A Brief History of Logic

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1 Trivia

The word trivial has an interesting entymology. It is composed of "tri" (meaning "3") and "via" (meaning "ways"). It originally referred to the *trivium*, the three fundamental curriculae: grammar, rhetorics, and logic. Mastery of these subjects was considered essential before study could continue with the quadrivium, which consisted of arithmetic, geometry, music, and astronomy.

Why was logic considered to be fundamental to one's education? To answer this question, it is necessary to explain what we mean by the term "logic".

Lewis Carroll, Through the Looking Glass:

> "Contrariwise," continued Tweedledee, "if it was so, it might be;¡ and if it were so, it would be; but as it isn't, it ain't. That's logic."

Some attempts by the class to define logic were:

1. The ability to determine correct answers through a standardized process.
2. The study of formal inference.
3. A sequence of verified statements.
4. Reasoning, as opposed to intuition.
5. The deduction of statements from a set of statements.

In a sense, all of these definitions are correct.

Logic was originally studied by the Sophists, who engaged in formal debates. Eventually, they sought to devise an objective system of rules to determine beyond any doubt who had won an argument. Logic was devised for this purpose.

As Francis Bacon put it, in 1605,

> "Logic differeth from rhetoric..in this, that logic handleth reason exact and in truth, and rhetoric handleth it as it is planted in popular opinions and manners."

So logic deals with a set of rules for reasoning and arguing. Thus, it deals with a fundamental problem in intellectual pursuits: how to distinguish what is true from what is false, what is right from what is wrong.

2 The First Age of Logic: Symbolic Logic (500 B.C. - 19th Century)

Originally, logic dealt with arguments in the natural languages used by humans. For example, it would be used to demonstrate the correctness of arguments like the following:

> All men are mortal.
>
> Socrates is a man.
>
> Therefore, Socrates is mortal.

If we change "all" to "some", the argument doesn't hold. But how could we demonstrate this? We could attempt to define words such as "all" and "some" in terms of what inferences we could draw from them. Then the demonstration would follow from these definitions. The problem is that natural language turns out to be very ambiguous. For example, consider the word "any". In the sentence

> "Eric does not believe that Mary can pass any test."

it could be taken to mean either "all" or "one". (Note: in this class, or in any technical context, don't use the term "any").

Also, consider the sentence

> "I only borrowed your car.".

Different inflections imply different meanings.

Another problem with natural language was that it led to many paradoxes. The most famous is The Liar's Paradox. Consider the sentence

> "This sentence is a lie."

If it were true, then it must be a lie, as it says, but this is impossible. Similarly, if it were a lie, then what it says is true!

Other examples are:

The Sophist's Paradox. A Sophist is sued for his tuition by the school that educated him. He argues that he must win, since, if he loses, the school didn't educate him well enough, and doesn't deserve the money. The school argues that he must lose, since, if he wins, he was educated well enough and therefore should pay for it.

The Surprise paradox. A logic professor announces to his class that there will be a test next week, but he won't tell which day. He promises, "When you get it, you'll be surprised.". The students deduce that the test can't be given on Friday: if it were, then, come Friday morning, they would know it had to be given on that day, and, therefore, they wouldn't be surprised. But since they now know it can't be on Friday, they reason that it can't be on Thursday either, since they would no longer be surprised on Thursday morning. Similarly, it couldn't be held on Wednesday, Tuesday, or Monday, But on Tuesday, the professor does give the test, and the students are very surprised.

These paradoxes, along with the ambiguities of natural language, eventually led to the effort to formulate logic in a symbolic language.

3 The 2nd Age of Logic: Algebraic Logic (Mid to late 19th Century)

In 1847, George Boole, in "The Mathematical Analysis of Logic" attempted to formulate logic in terms of a mathematical language. Rules of inference were modeled after various laws for manipulating algebraic expressions:

> "The design of the following treatise is to investigate the fundamental laws of the operations of the mind by which reasoning is performed; to give expressions to them in the symbolic language of a calculus, and upon this foundation to establish the science of logic and construct its methods."

The basis for his work was the similarity between the relationship of set union and intersection and that of numerical addition and multiplication, e.g.,

$a(b+c) = (ab) + (ac)$ is similar to $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.

