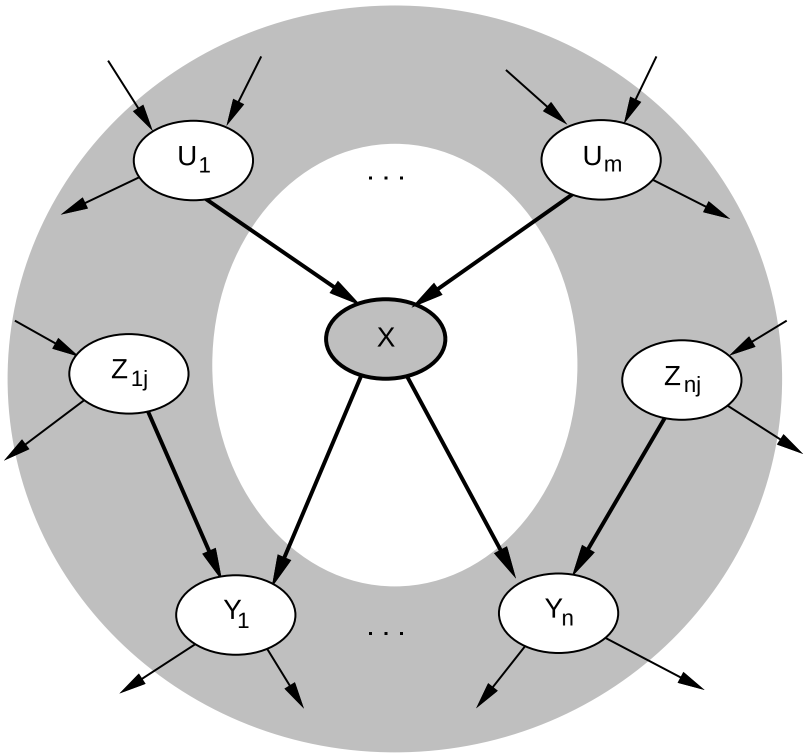
> *Given all its parents as evidence, each node in the network is conditionally independent from its non-descendants*



Belief Networks and Cond. Independence

Another consequence is:

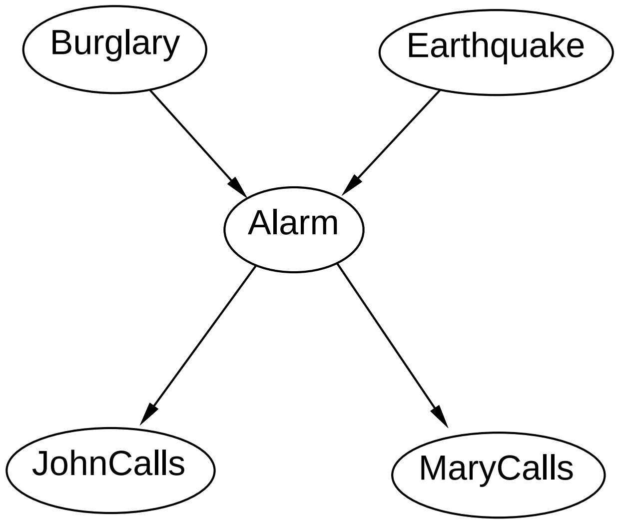
> *Given all its parents, children, and children's parents as evidence, each node in the network is conditionally independent from all the other nodes*



Belief Networks and Cond. Independence

$$P(X_1 = x_1 \mid X_2 = x_2 \wedge \cdots \wedge X_n = x_n) \; = \; P(X_1 = x_1 \mid Parents(X_1))$$

Examples:



$$P(b \mid e) = P(b)$$
$$P(j \mid m \wedge a) = P(j \mid a)$$
$$P(j \mid a \wedge e) = P(j \mid a)$$
$$P(j \mid a \wedge b \wedge e) = P(j \mid a)$$
$$P(j \mid m \wedge a \wedge b \wedge e) = P(j \mid a)$$

Exercise: Find all the conditional independences holding in this network

Constructing Belief Networks

General Procedure

1. Identify a set of random variables $\{X_i\}_i$ that describe the domain
2. Choose an ordering $X_1, \ldots, X_n$ of the variables
3. Start with an empty network
4. For $i = 1 \ldots n$:
   (a) add X_i to the network
   (b) select as parents of X_i nodes from $X_1, \ldots, X_{i-1}$ such that $\mathbf{P}(X_i \mid Parents(X_i)) = \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1})$
   (c) fill in the CPT for X_i

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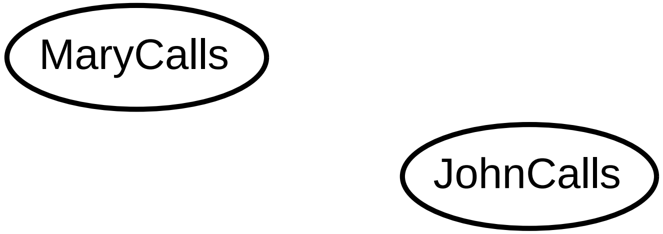
   (c) fill in the CPT for X_i

This choice of parents guarantees the network semantics:

$$\begin{aligned} \mathbf{P}(X_1, \ldots, X_n) &= \textstyle\prod_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1}) && \text{(chain rule)} \\ &= \textstyle\prod_{i=1}^{n} \mathbf{P}(X_i \mid Parents(X_i)) && \text{(by construction)} \end{aligned}$$

Example

Suppose we choose the ordering $MaryCalls$, $JohnCalls$, $Alarm$, $Burglary$, $Earthquake$



$P(j \mid m) = P(j)$?

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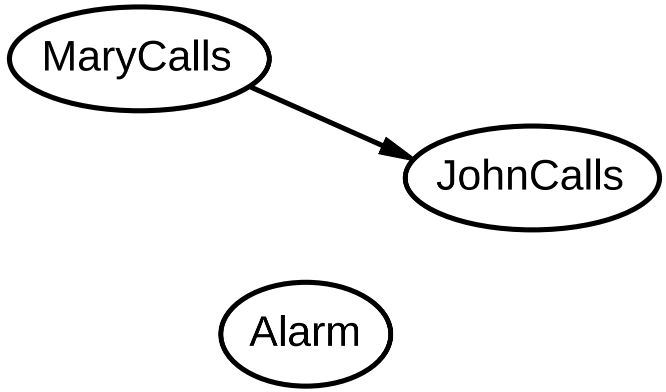

$P(j \mid m) = P(j)$? No
$P(a \mid j, m) = P(a \mid j)$?
$P(a \mid j, m) = P(a)$?

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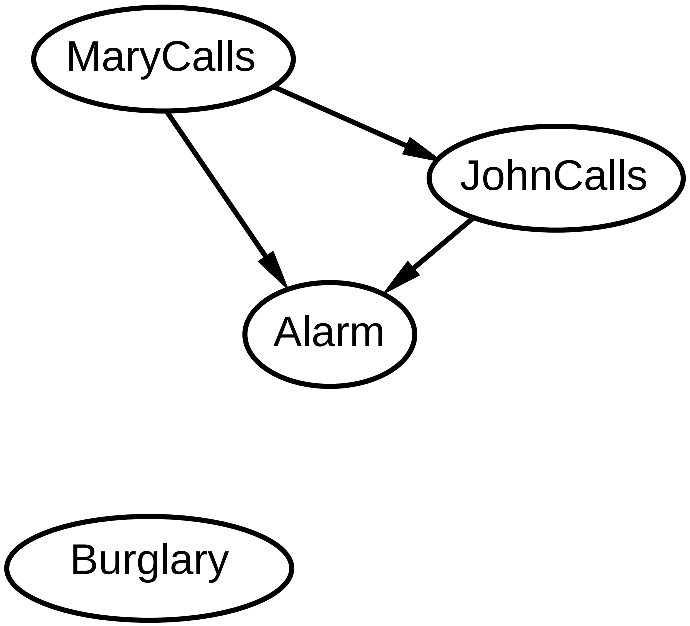

$P(j \mid m) = P(j)$? No
$P(a \mid j, m) = P(a \mid j)$? No
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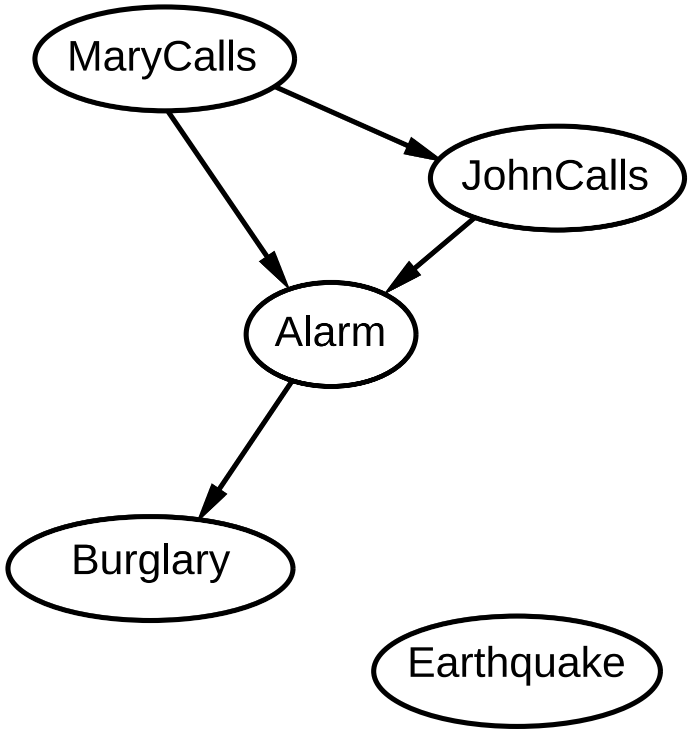

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$P(b \mid a, j, m) = P(b \mid a)$? Yes
$P(b \mid a, j, m) = P(b)$? No
$P(e \mid b, a, j, m) = P(e \mid a)$?
$P(e \mid b, a, j, m) = P(e \mid a, b)$?

