CHAPTER 5: Some Discrete Probability Distributions

Discrete Uniform Distribution: 5.2

Definition: If the random variable X assumes the values $x_1, x_2, \ldots, x_k$ with equal probabilities, then the discrete uniform distribution is given by

$$f(x; k) = \frac{1}{k}, \quad x = x_1, x_2, \ldots, x_k$$

Example: When a die is tossed, each element of the sample space $S = \{1, 2, 3, 4, 5, 6\}$ occurs with probability 1/6. Therefore, we have a uniform distribution, with

$$f(x; 6) = \frac{1}{6}, \quad x = 1, 2, 3, 4, 5, 6.$$

$$\mu = \frac{1 + 2 + \cdots + 6}{6} = 3.5 \quad \sigma_X^2 = \frac{35}{12}$$

Binomial Distribution: 5.3

NOTE: Multinomial Distribution is not required.

The Bernoulli Process

- The experiment consists of n repeated trials.
- Each trial results is an outcome that may be classified as a success or a failure. (i.e. heads/tails, correct/erroneous bits, good/defective items, active/silent speakers).
- The probability of success denoted by p, remains constant from trial to trial.
- The repeated trials are independent.

The number X of successes in n Bernoulli trials is called a **binomial random variable**. For example X could be the number of heads in n tosses of a coin.

Example: An early warning detection system for aircraft consists of four identical radar units operating **independently** of one another. Suppose that each has a probability of 0.95 of detecting an intruding aircraft. When an intruding aircraft enters the scene, let
$X =$ the number of radar units that do not detect the plane.

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The probability distribution of a binomial random variable:

Example
For the last example
$P(X = 3) = P(\text{SSSF, SSFS, SFSS, FSSS}) =.$

The probability distribution of a binomial random variable is called a **binomial distribution**, and its values will be denoted by $b(x; n, p)$.

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, \dots, n$$

where $q = 1 - p$.

Example: Find the probability that seven of 10 persons will recover from a tropical disease if we can assume independence and the probability is 0.80 that any one of them will recover from the disease.

Solution:

Example: Experience has shown that 30% of all persons afflicted by a certain illness recover. A drug company has developed a new medication. Ten people with the illness were selected at random and injected with the medication; nine recover shortly thereafter. Suppose that the medication was absolutely worthless. What is the probability that at least nine of ten injected with the medication will recover?

Solution

Frequently, we are interested in problems where it is necessary to find $P(X < r)$ or $P(a \le X \le b)$.
$B(r; n, p) = F(r) = P(X \le r) = \sum_{x=0}^{r} b(x; n, p)$
are available and are given in Table A.1 of the appendix for $n = 1, 2, \dots 20$, and selected values of p from 0.1 to 0.9.

Examples:

$B(2; 8, 0.1) = 0.9619$
$B(4; 8, 0.1) = 0.9996$
$B(2; 14, 0.25) = 0.2811$

$B(r; n, p)$ is the cumulative distribution function for $b(x; n, p)$.

<u>How to use Minitab to find $b(x; n, p)$ and $B(r; n, p)$</u>

Example: Find $b(x; 8, 0.1)$ and $B(r; 8, 0.1)$ for $x = 0, 1, \dots, 8$ and $r = 0, 1, \dots, 8$.

- Put the numbers $0, 1, , \dots, 8$ in C1
- Use the option Calc Probability Distribution Binomial.
- You can choose from the following:
  a. Probability (this is $b(x; n, p)$)
  b. Cumulative Probability ($B(r; n, p)$)
  c. Inverse cumulative probability

- Number of trials: 8
- Probability of success: 0.1
- Input column C1

Output:

$b(x; 8, 0.1)$

x	$P(X = x)$
0.00	0.4305
1.00	0.3826
2.00	0.1488
3.00	0.0331
4.00	0.0046
5.00	0.0004
6.00	0.0000
7.00	0.0000
8.00	0.0000

$B(x; 8, 0.1)$

x	$P(X \leq x)$
0.00	0.4305
1.00	0.8131
2.00	0.9619
3.00	0.9950
4.00	0.9996
5.00	1.0000
6.00	1.0000
7.00	1.0000
8.00	1.0000

Exercises on p. 124
7. One prominent physician claims that 70% of those with lung cancer are chain smokers. If his assertion is correct:
(a) find the probability that of 10 such patients recently admitted to a hospital, fewer than half are chain smokers.
(b) find the probability that of 20 such patients recently admitted to a hospital, fewer than half are chain smokers.

Solution:

10. A nationwide survey of seniors by the University of Michigan reveals that almost 70% disapprove of daily pot smoking according to a report in Parade. If 12 seniors are selected at random and asked their opinion, find the probability that the number who disapprove of smoking pot daily is
(a) anywhere from 7 to 9;
(b) at most 5;
(c) not less than 8.