Unfortunately, Boole would sometimes take the analogy too far. For example, he tried to find a set operation analogous to division, even though no such operation exists in logic.

Soon to follow Boole's work was that of Charles Ludwig Dodgeson, known as Lewis Carol, who published several texts on the subject, and developed Venn Diagrams as a means of reasoning about sets.

It was also during this period that Ernst Schröeder anticipated the importance of developing fast algorithms to decide various logical and mathematical problems. In "The Algebra of Logic" he writes:

> "Getting a handle on the consequences of any premisses, or at least the fastest method for obtaining these consequences, seems to me to be one of the noblest, if not the ultimate goal of mathematics and logic."

Once symbolic logic matured, it became tremendously useful in resolving many serious problems developing in mathematics.

4 The 3rd Age of Logic: Mathematical Logic (late 19th to mid 20th Century)

As mathematical proofs became more sophisticated, paradoxes began to show up in them just as they did in natural language.

For example, in 1820: Cauchy "proved" that for all infinite sequences

$$f_1(x), f_2(x), f_2(x), \cdots$$

of continous functions, the sum

$$f(x) = \sum_{i=1}^{\infty} f_i(x)$$

was also continuous. But in 1826 Abel found a counterexample!

To deal with issues such as these, in 1879, Frege proposed logic as a language for mathematics. This development was anticipated centuries earlier by Leibnitz, but it wasn't until the late 19th Century that the logical tools existed to allow for it. The rigor of mathematical proofs increased dramatically. In the field of analysis, the now standard epsilon-delta definition of limits resolved ambiguities such as the one above.

With the rigor of this new foundation, Cantor was able to analyze the notion of infinity in ways that were previously impossible. He concluded that, instead of there being just one infinity, there was a whole hierarchy of them. His argument was follows:

Consider the set 2^N of all subsets natural numbers. Suppose, for a contradiction, that there is only one infinity. Then this set, which is infinitely large, must be of the same size as the natural numbers, which is also infinitely large. Therefore, we could assign each subset of N a distinct $n \in N$, forming an infinite sequence $P_0, P_1, P_2, \ldots$ of subsets of N. Now consider the set Q of natural numbers n such that $n \notin P_n$. Since $Q \in 2^N$, it must have been assigned some number j. But then consider the question of whether $j \in Q$:

$$j \in Q \Leftrightarrow j \in P_j \Leftrightarrow j \notin Q$$

A contradiction.

Therefore, our original supposition is incorrect, and 2^N is strictly larger than N. (Note that this leaves open the question of the existence of other sets with sizes falling in between these constructed sizes. See below.) We could further devise an analogous argument that 2^{2^N} is strictly larger than 2^N, and so on, forming an infinite hierarchy of infinities.

So how do we determine the size of infinite sets? Cantor defined the relative size of two sets as follows:

> Definition: $|A| \leq |B|$ iff there exists a one-to-one function from A to B.

This leads to a range of all possible sizes known as the cardinal numbers. It starts with a size for each of the natural numbers, and then continues with sizes for infinitely large sets. These new sizes are denoted with the symbol $\aleph$ (read as "aleph"). The first such size ($\aleph_0$) is the size of the natural numbers.

$$0, 1, ...\aleph_0, \aleph_1, \ldots$$

Then, at the turn of the 20th Century, David Hilbert, the most prominent mathematician of his time, proposed a grand program to devise a single formal procedure that would derive all mathematical truth:

> "Once a logical formalism is established one can expect that a systematic, so-to-say computational, treatment of logic formulas is possible, which would somewhat correspond to the theory of equations in algebra."

The first major stumbling block came when Bertrand Russell discovered that naive set theory, again like the natural language logics, led to paradoxes. Russell wondered whether the collection of all sets is a set. He concluded that this leads to the so-called *Russell's Paradox*: Consider the set

$$T = \{S | S \notin S\}$$

Then $T \in T \Leftrightarrow T \notin T$ (a contradiction).

Russell worked around this problem by reformulating mathematics in terms of a hierarchy of sets. Sets of a given hierarchy could only contain sets from lower hierarchies. His work culminated in the Principia Mathematica, a joint work with Alfred North Whitehead. This work formally proved (i.e., with sheer symbolic manipulation) the bulk of the mathematical knowledge of its time.