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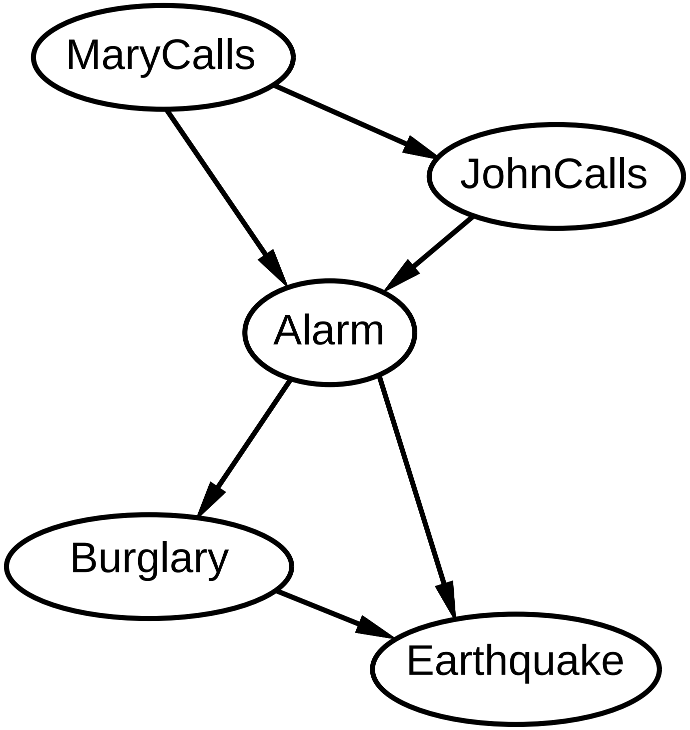

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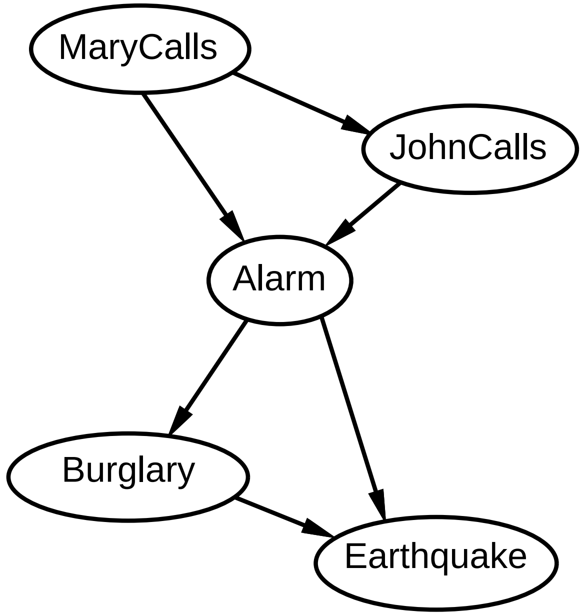
$P(e \mid b, a, j, m) = P(e \mid a)$? No
$P(e \mid b, a, j, m) = P(e \mid a, b)$? Yes

Example contd.



Deciding conditional independence is hard in non-causal directions

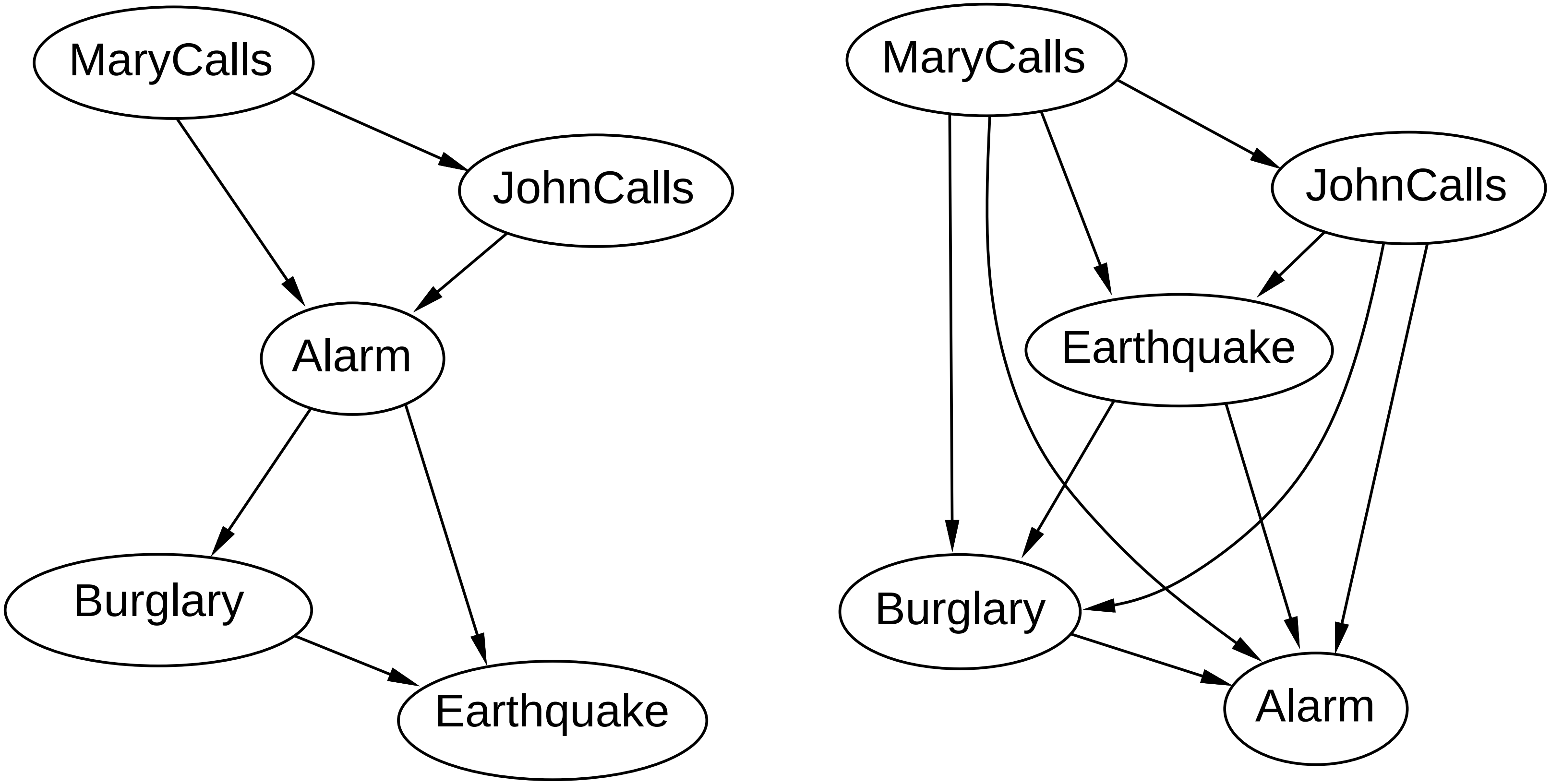
Causal models and conditional independence seem hardwired for humans!

Assessing conditional probabilities is hard in non-causal directions

Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ probabilities needed

Ordering the Variables

The order in which add the variables to the network is important

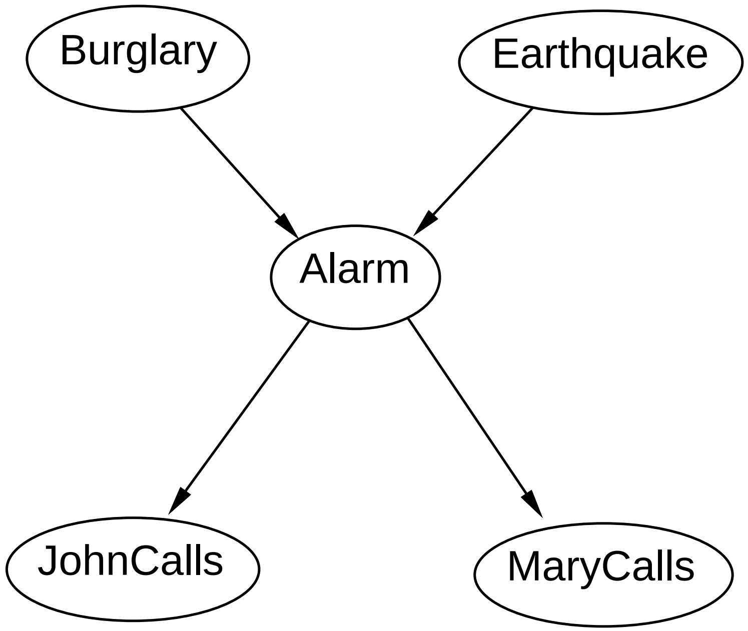


“Wrong” orderings produces more complex networks

Ordering the Variables Right

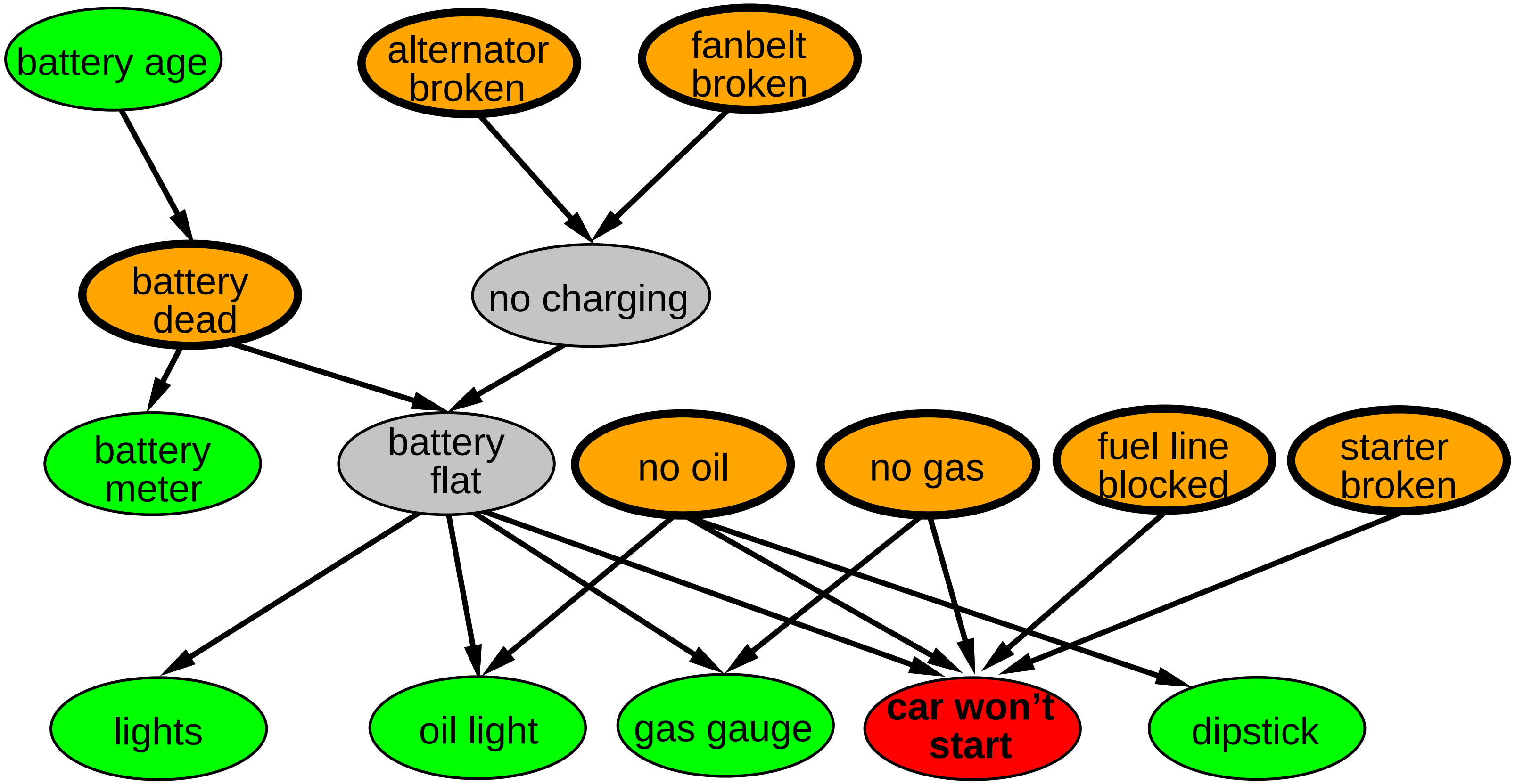
A general, effective heuristic for constructing simpler belief networks is to exploit causal links between random variables whenever possible

