CS:5810 Formal Methods in Software Engineering

Sets and Relations

These Notes

- Review the concepts of sets and relations required to work with the Alloy language
- Focus on the kind of set operation and definitions used in specifications
- Give some small examples of how we will use sets in specifications

Sets

A set is a *collection* of distinct objects

A set's objects are drawn from a larger *domain* of objects all of which have the same type --- sets are homogeneous

Examples:

{ 2, 4, 5, 6, … }	set of integers
{ red, yellow, blue }	set of colors
{ true, false }	set of boolean values
{ red, true, 2 }	for us, not a set!

Set Values

The value of a set is the collection of its members

Two sets A and B are equal iff

- every member of A is a member of B
- every member of B is a member of A

Notation:

- x ∈ S denotes "x is a member of S"
- ∅ denotes the empty set

Defining Sets

We can define a set by *enumeration*

- PrimaryColors := { red, yellow, blue }
- Boolean := { true, false }
- Evens := { …, -4, -2, 0, 2, 4, … }

This works fine for finite sets, but

- what do we mean by "…" ?
- remember, we want to be precise

Defining Sets

We can define a set by *comprehension*, that is, by describing a property that *all and only* its elements share

Notation: $\{ x : D \mid P(x) \}$

Form a new set of elements drawn from domain D by including exactly the elements that satisfy predicate (i.e., Boolean function) P

Examples:

$\{ x : \mathbb{N} \mid x < 10 \}$	*Natural numbers less than 10*
$\{ x : \mathbb{Z} \mid (\exists\, y : \mathbb{Z} \mid x = 2y) \}$	*Even integers*
$\{ x : \mathbb{N} \mid x > x \}$	*Empty set of natural numbers*

Cardinality

The *cardinality* (#(_) or |_|) of a finite set is the number of its elements

Examples:

- #({ 1, 23 }) = |{ 1, 23 }| = 2
- #({ red, yellow, blue }) = 3
- #($\mathbb{Z}$)= ?

Cardinalities are defined for infinite sets too, but we'll be mostly concerned with the cardinality of finite sets

Set Operations

Union (X, Y sets over domain D): $X \cup Y \equiv \{ e: D \mid e \in X \text{ or } e \in Y \}$

– {red} ∪ {blue} = {red, blue}

Intersection: $X \cap Y \equiv \{ e: D \mid e \in X \text{ and } e \in Y \}$

– {red, blue} ∩ {blue, yellow} = {blue}

Difference : $X \setminus Y \equiv \{ e: D \mid e \in X \text{ and } e \notin Y \}$

– {red, yellow, blue} \ {blue, yellow} = {red}

Subsets

A *subset* holds elements drawn from another set

$X \subseteq Y$ iff every element of X is in Y

Example: $\{ 1, 7, 24 \} \subseteq \{ 1, 7, 17, 24 \} \subseteq \mathbb{Z}$

A *proper subset* is a non-equal subset

Another view of set equality: $A = B$ iff ($A \subseteq B$ and $B \subseteq A$)

Power Sets

The *power set* of set S, denoted Pow (S), is the set of all subsets of S:

$$\text{Pow}(S) \equiv \{ e \mid e \subseteq S \}$$

Example:

Pow ({a,b,c}) = { ∅, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c} }

Note: for every S, ∅ ⊆ S and thus ∅ ∈ Pow (S)

Note: for every A and S, A ∈ Pow (S) iff A ⊆ S

Exercises

These slides include questions that you should be able to solve at this point

They may require you to think some

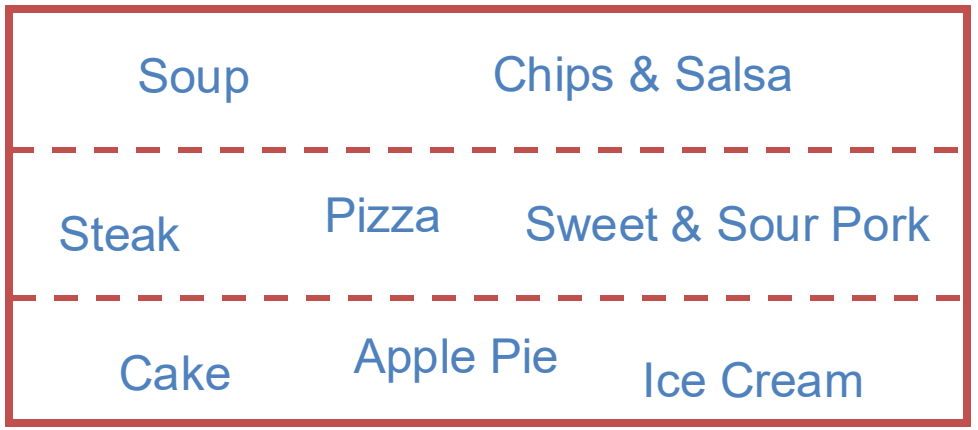
You should spend some effort in solving them
… and may in fact appear on exams