In addition, mathematicians soon took advantage of the fact that symbolic logic, being a formal system, could itself be the object of mathematical investigation. To the limited extent that logic formed a foundation for mathematics, the results of this investigation yielded results on the nature of mathematics itself.

For example, a natural question that arose from Cantor's proof of higher infinities was: what is the relationship between $2_0^{\aleph}$ and $\aleph_1$? It was conjectured that $2_0^{\aleph} = \aleph_1$. This was known as the Continuum Hypothesis (CH). Despite many attempts to prove and disprove it, nobody succeeded. Later, in the 1930's, Kurt Gödel proved that, within the framework of logic and formal set theory as the foundation of mathematics, supposing CH to be true will never lead to a contradiction. Then, in the 1960's, Cohen proved that supposing the negation of CH wouldn't lead to a contradiction either. This established the independence of CH from mathematics as based on set theory.

Although the analysis of logic as a mathematical object proved to be a very powerful technique, it was soon to yield two results that proved devastating to Hilbert's Program:

1. Gödel's First Incompleteness Theorem. Kurt Gödel proved that, in a formal system powerful enough to form statements about what it can

prove, there will always be true statements that the system can express but can't prove. These statements could be proven with even more powerful systems, but these new systems could then express new statements that they couldn't prove, and so on.

2. Gödel's Second Incompleteness Theorem. Kurt Gödel proved that a formal system powerful enough to form statements about arithmetics cannot prove its own consistency.

3. Alonzo Church and Alan Turing showed that there are some problems that no algorithm could ever solve. If such problems exist, then there could be no hope of finding a single algorithm to produce all mathematical truth.

Despite these results, logic continued to flourish, not as the universally accepted ultimate foundation of all mathematics, but simply as another branch of it. Also, various independent formal systems could still serve as the foundations for the individual, well-defined, branches of mathematics.

5 The 4th Age of Logic: Logic in Computer Science

Logic gave us characterization of computability or solvability. Before 1920's people did computing in their heads, which became incredibly complicated with time. Today, with the advent of the electronic computer, a new home has been found for logic: computer science. In computer science, we design and study systems through the use of formal languages that can themselves be interpreted by a formal system. In short, "Description is our business". So it is quite natural that the formal descriptive languages of modern logic could serve as a working tool for computer science. Some of the most basic applications of this tool are:

1. Boolean circuits: The design of hardware built out of gates that implement Boolean logic primitives. This forms the foundation of modern digital design. ENIAC - an early digital computer implemented in decimal digits. But it was found that working in boolean logic is much easier. Billions of dollars are spent yearly in this industry. Boole intended to discovered the "laws of the mind", but his biggest impact has been on the computer industry. In Electrical Engineering boolean logic is hardware consisting of gates.

2. Some problems seem to be so hard that computers cannot solve them no matter how fast they are. The reason for the difficulty is a combinatorial explosion that seem to be inherent in this problems. Logic played a crucial role in the development of the theory of NP-completeness, which formalize the concept of combinatorial explosion.

3. A computer has to be told what to do, in very precise and formal way. An application of this exact description is SQL (Standard Query Language): a language used for interfacing with databases. Although the syntax is different, it is essentially equivalent to standard first-order logic.

4. Semantics: To make sure that different implementation of a programming language yield the same results, programming languages need to have a formal semantics. Logic provides the tool to develop such a semantics.

5. Design Validation and verification: to verify the correctness of a design with a certainty beyond that of conventional testing. It uses temporal loic. The famous Intel bug of 1995, involving faulty floating point divisions cost 500 million dollars.

6. AI: mechanized reasoning and expert systems. There are many domains in which expertise, acquired by humans over decades, can be described with a formal system. The resulting system can often yield results on par with the human expert, and sometimes better. It tries to capture "common-sense reasoning" in a formal way.

7. Security: With increasing use of network, security has become a big issue. Hence, the concept of *proof-carrying code*. If an applet came with it's own proof of safety and correctness, then it would be really nice.

As our understanding of computer science, and logic, improve, increasingly deeper connections are made. As a result, logic is sometimes described as the "calculus of computer science". We will learn propositional and First-Order Logic, with applications in Complexity, Database, and Verification.