This is done by adding variables to the network so that causes get added before effects



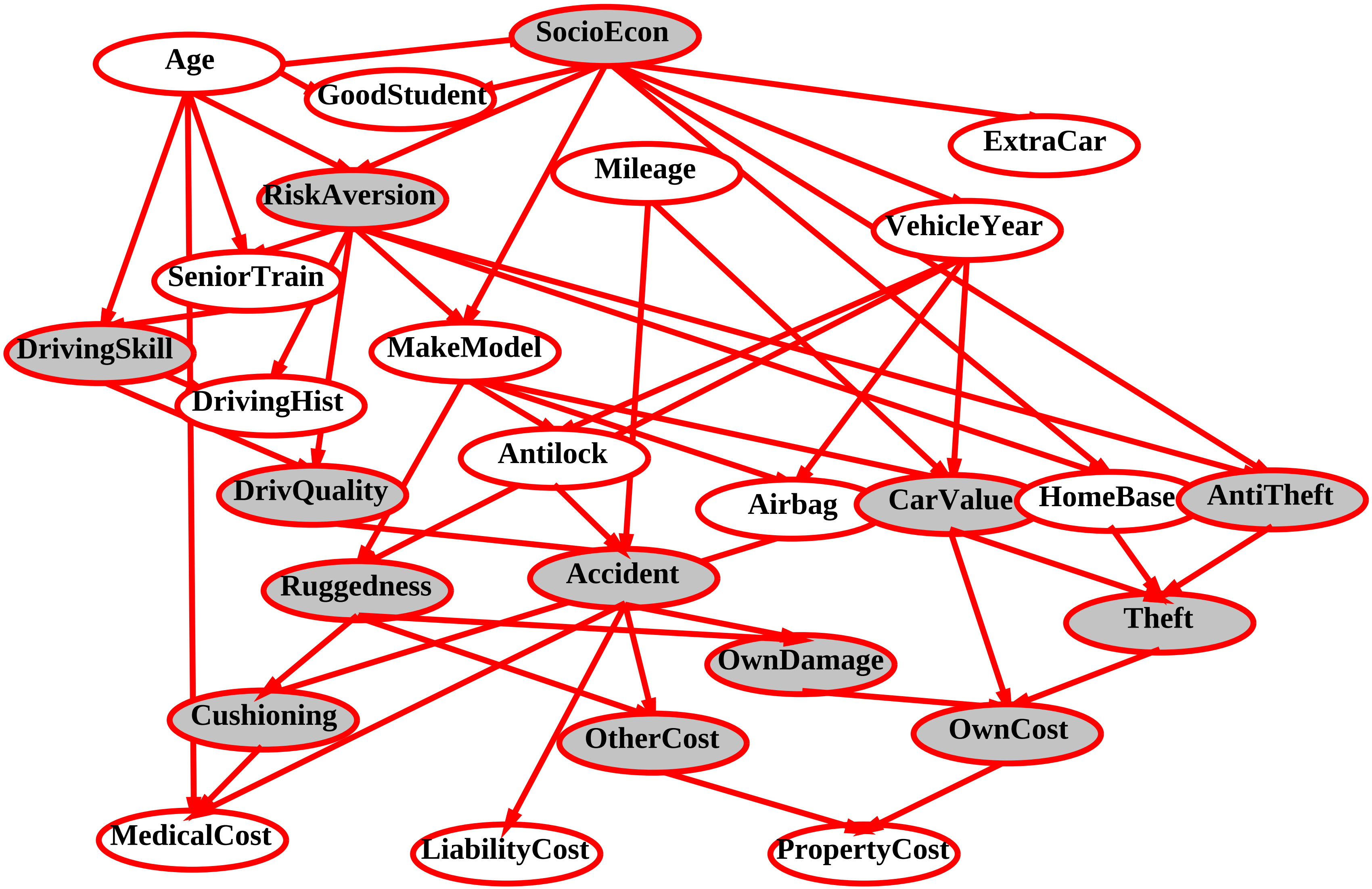
Example: Car diagnosis

Initial evidence: car won't start



Testable variables (green), actionable variables (orange)
Hidden variables (gray) ensure sparse structure, reduce parameters

Example: Car insurance



Compactness of Belief Networks

A belief network is a complete and non-redundant representation of a full joint probability distribution

In addition, its is typically more compact than a full joint probability table

The reason is that probabilistic domains are often representable as a locally structured system

In a locally structured system, each subcomponent interacts with only a bounded number of other components, regardless of the size of the system

The complexity of local structures generally grows linearly, instead of exponentially

Locally Structured Systems

In many real-world domains, each random variable is influenced by at most k others, for some fixed constant k

With n variables, all Boolean, a JPT will have 2^n entries

In a well constructed belief network, each node will have at most k parents

Hence each node will have a CPT with at most 2^k entries, for a total of $n2^k$ entries.

Example: $n = 20$, $k = 5$

$$\begin{array}{lcrcr} \text{entries in network CPTs} & \leq & 20 \times 2^5 & = & 640 \\ \text{entries in JPT} & = & 2^{20} & = & 1{,}048{,}576 \end{array}$$

Representing CPTs

Even with a fairly small number of parents per node, constructing the CPTs for a belief network may require a lot a work

However, if the network is built with the right topology, the relationship between parent and children nodes will typically fall into a category with some *canonical distribution*

Examples:

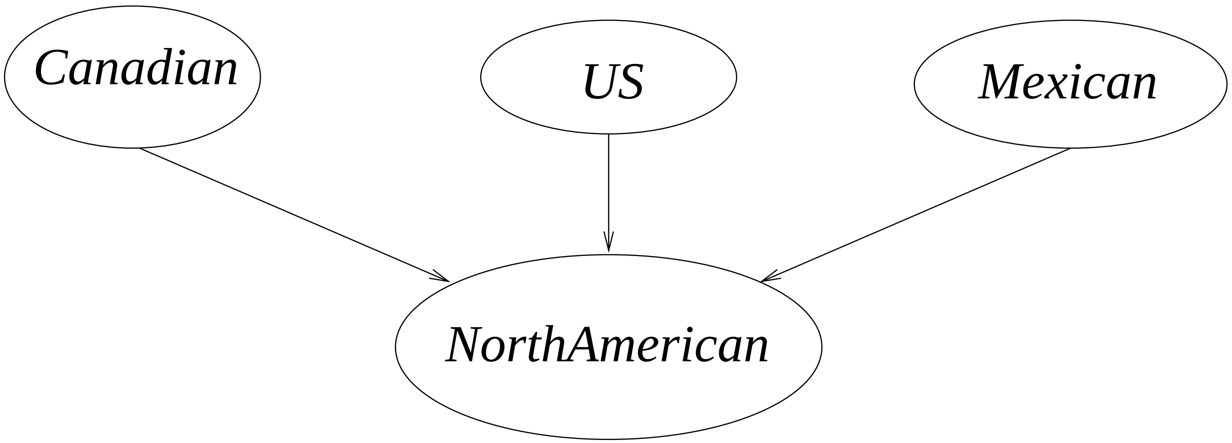
- Deterministic nodes
- Noisy-OR relationships

Deterministic Nodes

A node is *deterministic* if its value is a function of the values of its parents, with no uncertainty

- Logical implications or equivalences:

$$NorthAmerican \Leftrightarrow Canadian \vee US \vee Mexican$$



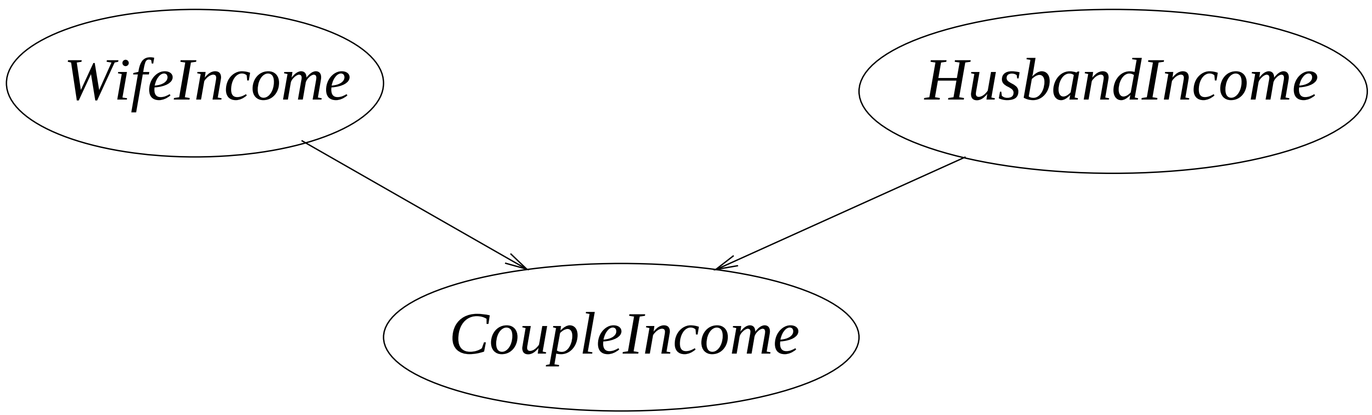
$$P(n \mid \neg c \wedge \neg u \wedge \neg m) = 0$$

$$\begin{aligned} P(n \mid u) &= P(n \mid c \wedge u) = \ldots = P(n \mid u \wedge \neg m) \\ &= P(n \mid c \wedge \neg u \wedge \neg m) = \ldots = 1 \end{aligned}$$

Deterministic Nodes (cont.)