Exercises

1. Specifying using comprehension notation
   a) Odd positive integers
   b) The squares of integers, i.e., { 0, 1, 4, 9, 16, … }
2. Express the following logic properties on sets without using the # operator
   a) Set S has no elements
   b) Set S has exactly one element
   c) Set S has at least one element
   d) Set S has exactly two elements
   e) Set S has at least two elements

Set Partitioning

- Sets are *disjoint* if they share no elements
- We will often take some set S and divide its members into disjoint subsets called *blocks* or *parts*
- We call this division a *partition*
- Each member of S belongs to exactly one block of the partition



Partition Example

Model residential scenarios

Basic domains: *Person*, *Residence*

Partitions:

- Partition *Person* into *Child*, *Adult*
- Partition *Residence* into *Home*, *DormRoom*, *Apartment*

Exercises

1. Express the following properties of pairs of sets
   a) Two sets are disjoint
   b) Two sets form a partition of a third set

Expressing Relationships

It's useful to be able to refer to structured values

- a group of values that are bound together
- e.g., struct, record, object fields

Alloy is a calculus of *relations* (sets of tuples)

All of our Alloy models will be built using relations

… but first some basic definitions

Products

Given two sets A and B, the product of A and B, usually denoted A x B, is the set of all possible pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B \equiv \{ (a, b) \mid a \in A, b \in B \}$$

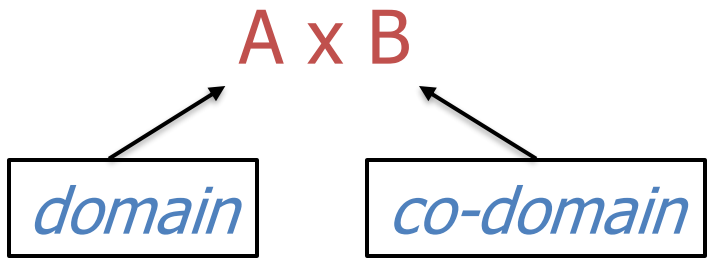
Example:

PrimaryColor x Boolean =

(red, true),	(red, false),
(blue, true),	(blue, false),
(yellow, true),	(yellow, false)

Binary Relations

A *binary relation* R between A and B is an element of *Pow*(A x B), i.e., R ⊆ A x B



Examples:

Parent : Person x Person =	{ (John, June), (John, Sam) }
Square : Z x N =	{ (1, 1), (-1, 1), (-2, 4) }
ClassGrades : Person x { A, B, C, D, F } =	{ (Kim, A), (Alex, B) }

Binary Relations

The set of first elements is the *definition domain* of the relation

- Parent : Person x Person = { (John, Autumn), (John, Sam) }
- *defdomain* (Parent) = { John } not Person!

The set of second elements is the *image* of the relation

- *image* ({ (1, 1), (-1, 1), (-2, 4) }) = { 1, 4 } not $\mathbb{N}$!

What about { (1, blue), (2, blue), (1, red) } ?

- definition domain? image?

