CNF

- A formula A is in conjunctive normal form, or simply CNF, if it is either $\top$, or $\bot$, or a conjunction of disjunctions of literals:

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

(That is, a conjunction of clauses.)

- A formula B is called a conjunctive normal form of a formula A if B is equivalent to A and B is in conjunctive normal form.

Satisfiability on CNF

- An interpretation I satisfies a formula in CNF

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

if and only if it satisfies every clause

$$\bigvee_j L_{i,j}.$$

in it.

- An interpretation I satisfies a clause

$$L_1 \vee \ldots \vee L_k$$

if and only if it satisfies at least one literal L_m in this clause.

CNF transformation

$$
\begin{aligned}
A \leftrightarrow B &\;\Rightarrow\; (\neg A \vee B) \wedge (\neg B \vee A), \\
A \rightarrow B &\;\Rightarrow\; \neg A \vee B, \\
\neg(A \wedge B) &\;\Rightarrow\; \neg A \vee \neg B, \\
\neg(A \vee B) &\;\Rightarrow\; \neg A \wedge \neg B, \\
\neg\neg A &\;\Rightarrow\; A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n &\;\Rightarrow\; (A_1 \vee B_1 \vee \ldots \vee B_n) \quad \wedge \\
&\qquad\qquad \ldots \quad \wedge \\
&\quad\; (A_m \vee B_1 \vee \ldots \vee B_n).
\end{aligned}
$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$

$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$

$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$

$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$

$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$

$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$

$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$

$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$

$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

Therefore, the formula

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r))$$

has the same models as the set consisting of four clauses

$$\begin{array}{l} \neg p \lor q \\ \neg p \lor \neg q \lor r \\ p \\ \neg r \end{array}$$

The CNF transformation allows one to reduce the satisfiability problem for formulas to the satisfiability problem for sets of clauses.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

The first step yields:

$$\begin{aligned} & p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\ & \quad (\neg p_1 \lor (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \\ & \land (\neg (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \lor p_1). \end{aligned}$$

If we continue, the formula will grow exponentially.

Idea

Replace subformulas

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

by names:

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n)))$$
$$n \leftrightarrow (p_5 \leftrightarrow p_6)$$

After several steps

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

$$p_1 \leftrightarrow (p_2 \leftrightarrow n_3);$$

$$n_3 \leftrightarrow (p_3 \leftrightarrow n_4);$$

$$n_4 \leftrightarrow (p_4 \leftrightarrow n_5);$$

$$n_5 \leftrightarrow (p_5 \leftrightarrow p_6).$$

Naming

Lemma. Suppose that S is a set of formulas and B is a formula. Let n be a boolean variable not occurring in $S \cup \{B\}$. Then S is satisfiable if and only if so is the set of formulas $S \cup \{n \leftrightarrow B\}$.

Clausal Form

- ▷ **Clausal form of a formula A**: a set of clauses which is satisfiable if and only if A is satisfiable.
- ▷ **Clausal form of a set S of formulas:** a set of clauses which is satisfiable if and only if so is S.

Definitional Clause Form Transformation

This algorithm converts a formula A into a set of clauses S such that S is a clausal normal form of A.

If A has the form $C_1 \wedge \ldots \wedge C_n$, where $n \geq 1$ and each C_i is a clause, then $S \stackrel{\text{def}}{=} \{C_1, \ldots, C_n\}$.

Otherwise, introduce a name for each subformula B of A such that B is not a literal and use this name instead of the formula.

Example

name	subformula	formula to be transformed to CNF	clauses
			p_1
p_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$p_1 \leftrightarrow \neg p_2$	$\neg p_1 \vee \neg p_2$ $p_1 \vee p_2$
p_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$p_2 \leftrightarrow (p_3 \rightarrow p_7)$	$\neg p_2 \vee \neg p_3 \vee p_7$ $p_3 \vee p_2$ $\neg p_7 \vee p_2$
p_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$p_3 \leftrightarrow (p_4 \wedge p_5)$	$\neg p_3 \vee p_4$ $\neg p_3 \vee p_5$ $\neg p_4 \vee \neg p_5 \vee p_3$
p_4	$p \rightarrow q$	$p_4 \leftrightarrow (p \rightarrow q)$	$\neg p_4 \vee \neg p \vee q$ $p \vee p_4$ $\neg q \vee p_4$

Example

name	subformula	formula to be transformed to CNF	clauses
p_5	$p \land q \to r$	$p_5 \leftrightarrow (p_6 \to r)$	$\neg p_5 \lor \neg p_6 \lor r$ $p_6 \lor p_5$ $\neg r \lor p_5$
p_6	$p \land q$	$p_6 \leftrightarrow (p \land q)$	$\neg p_6 \lor p$ $\neg p_6 \lor q$ $\neg p \lor \neg q \lor p_6$
p_7	$p \to r$	$p_7 \leftrightarrow (p \to r)$	$\neg p_7 \lor \neg p \lor r$ $p \lor p_7$ $\neg r \lor p_7$