- Functional relationships:

$$CoupleIncome = WifeIncome + HusbandIncome$$



$$P(C = 80K \mid H = 50K \wedge W = 30K) = 1$$

$$P(C = 95K \mid H = 50K \wedge W = 30K) = 0$$

Noisy-OR

A generalization of logical OR. Adds uncertainty to statements like

$$Fever \Leftrightarrow Cold \lor Flu \lor Malaria$$

Three assumptions are needed:

1. Each cause has an independent chance of producing the effect
2. All possible causes are listed
3. The reason for a cause not to produce the effect is independent from the reason for another cause not to produce the effect:

$$P(\neg Effect \mid Cause_i \land OtherCauses)$$
$$= P(\neg Effect \mid Cause_i)\, P(\neg Effect \mid OtherCauses)$$

The possibility that a cause does not produce an effect is given by a *noise-parameter*

CPTs for Noisy-ORs

Knowing the noise parameters (in boldface below) is enough to compute the whole CTP

CTP of $Fever$:

$Cold$	Flu	$Malaria$	$P(Fever)$	$P(\neg Fever)$
F	F	F	0.00	1.00
F	F	T	0.90	**0.10**
F	T	F	0.80	**0.20**
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.40	**0.60**
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

A noisy-OR with k causes can be specified with k values, the noise parameters, instead of the 2^k values of a full CPT

Inference in Belief Networks

Main task of a belief network: Compute the conditional probability of a set of query variables, given exact values for some evidence variables

$$P(Query \mid Evidence)$$

Belief networks are flexible enough so that any node can serve as either a query or an evidence variable

In general, to decide what actions to take, an agent

1. first gets values for some variables from its percepts, or from its own reasoning
2. then asks the network about the possible values of the other variables

Probabilistic Inference with BNs

Belief networks are a very flexible tool for probabilistic inference because they allow several kinds of inference:

Diagnostic inference (from effects to causes)

E.g. $P(Burglary \mid JohnCalls)$

Causal inference (from causes to effects)

E.g. $P(JohnCalls \mid Burglary)$

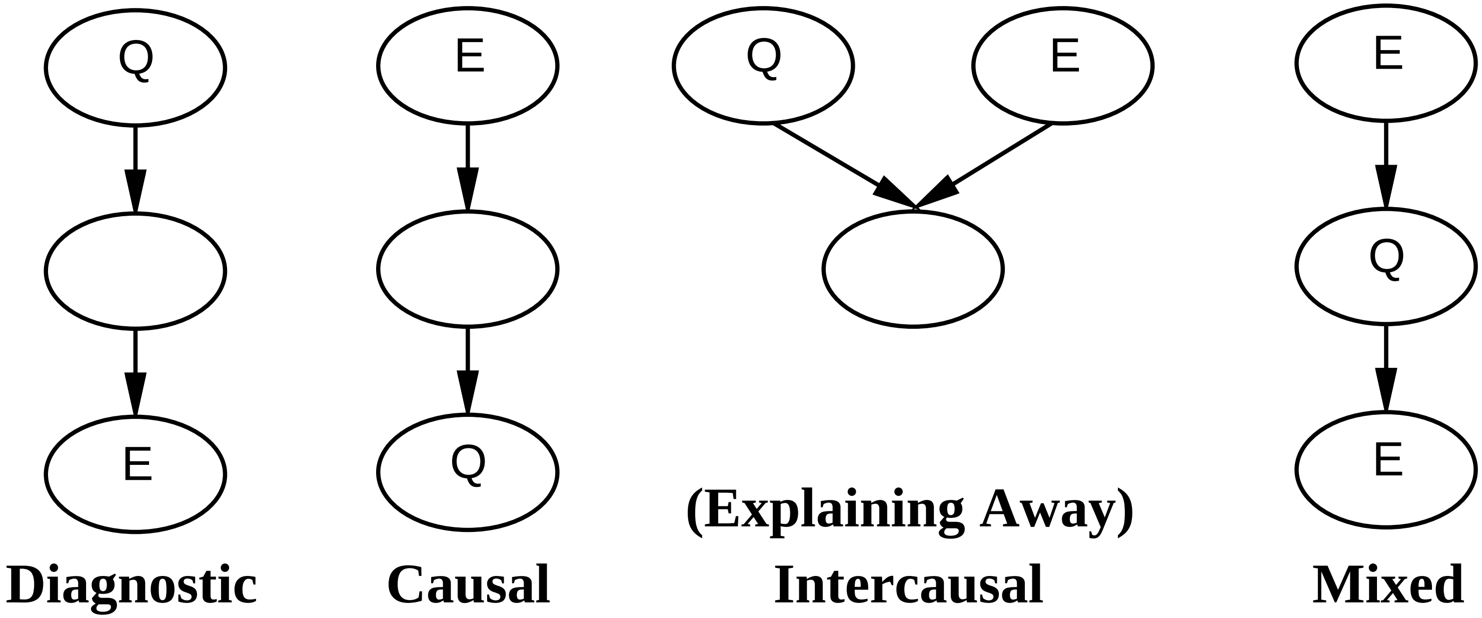
Intercausal inference (between causes of a common effect)

$P(Burglary \mid Alarm \wedge Earthquake)$

Mixed inference (combination of the above)

$P(Alarm \mid JohnCalls \wedge \neg Earthquake)$

Types of Inference in Belief Networks



Q = query variable

E = evidence variable

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i \mid \mathbf{E} = \mathbf{e})$

E.g., $P(NoGas \mid Gauge = empty, Lights = on, Starts = false)$

Conjunctive queries:

$$\mathbf{P}(X_i, X_j \mid \mathbf{E} = \mathbf{e}) = \mathbf{P}(X_i \mid \mathbf{E} = \mathbf{e})\mathbf{P}(X_j \mid X_i, \mathbf{E} = \mathbf{e})$$

Optimal decisions: decision networks include utility information; probabilistic inference required for $P(outcome \mid action, evidence)$

Value of information: which evidence to seek next?

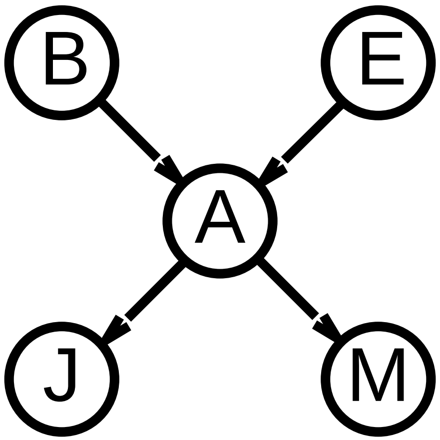
Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:



$$\begin{aligned}&\mathbf{P}(B \mid j, m)\\ &\quad = \mathbf{P}(B, j, m)/P(j, m)\\ &\quad = \alpha\mathbf{P}(B, j, m)\\ &\quad = \alpha \textstyle\sum_e \sum_a \mathbf{P}(B, e, a, j, m)\end{aligned}$$

Rewrite full joint entries using product of CPT entries:

$$\begin{aligned}&\mathbf{P}(B \mid j, m)\\ &\quad = \alpha \textstyle\sum_e \sum_a \mathbf{P}(B)P(e)\mathbf{P}(a \mid B, e)P(j \mid a)P(m \mid a)\\ &\quad = \alpha\mathbf{P}(B) \textstyle\sum_e P(e) \sum_a \mathbf{P}(a \mid B, e)P(j \mid a)P(m \mid a)\end{aligned}$$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Enumeration algorithm

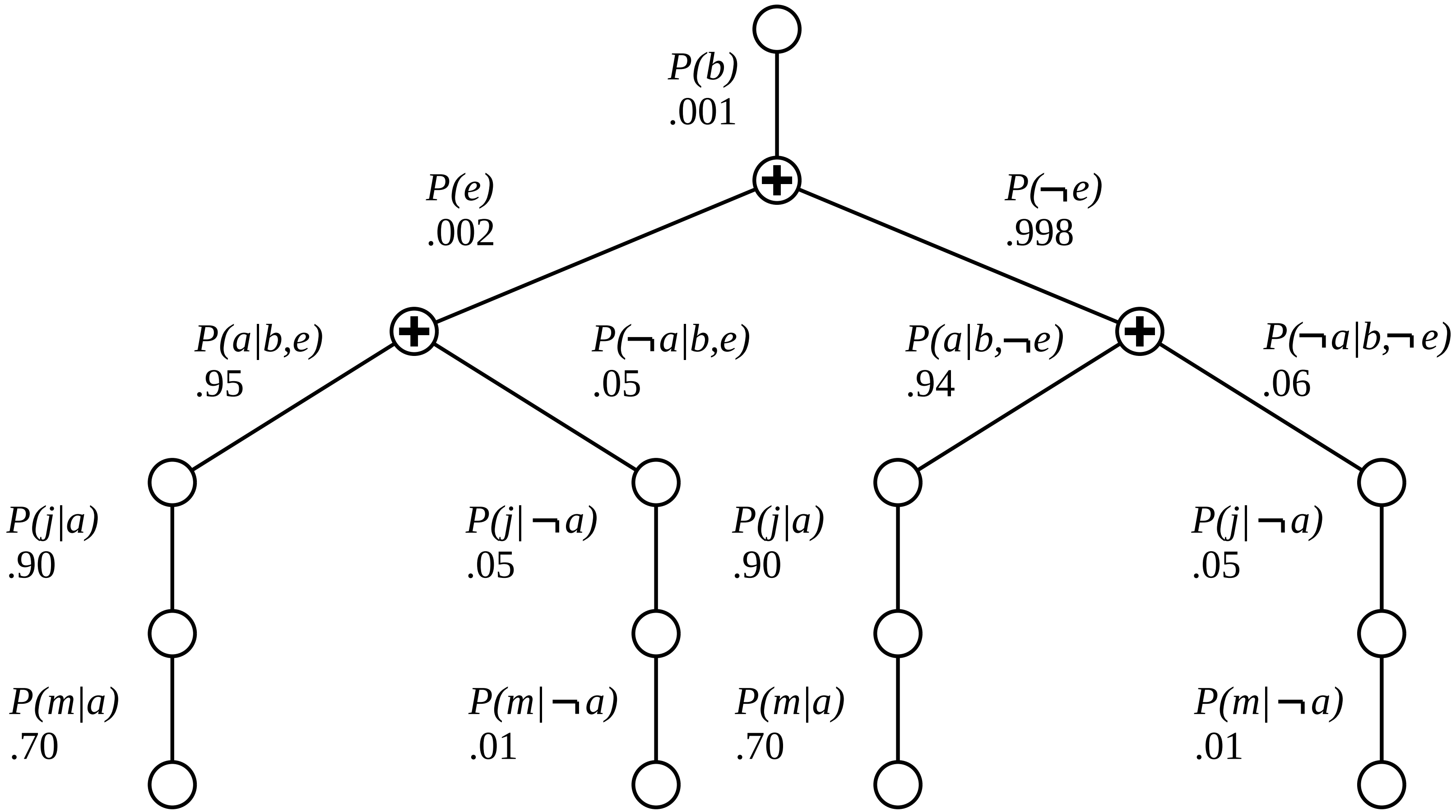
```
function ENUMERATION-ASK(X, e, bn) returns a distribution over X
   inputs: X,  the query variable
           e, observed values for variables E
           bn, a Bayesian network with variables {X} ∪ E ∪ Y

   Q(X) ← a distribution over X, initially empty
   for each value x_i of X do
       extend e with value x_i for X
       Q(x_i) ← ENUMERATE-ALL(VARS[bn], e)
   return NORMALIZE(Q(X))
```

```
function ENUMERATE-ALL(vars, e) returns a real number
   if EMPTY?(vars) then return 1.0
   Y ← FIRST(vars)
   if Y has value y in e
       then return P(y | Pa(Y)) × ENUMERATE-ALL(REST(vars), e)
       else return Σ_y P(y | Pa(Y)) × ENUMERATE-ALL(REST(vars), e_y)
            where e_y is e extended with Y = y
```