N-ary Relations

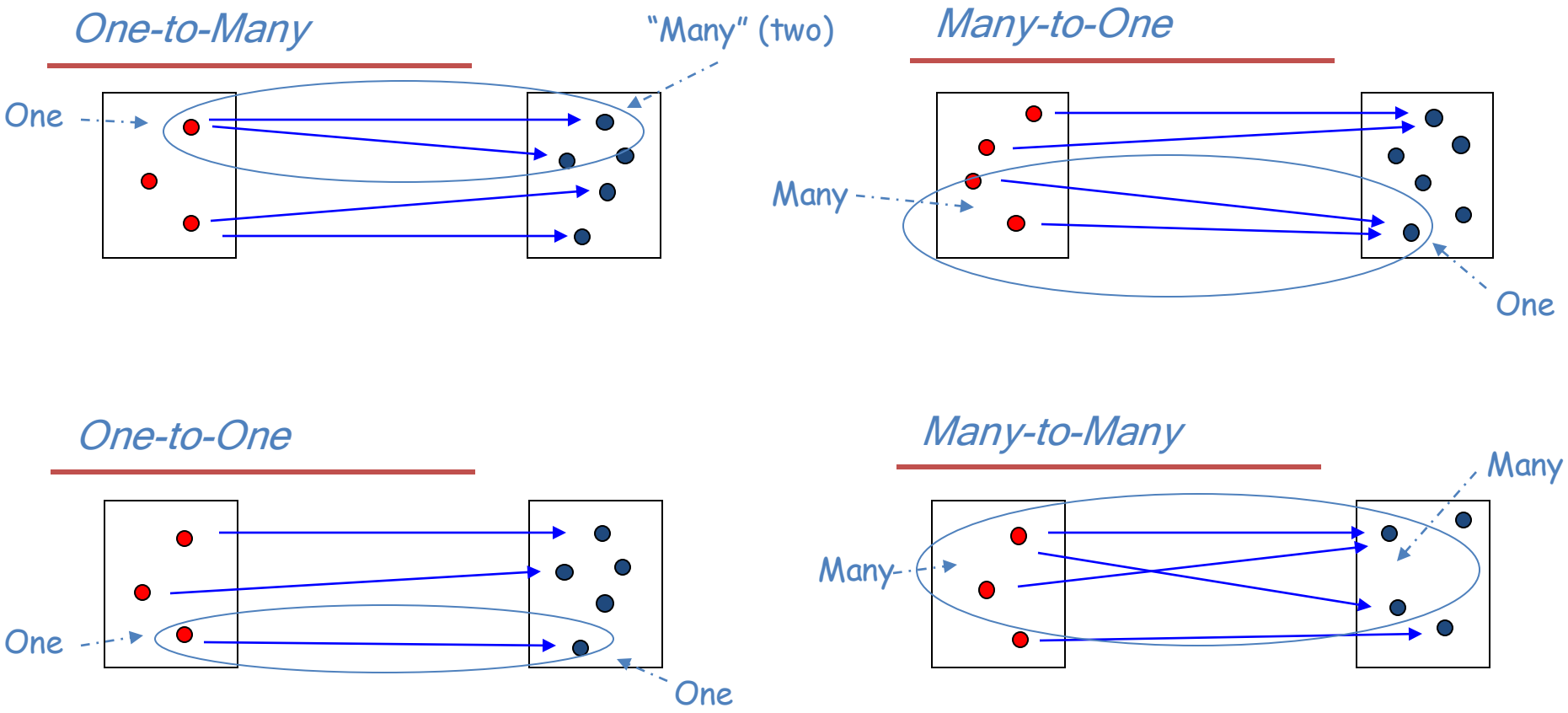
A *ternary relation* R between A, B and C is an element of *Pow*(A x B x C)

Example:

FavoriteBeerAtPrice : Person x Beer x Price
= { (John, Miller, $2), (Ted, Heineken, $4), (Steve, Miller, $2) }

n-ary relations with n > 3 are defined analogously
(n is the *arity* of the relation)

Common Relation Structures



Functional Relations

A *function* is a relation F of arity n+1 containing no two distinct tuples with the same first n elements,

– i.e., for n = 1, there is no $(a, b_1), (a, b_2) \in F$ s.t. $b_1 \neq b_2$

Examples:

– { (2, red), (3, blue), (5, red) } ✓

– { (4, 2), (6, 3), (8, 4) } ✓

– { (2, red), (3, blue), (2, blue) } ✗

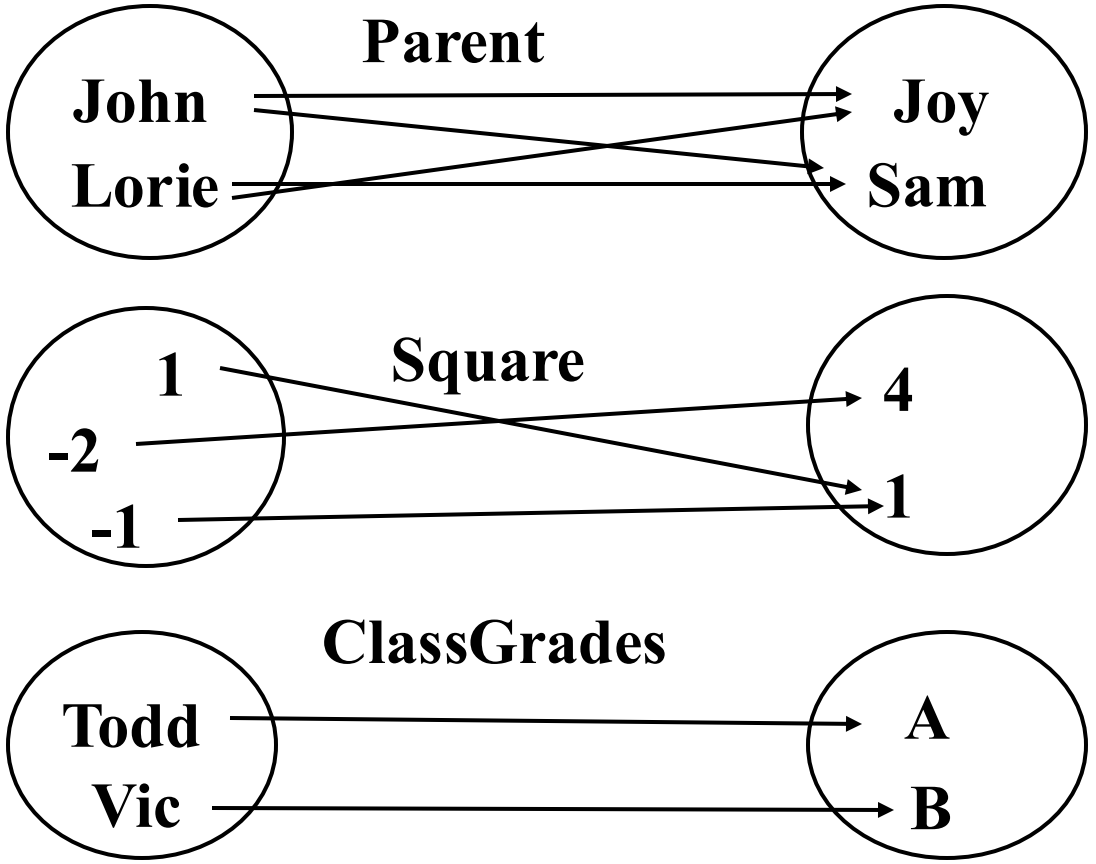
Instead of $F: A_1 \times A_2 \times \ldots \times A_n \times B$ we write $F: A_1 \times A_2 \times \ldots \times A_n \rightarrow B$

Exercises

Which of the following relations are functions?

1. Parent = { (John, Ann), (John, Sam), (Sam, Joy) }
2. Square = { (1, 1), (-1, 1), (-2, 4) }
3. ClassGrades = { (Todd, A), (Vic, B) }

Relations vs. Functions



Many-to-many

Many-to-one

One-to-one

A function is an X-to-one relation

Special Kinds of Functions

Consider a function f from S to T

f is *total* if defined for all values of S

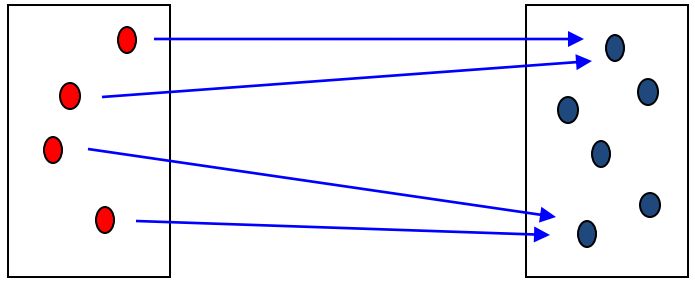
f is *partial* if undefined for some values of S

Examples:

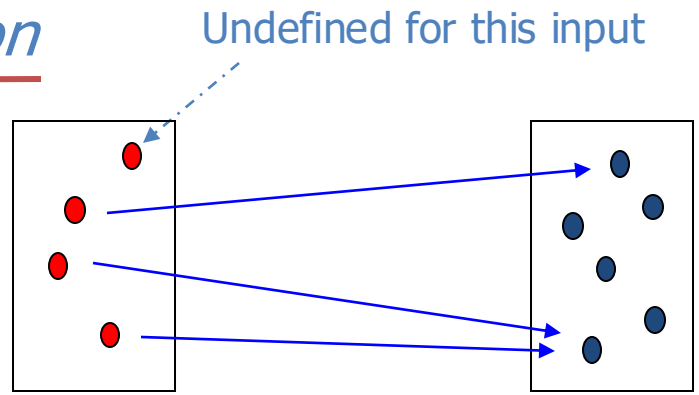
- Square : Z -> N = {…, (-1,1), (0,0), (1, 1), (2,4), …} total
- SquareRoots : N -> N = { (x, y) : N x N | $y^2 = x$) } partial