Solution:

16. Suppose that airplane engines operate independently and fail with probability equal to 0.4. Assuming that a plane makes a safe flight if at least one-half of its engines run, determine whether a $4-$engine plane or a 2-engine plane has the higher probability for a successful flight.

Solution

Theorem 5.2: The mean and variance of the binomial distribution $b(x; n, p)$ are

$$\mu = np \quad \text{and} \quad \sigma^2 = npq$$

The proof is NOT required

Example: A nationwide survey of seniors by the University of Michigan reveals that almost 70% disapprove of daily pot smoking according to a report in Parade. If 12 seniors are selected at random and asked their opinion, find the mean and the standard deviation of the number of seniors who disapprove of smoking pot daily

Solution:

Poisson Distribution and the Poisson Process: 5.6

In many applications, we are interested in counting the number of occurrences of an event in a certain time period or in a certain region in space.

The Poisson random variable arises in situations where the events occur "completely at random" in time or space.

X, the number of outcomes occurring during a given time interval or in a specified region is called **Poisson experiment**.

Examples:

1. X is the number of telephone calls per hour received by an office.
2. The number of particles emitted by a radioactive mass during a fixed time period.

Poisson Distribution: The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region denoted by t, is

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x - 0, 1, 2, \ldots$$

where λ is the average number of outcomes per unit time or region, and $e = 2.71828\ldots$

- $p(x; \lambda t)$ is called the **Poisson distribution**.
- The mean number of outcomes is

$$\mu = \lambda t$$

Example: Messages arrive at a computer at an average rate of 15 messages per second. The number of messages that arrive in 1 second is known to be Poisson random variable. Find the probability that no messages arrive in 1 second.

Solution:

Table A.2 contains Poisson probability sum

$$P(r; \mu = \lambda t) = \sum_{x=0}^{r} p(x; \mu = \lambda t)$$

for a few selected values of λt ranging from 0.1 to 18.

Example: On a certain section of an interstate there is an average of three deaths in a typical week. Assuming a Poisson distribution, what is the probability of having 5 or more deaths during some given week?

Solution:

Exercise 12(a) on p. 139: The average number of field mice per acre in a 5-acre wheat field is estimated to be 12. Find the probability that fewer than 7 field mice are found on a given acre.

Solution:

Example: A certain kind of sheet metal has, on the average, five defects per 10 square feet. If we assume a Poisson distribution, what is the probability that a 15-square-foot sheet of metal will have at least six defects?

Theorem 5.5: The mean and variance of the Poisson distribution $p(x; \lambda t)$ both have value λt.

The Poisson Distribution As a Limiting Form of the Binomial

Theorem 5.6: When $n \to \infty$, $p \to 0$ and $\mu = np$ remains constant,

$$b(x; n, p) \to p(x; \mu).$$

Example: The probability of a bit error in a communication line is 10^{-3}. Find the probability that a block of 1000 bits has five or more errors.

Exercise 16 on p. 139: The probability that a student fails the screening test for scoliosis (curvature of the spine) at a local high school is known to be 0.0004. Of the next 1875 students who are screened for scoliosis, find the probability that
(a) fewer than 5 fail the test;

(b) $8, 9$, or 10 fail the test.

Solution:

Output from Minitab

Cumulative Distribution Function:
Poisson with $\mu = 7.5$

x	$P(X \leq x)$
10.0	0.8622
7.0	0.5246

Exercise 22 on p. 140: Consider Exercise 16. What is the mean number of students who fail this test?

Solution:

Ex 8 on p. 141:
A local drugstore owner knows that, on the average, 100 people per hour stop by his store.
(a) Find the probability that in a given 3-minute period nobody enters the store.
(b) Find the probability that in a given 3-minutes period more than 5 people enter the store.

Solution:

Review Questions

1. A distributor receives a large shipment of components. The distributor would like to accept the shipment if 10% or fewer of the components are defective and to return it if more than 01% of the components are defective. She decides to sample 10 components, and to return the shipment if more than 1 of the 10 is defective.
a. If the proportion of defectives in the batch is in fact 10%, what is the probability that she will return the shipment?

b. If the proportion of defectives in the batch is in fact 20%, what is the probability that she will return the shipment?

c. If the proportion of defectives in the batch is in fact 2%, what is the probability that she will return the shipment?

d. The distributor decides that she will accept the shipment only if none of the sampled items are defective. What is the minimum number of items she should sample if she wants to have a probability no greater than 0.01 of accepting the shipment if 2% of the components in the shipment are defective?

e. Find the mean number and the standard deviation of the number of defective component in the batch.

2. Particles are suspended in a liquid medium at a concentration of 6 particles per mL. A large volume of the suspension is thoroughly agitated, and then 3 mL are withdrawn. What is the probability that exactly 15 particles are withdrawn?