Evaluation tree

$\alpha \mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a \mid B, e) P(j \mid a) P(m \mid a)$



Enumeration is inefficient: repeated computation. E.g., computes $P(j \mid a) P(m \mid a)$ for each value of e

Inference by variable elimination

Variable elimination: carry out summations right-to-left, storing intermediate results, *factors*, to avoid recomputation

Use matrix operations

$$
\begin{aligned}
\mathbf{P}(B \mid j, m) &= \alpha \, \sum_e \, \sum_a \, \mathbf{P}(B, e, a, j, m) \\
&= \alpha \underbrace{\mathbf{P}(B)}_{\mathbf{f}_1(B)} \sum_e \underbrace{P(e)}_{\mathbf{f}_2(E)} \sum_a \underbrace{\mathbf{P}(a \mid B, e)}_{\mathbf{f}_3(A,B,E)} \underbrace{P(j \mid a)}_{\mathbf{f}_4(A)} \underbrace{P(m \mid a)}_{\mathbf{f}_5(A)} \\
&= \alpha \, \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \sum_a \mathbf{f}_3(A, B, E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A) \\
&= \alpha \, \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \sum_a \mathbf{f}_{3,4,5}(A, B, E) \\
&= \alpha \, \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B, E) \\
&= \alpha \, \mathbf{f}_1(B) \times \sum_e \mathbf{f}_{2,6}(B, E) \\
&= \alpha \, \mathbf{f}_1(B) \times \mathbf{f}_7(B) \\
&= \alpha \, \mathbf{f}_{1,7}(B)
\end{aligned}
$$

Variable elimination: Basic operations

Summing out a variable from a product of factors:

1. move any constant factors outside the summation
2. add up submatrices in pointwise product of remaining factors

$$\begin{aligned} \textstyle\sum_x f_1 \times \cdots \times f_k &= f_1 \times \cdots \times f_i \textstyle\sum_x f_{i+1} \times \cdots \times f_k \\ &= f_1 \times \cdots \times f_i \times f_{\bar{X}} \end{aligned}$$

assuming $f_1, \ldots, f_i$ do not depend on X

Pointwise product of factors f_1 and f_2:

$$\begin{aligned} & f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) \times f_2(y_1, \ldots, y_k, z_1, \ldots, z_l) \\ & \quad = f(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_l) \end{aligned}$$

E.g., $f_1(A, B) \times f_2(B, C) = f(A, B, C)$

Pointwise Multiplication

A	B	$\mathbf{f}_1(A,B)$	B	C	$\mathbf{f}_2(B,C)$	A	B	C	$\mathbf{f}_3(A,B,C)$
T	T	0.3	T	T	0.2	T	T	T	$0.3 \times 0.2 = 0.06$
T	F	0.7	T	F	0.8	T	T	F	$0.3 \times 0.8 = 0.24$
F	T	0.9	F	T	0.6	T	F	T	$0.7 \times 0.6 = 0.42$
F	F	0.1	F	F	0.4	T	F	F	$0.7 \times 0.4 = 0.28$
						F	T	T	$0.9 \times 0.2 = 0.18$
						F	T	F	$0.9 \times 0.8 = 0.72$
						F	F	T	$0.1 \times 0.6 = 0.06$
						F	F	F	$0.1 \times 0.4 = 0.04$

Variable elimination algorithm

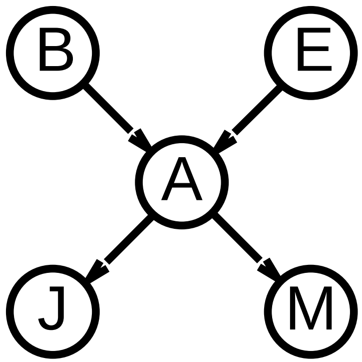
function ELIMINATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, evidence specified as an event
    bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $factors \leftarrow [\,]$; $vars \leftarrow$ REVERSE(VARS[bn])
  for each var **in** $vars$ **do**
    $factors \leftarrow$ [MAKE-FACTOR($var, \mathbf{e}$)|$factors$]
    if var is a hidden variable **then** $factors \leftarrow$ SUM-OUT(var, $factors$)
  return NORMALIZE(POINTWISE-PRODUCT($factors$))

Any ordering of the variables will do for correctness

Following topological order over BN is usually most efficient (although finding optimal ordering is NP-hard)

Irrelevant variables



Consider the query $\mathbf{P}(JohnCalls \mid Burglary = true)$

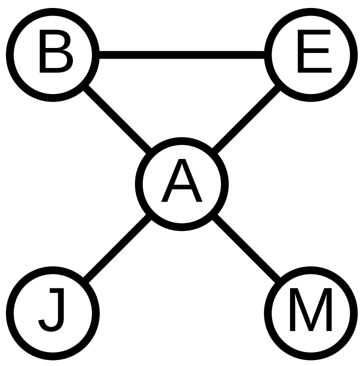
$$\mathbf{P}(J \mid b) = \alpha P(b) \sum_e P(e) \sum_a P(a \mid b, e) \mathbf{P}(J \mid a) \sum_m P(m \mid a)$$

Sum over m is identically 1; $Mary$ is *irrelevant* to the query

Thm 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = J$, $\mathbf{E} = \{B\}$, and $Ancestors(\{X\} \cup \mathbf{E}) = \{A, B, E\}$
so M is irrelevant

Irrelevant variables contd.



The *moral graph* of a belief network is obtained by marrying all parents of the same node and then ignoring edge directions

A set $\mathbf{A}$ of notes is *m-separated* from a set $\mathbf{B}$ by a set $\mathbf{C}$ iff it is separated by $\mathbf{C}$ in the moral graph

Thm 2: Y is irrelevant if m-separated from X by $\mathbf{E}$

For $P(JohnCalls \mid Alarm = true)$, both $Burglary$ and $Earthquake$ are irrelevant

Complexity of exact inference

Singly connected networks (or polytrees):

- any two nodes are connected by at most one (undirected) path
- time and space cost of variable elimination are $O(d^k n)$

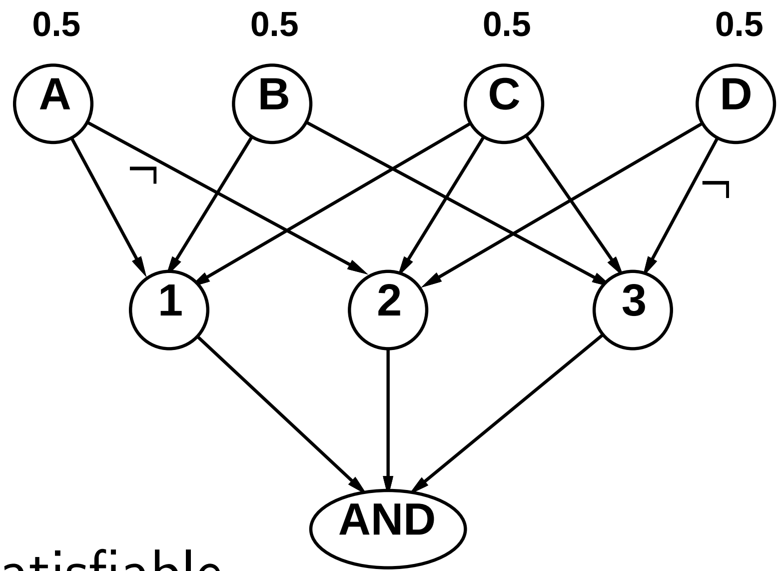
Multiply connected networks:

- can reduce 3SAT to exact inference $\Longrightarrow$ NP-hard
- equivalent to counting 3SAT models $\Longrightarrow$ #P-complete

1. A v B v C

2. C v D v ¬A

3. B v C v ¬D



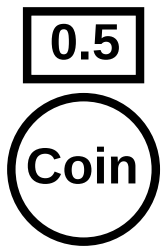
$P(AND) > 0$ iff $\{1, 2, 3\}$ is satisfiable

Approximate inference in belief networks

Inference by stochastic simulation

Basic idea:

1. Draw N samples from a sampling distribution S



2. Compute an approximate posterior probability $\hat{P}$
3. Show this converges to the true probability P

Inference by stochastic simulation

Direct Sampling

- Basic sampling: sampling with no evidence
- Rejection sampling: reject samples disagreeing with evidence
- Likelihood weighting: use evidence to weight samples

Markov chain simulation

- Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

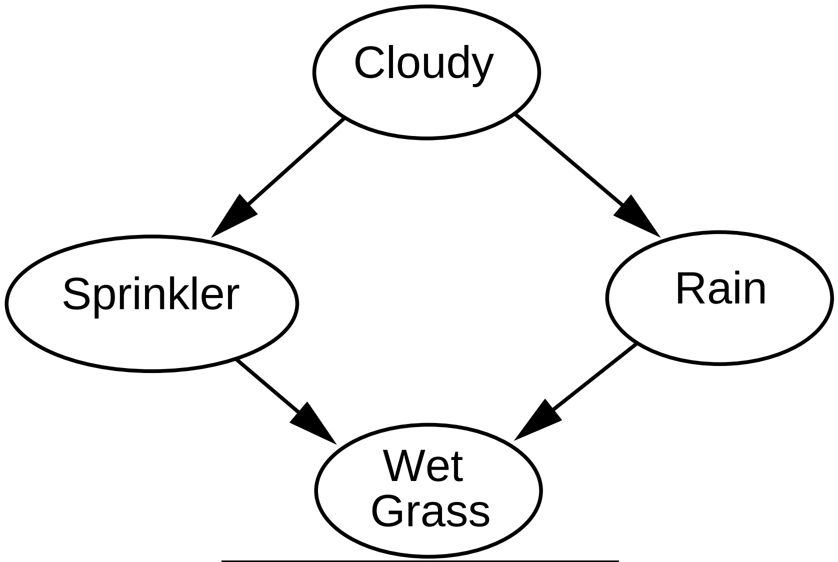
Sampling with no evidence

function PRIOR-SAMPLE(bn) **returns** an event sampled from bn
  inputs: bn, a belief network specifying jpd $\mathbf{P}(X_1, \ldots, X_n)$

  $\mathbf{x} \leftarrow$ an event with n elements
  for $i = 1$ **to** n **do**
    $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid Parents(X_i))$
  return $\mathbf{x}$