Function Structures

Total Function



Partial Function



Note:
the empty relation over a non-empty domain is a partial function

Special Kinds of Functions

A function f: S -> T is

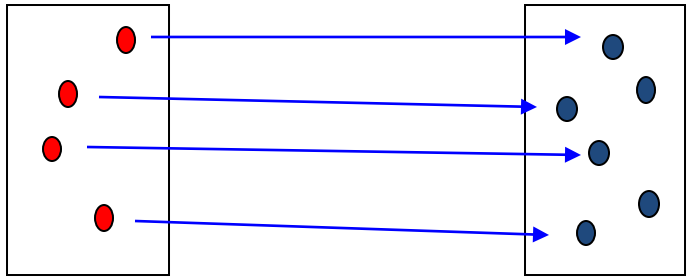
- *injective* (*one-to-one*) if no image element is associated with multiple domain elements
- *surjective* (*onto*) if its image is T
- *bijective* if it is both injective and surjective

We'll see that these come up frequently

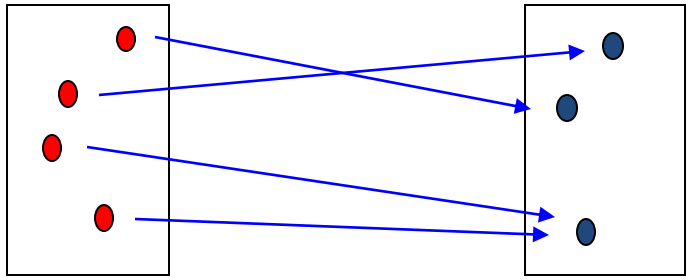
  – can be used to define properties concisely

Function Structures

Injective Function



Surjective Function



Exercises

1. What kind of function/relation is Abs?

   Abs : $\mathbb{Z}$ x $\mathbb{N}$ = { (x, y) : $\mathbb{Z}$ x $\mathbb{N}$ | (x < 0 and y = -x) or (x ≥ 0 and y = x) }

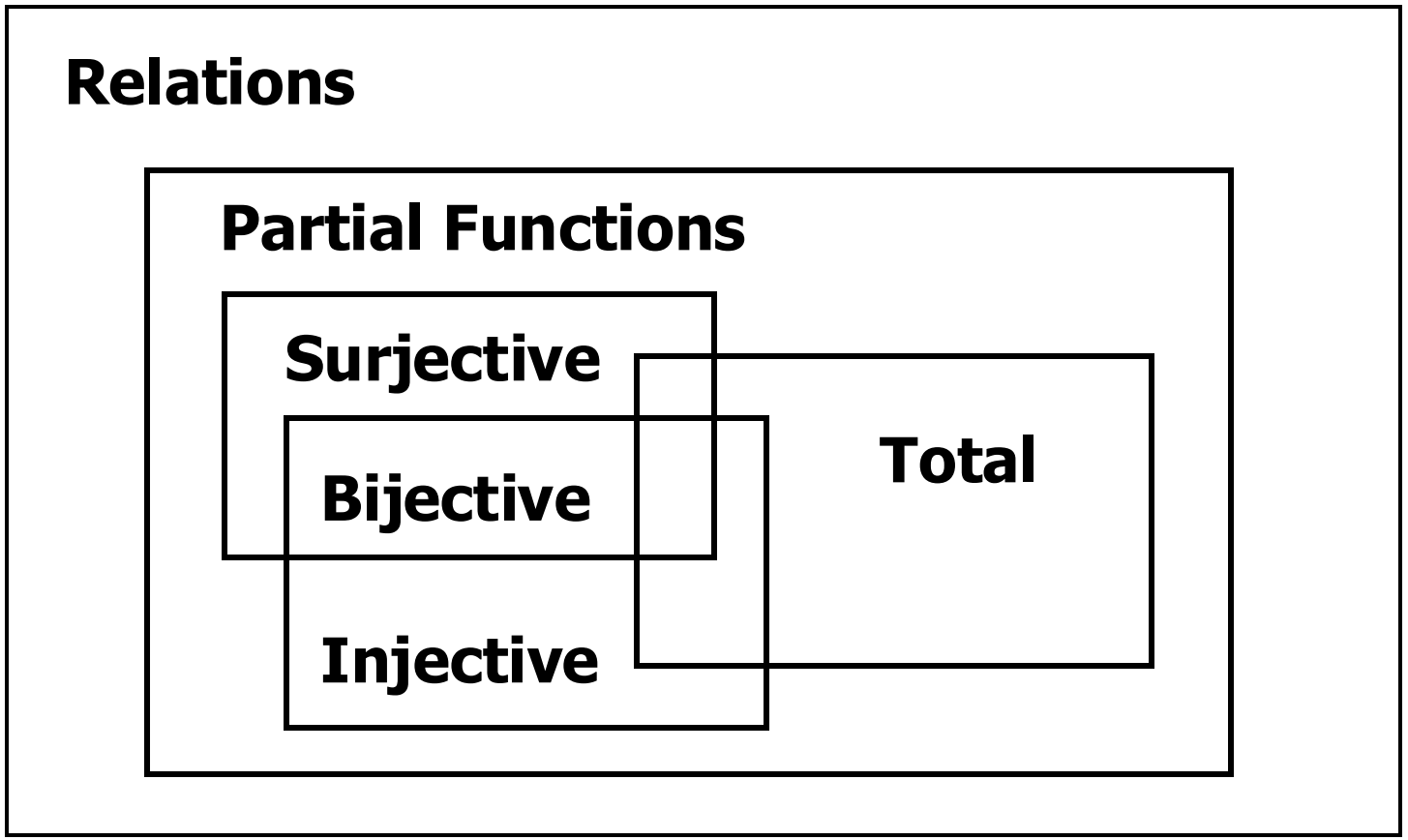
2. How about Squares?