Example

P(C)
.50



C	P(S|C)
T	.10
F	.50

C	P(R|C)
T	.80
F	.20

S	R	P(W|S,R)
T	T	.99
T	F	.90
F	T	.90
F	F	.01

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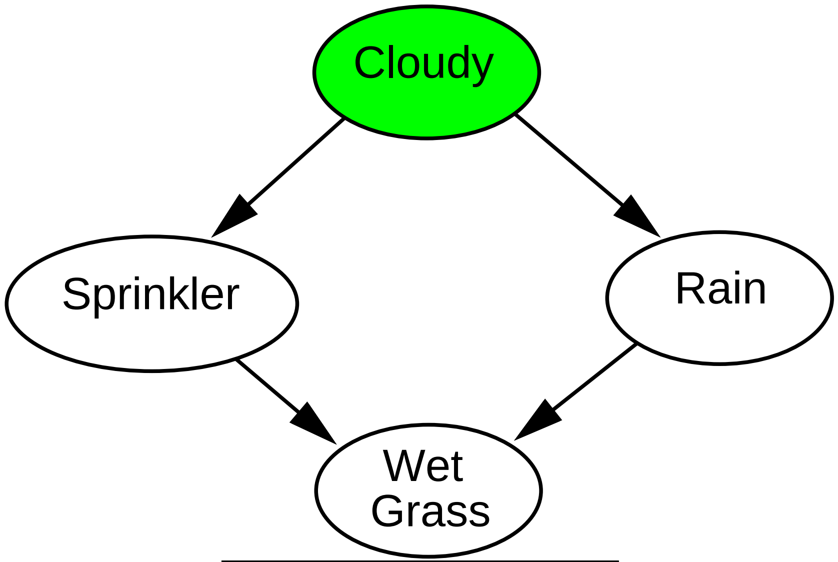

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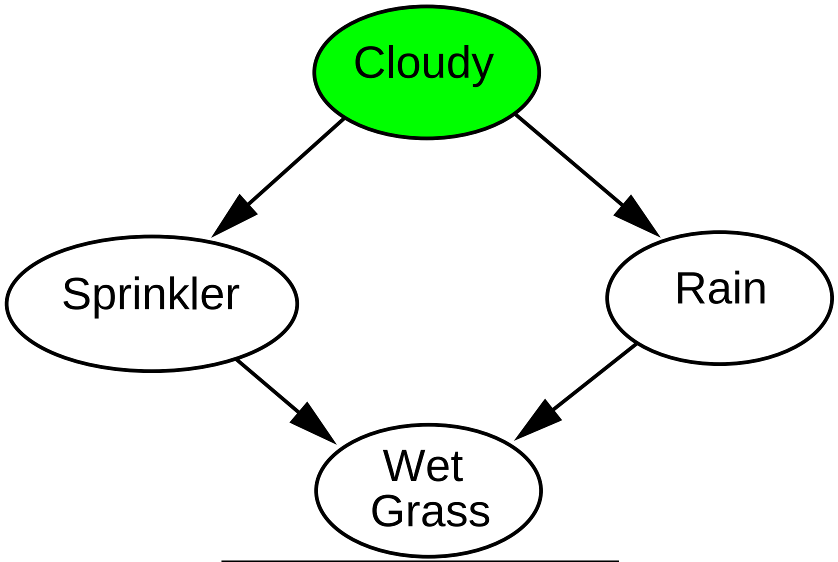

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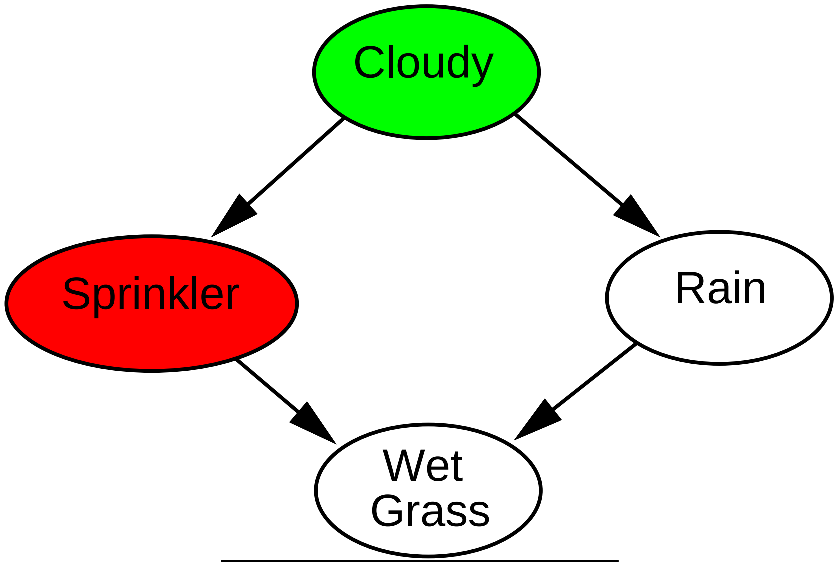

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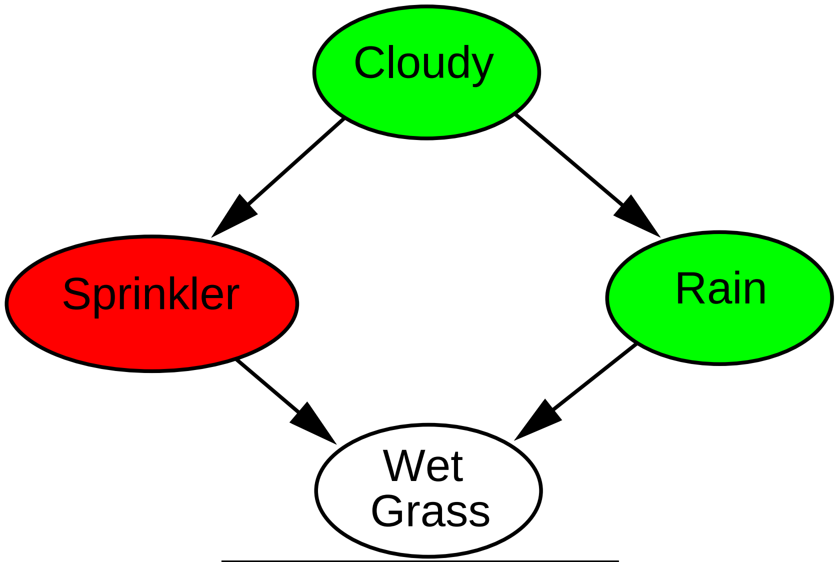

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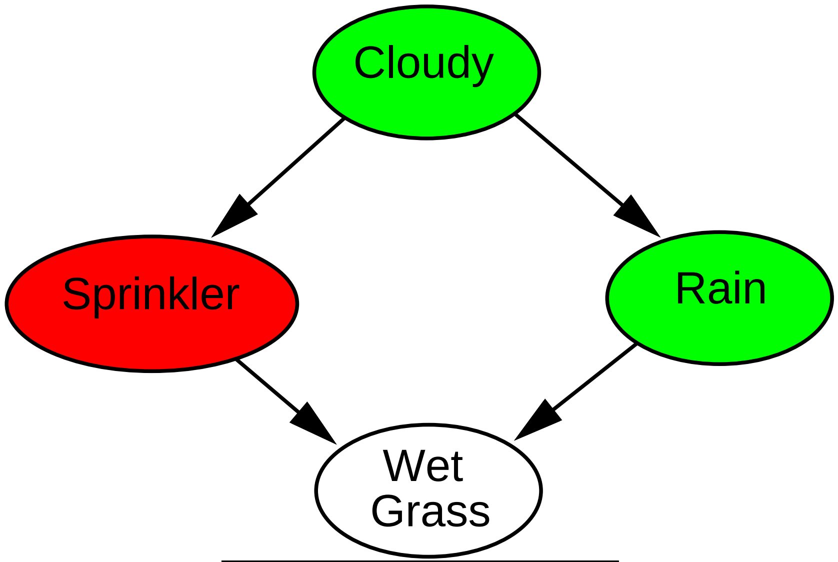

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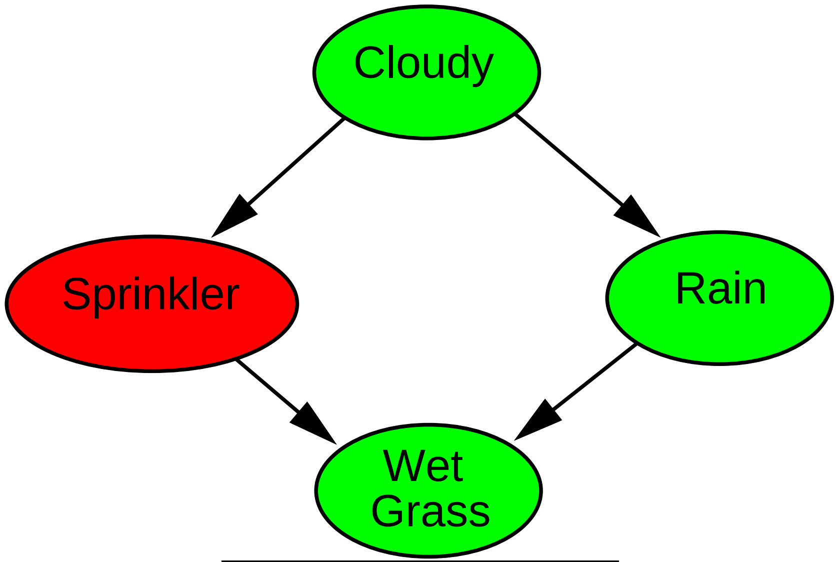

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Generated event:

$c \wedge \neg s \wedge r \wedge w$

Sampling from an empty network contd.