   Squares : $\mathbb{Z}$ x $\mathbb{N}$ = { (x, y) : $\mathbb{Z}$ x $\mathbb{N}$ | y = x·x }

3. How about Rel?

   Rel : $\mathbb{Z}$ x $\mathbb{N}$ = { (x, y) : $\mathbb{Z}$ x $\mathbb{N}$ | y = 2·x if x >= 0,
   y = 2·(-x) - 1 if x < 0 }

Special Cases



Functions as Sets

Functions are relations and hence sets

We can apply to them all the usual operators

- ClassGrades = { (Todd, A), (Jane, B) }
- #(ClassGrades ∪ { (Matt, C) }) = 3

Exercises

In the following, problems if an operator fails to preserve a property give an example

1. What operators preserve function-ness?
   a) $\cap$?
   b) $\cup$?
   c) $\setminus$?
2. What operators preserve surjectivity?
3. What operators preserve injectivity?

Relation Composition

Use two relations to produce a new one

- map domain of first to image of second
- Given s: A x B and r: B x C then s ; r : A x C

$$s ; r \equiv \{ (a,c) \mid \exists b \text{ s.t. } (a,b) \in s \text{ and } (b,c) \in r \}$$

Example:

- s = { (red,1), (blue,2) }
- r = { (1,2), (2,4), (3,6) }
- s ; r = { (red,2), (blue,4) }

Not limited to binary relations

Transitive Closure of a Relation

Intuitively, the *transitive closure* r^+ of a binary relation $r: S \times S$ is the result of adding a direct link (a,b) to r for every a and b where b is reachable from a along r :

$$r^+ \equiv r \cup (r ; r) \cup (r ; r ; r) \cup \ldots$$

Formally, $r^+ \equiv$ smallest transitive relation containing r

Example:

- GrandParent = Parent ; Parent
- GrandGrandParent = Parent ; GrandParent
- Ancestor = Parent ∪ GrandParent ∪ GrandGrandParent ∪ … = Parent^+

Relation Transpose

Intuitively, the *transpose* ~r: T x S of a relation r: S x T is the relation obtained by reversing all the pairs in r

$$\sim r \equiv \{ (b,a) \mid (a,b) \in r \}$$

Example:

- Child = ~Parent
- Descendant = $(\sim Parent)^{+}$

Exercises

1. What properties, i.e., function-ness, onto-ness, 1-1-ness, are preserved by these relation operators?
   a) composition (;)
   b) closure ($^+$)
   c) transpose ($^\sim$)

2. If an operator fails to preserve a property, provide a conterexample

Acknowledgements

Some of these slides are adapted from

David Garlan's slides from Lecture 3 of his course of Software Models entitled "Sets, Relations, and Functions" (http://www.cs.cmu.edu/afs/cs/academic/class/15671-f97/www/)