Probability that $\textsc{PriorSample}$ generates a particular event:

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i \mid Parents(X_i)) = P(x_1 \ldots x_n)$$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of times event $x_1, \ldots, x_n$ was generated and N the total number of samples. Then,

$$\begin{aligned}
\lim_{N\to\infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N \\
&= S_{PS}(x_1, \ldots, x_n) \\
&= P(x_1 \ldots x_n)
\end{aligned}$$

That is, estimates derived from $\textsc{PriorSample}$ are *consistent*

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection sampling (with evidence e)

$\hat{\mathbf{P}}(X \mid \mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function REJ-SAMPLING(X, e, bn, N) returns an estimate of P(X|e)
    local vars: N, a vector of counts over X, initially zero
    for j = 1 to N do
        x ← PRIOR-SAMPLE(bn)
        if x is consistent with e then
            N[x] ← N[x]+1 where x is the value of X in x
    return NORMALIZE(N[X])
```

E.g., estimate $\mathbf{P}(Rain \mid Sprinkler = true)$ using 100 samples

27 samples have $Sprinkler = true$. Of these, 8 have $Rain = true$

$\hat{\mathbf{P}}(Rain \mid Sprinkler = true) = \text{NORMALIZE}(\langle 8, 19 \rangle) = \langle 0.296, 0.704 \rangle$

Similar to a basic real-world empirical estimation procedure

Analysis of rejection sampling

$$\begin{array}{rcll}
\hat{\mathbf{P}}(X \mid \mathbf{e}) & = & \alpha \mathbf{N}_{PS}(X, \mathbf{e}) & \text{(algorithm defn.)} \\
& = & \mathbf{N}_{PS}(X, \mathbf{e}) / N_{PS}(\mathbf{e}) & \text{(normalized by } N_{PS}(\mathbf{e})\text{)} \\
& \approx & \mathbf{P}(X, \mathbf{e}) / P(\mathbf{e}) & \text{(property of } \textsc{PriorSample}\text{)} \\
& = & \mathbf{P}(X \mid \mathbf{e}) & \text{(defn. of conditional probability)}
\end{array}$$

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

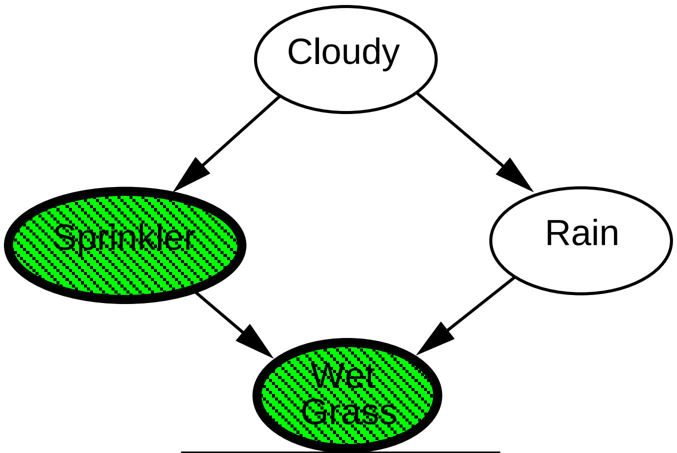
Likelihood weighting (with evidence $\mathbf{e}$)

Idea:

- fix evidence variables
- sample only non-evidence variables,
- weight each sample by the likelihood it accords the evidence

Likelihood weighting example

Query: $\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$



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$weight = 1.0$

Likelihood weighting example

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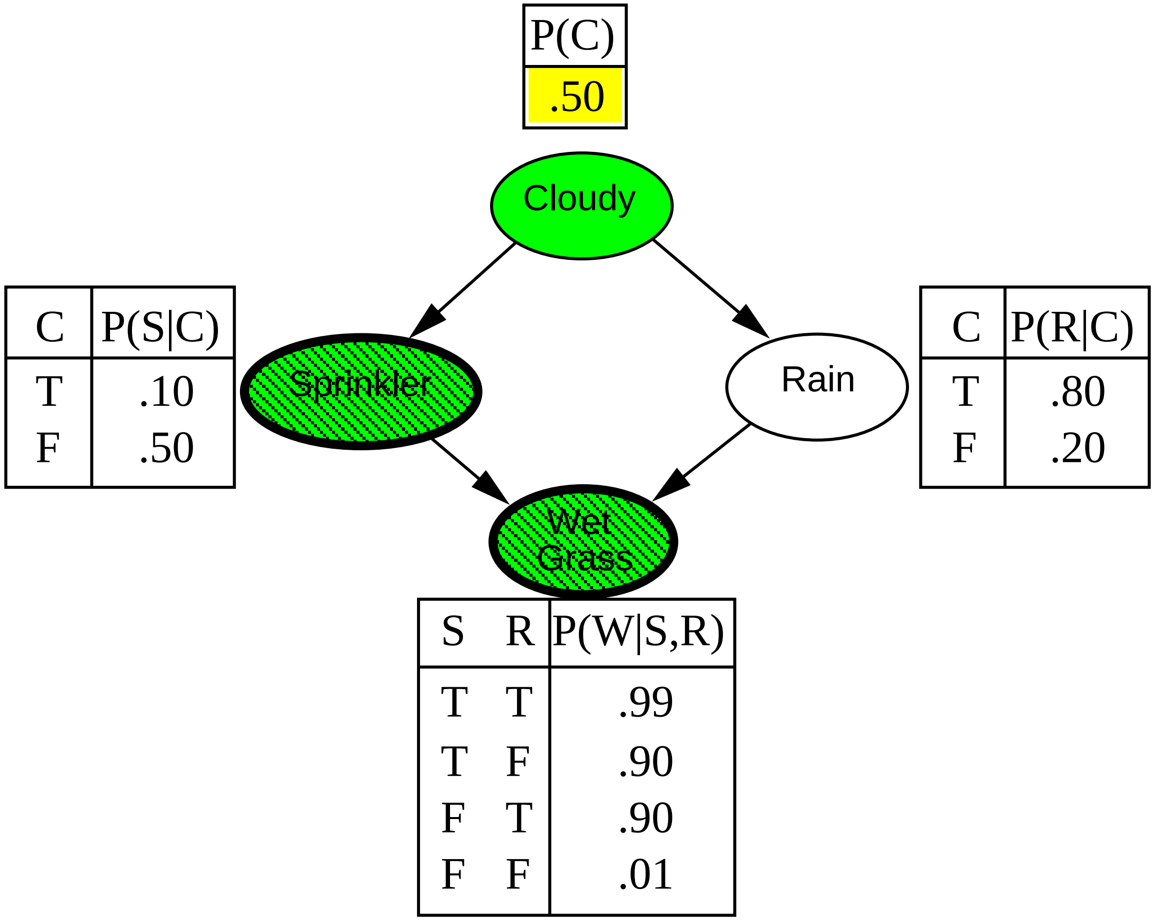

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Likelihood weighting example

Query: $\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$

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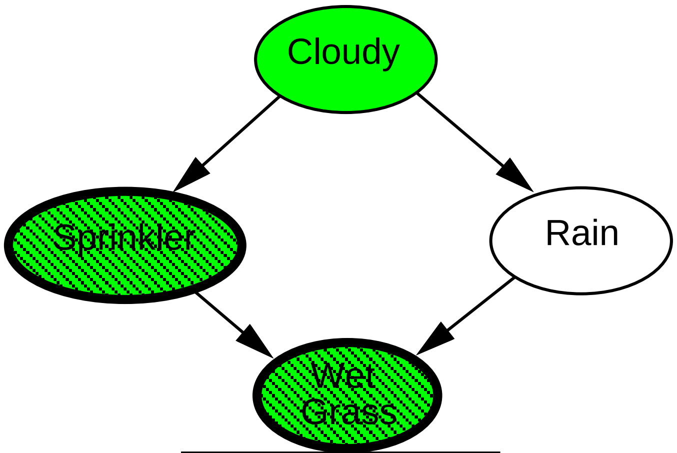

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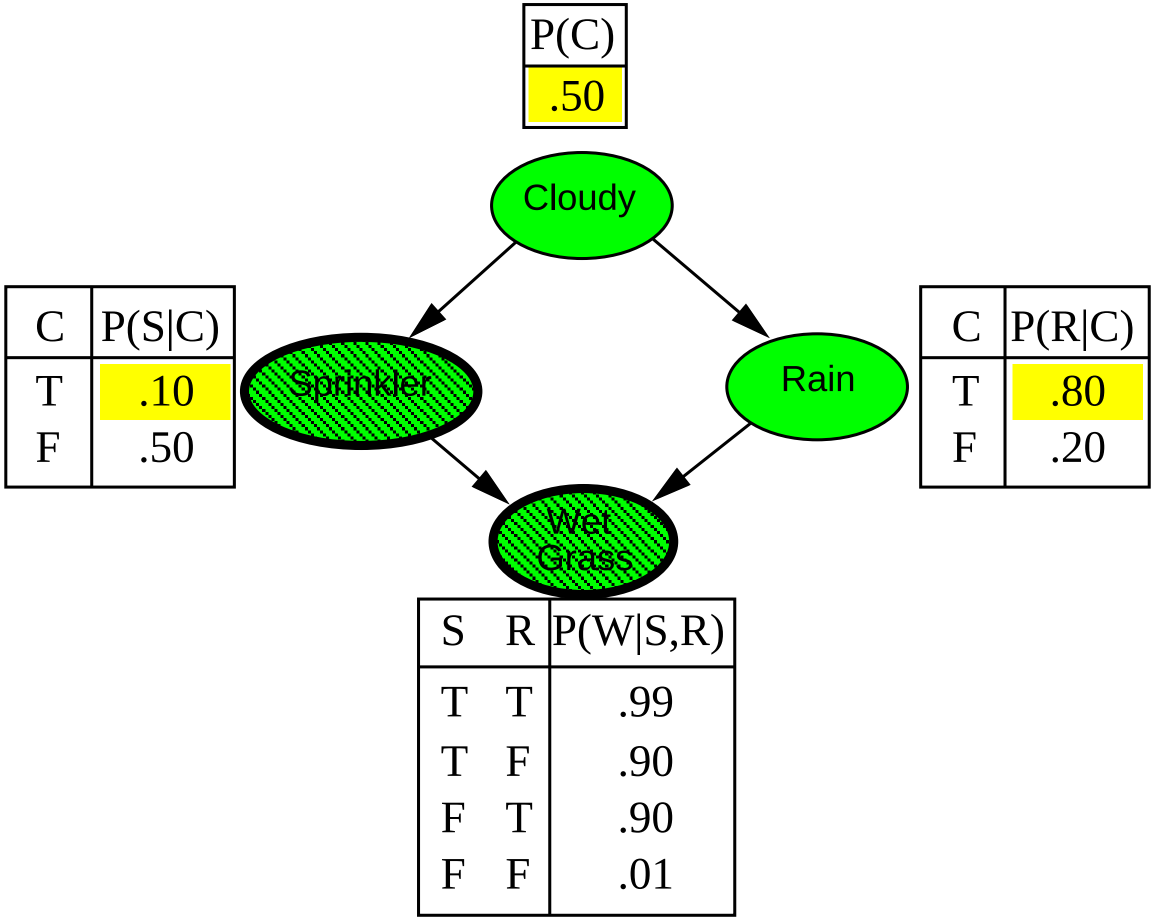
S	R	P(W|S,R)
T	T	.99
T	F	.90
F	T	.90
F	F	.01

$weight = 1.0 \times 0.1$

Likelihood weighting example

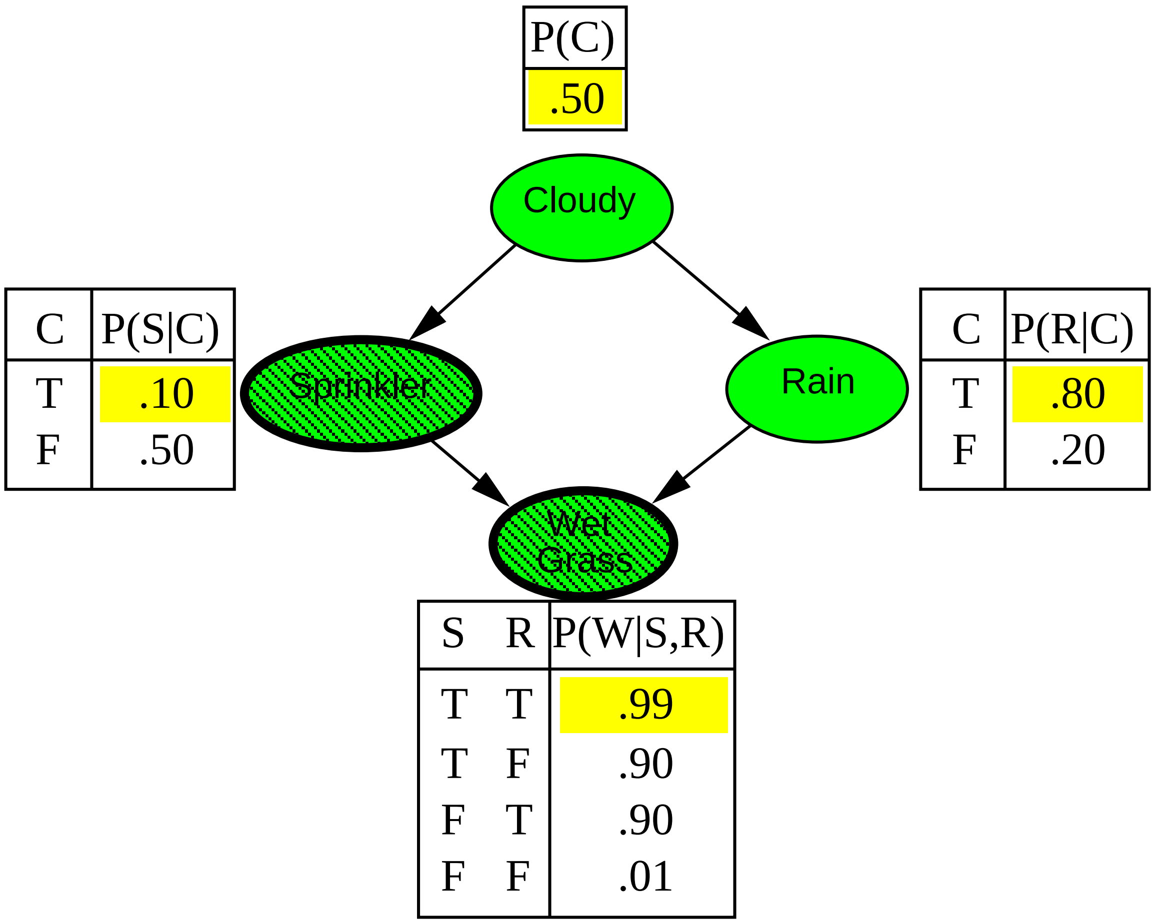
Query: $\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$



$weight = 1.0 \times 0.1$

Likelihood weighting example

Query: $\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$



$weight = 1.0 \times 0.1 \times 0.99 = 0.099$
$event = c \wedge s \wedge r \wedge w$

Likelihood weighting (with evidence $\mathbf{e}$)

function LIKELYHOOD-W(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local vars: $\mathbf{W}$, a vector of weighted counts over X, initially zero
  for $j = 1$ to N **do**
    $\mathbf{x}$, $w \leftarrow$ WEIGHTED-SAMPLE(bn)
    $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{W}[X]$)

function WEIGHTED-SAMPLE(bn, $\mathbf{e}$) **returns** an event and a weight
  $\mathbf{x} \leftarrow$ an event with n elements; $w \leftarrow 1$
  for $i = 1$ **to** n **do**
    if X_i has a value x_i in $\mathbf{e}$
      then $w \leftarrow w \times\ P(X_i = x_i \mid Parents(X_i))$
      else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid Parents(X_i))$
  return $\mathbf{x}$, w

Likelihood weighting analysis

Sampling probability for WEIGHTED-SAMPLE is

$$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{l} P(z_i \mid Parents(Z_i))$$

Note: pays attention to evidence in ancestors only $\Longrightarrow$ somewhere "in between" prior and posterior distribution

Weight for a given sample $\mathbf{z},\mathbf{e}$ is

$$w(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{m} P(e_i \mid Parents(E_i))$$

Weighted sampling probability is

$$\begin{aligned} & S_{WS}(\mathbf{z},\mathbf{e}) w(\mathbf{z},\mathbf{e}) \\ & \quad = \prod_{i=1}^{l} P(z_i \mid Parents(Z_i)) \ \prod_{i=1}^{m} P(e_i \mid Parents(E_i)) \\ & \quad = P(\mathbf{z},\mathbf{e}) \text{ (by standard global semantics of network)} \end{aligned}$$

Hence likelihood weighting returns consistent estimates
but performance still degrades with many evidence variables
because a few samples have nearly all the total weight

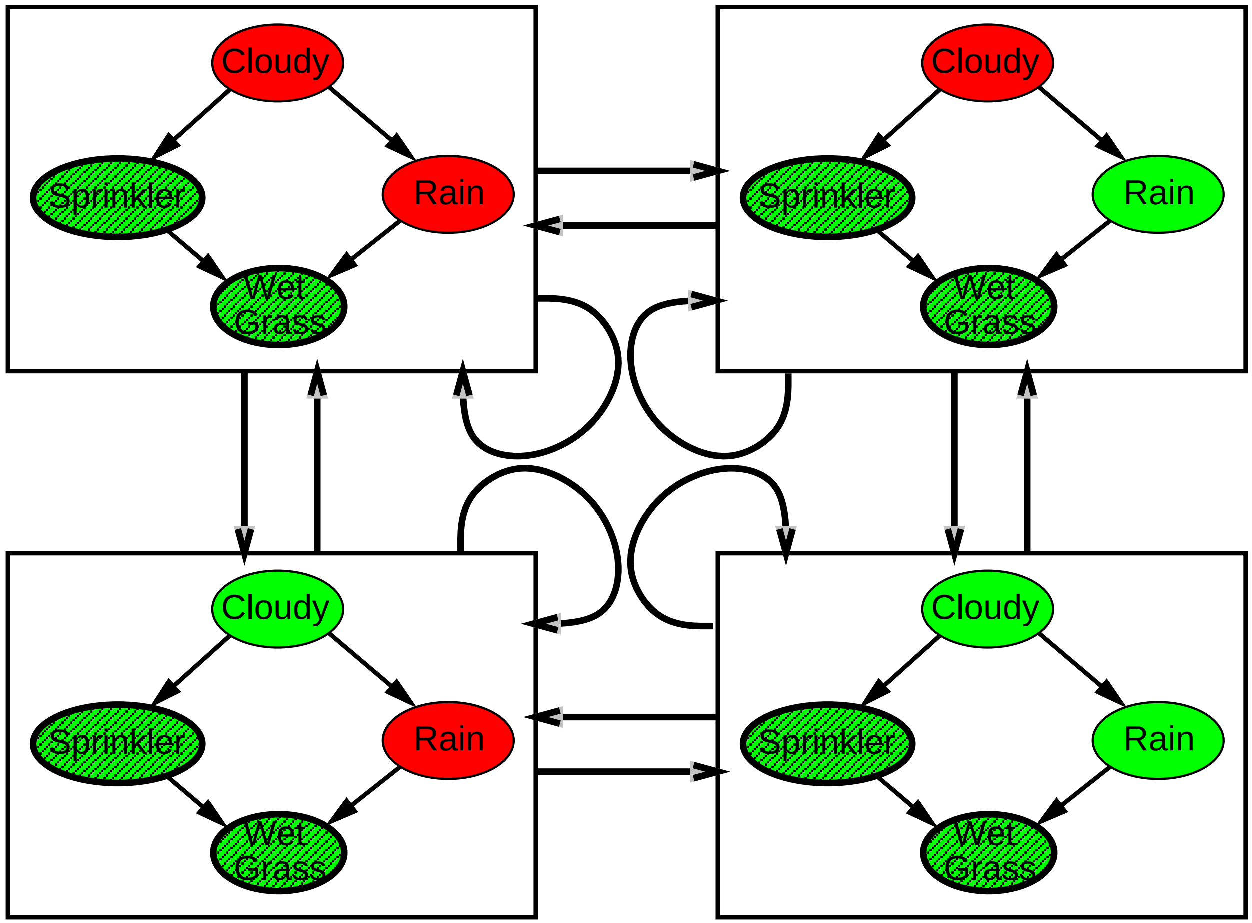
Approximate inference using MCMC

- "State" of network = current assignment to all of its variables
- Generate next state by sampling one var. given its Markov Blanket
- Sample each variable in turn, keeping evidence fixed

function GIBBS-ASK($X, \mathbf{e}, bn, N$) **returns** an estimate of $P(X|\mathbf{e})$
    local vars: $\mathbf{N}$, a vector of counts for each value of X, initially zero
                $\mathbf{Z}$, the nonevidence variables in bn
                $\mathbf{x}$, the current state of the network, initially copied from $\mathbf{e}$
    initialize $\mathbf{x}$ with random values for the variables in $\mathbf{Z}$
    for $j = 1$ to N **do**
        for each Z_i in $\mathbf{Z}$ **do**
            set the value of Z_i in $\mathbf{x}$ by sampling from $\mathbf{P}(Z_i \mid MB(Z_i))$
            $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where x is the value of X in $\mathbf{x}$
    return NORMALIZE($\mathbf{N}$)

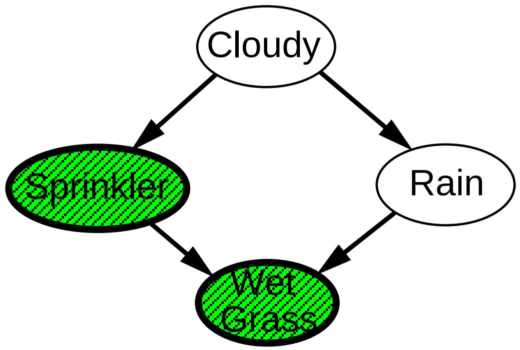
The Markov chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

Markov blanket sampling



$$
\begin{aligned}
MB(Cloudy) &= \{Sprinkler, Rain\} \\
MB(Rain) &= \{Cloudy, Sprinkler, WetGrass\}
\end{aligned}
$$

Probability given the Markov blanket is calculated as

$P(x_i \mid mb(X_i))$
$\quad = \alpha P(x_i \mid Parents(X_i)) \prod_{Z_j \in Children(X_i)} P(z_j \mid Parents(Z_j))$

where $mb(X_i)$ denotes the values (in the current state) of the variables in X_i's Markov blanket $MB(X_i)$

MCMC example

To estimate $\mathbf{P}(Rain \mid Sprinkler = true, WetGrass = true)$

1. Apply the Gibbs sampling algorithm with $Sprinkler$ and $WetGrass$ both fixed to $true$
2. Count number of times $Rain$ is $true$ and $false$ in the samples

Example:
Visit 100 states; 31 have $Rain = true$, 69 have $Rain = false$

$$\hat{\mathbf{P}}(Rain \mid Sprinkler = true, WetGrass = true)$$
$$= \textsc{Normalize}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$$

Theorem: Markov chain approaches *stationary distribution*, i.e., over the long run, the fraction of time spent in each state is exactly proportional to the state's posterior probability