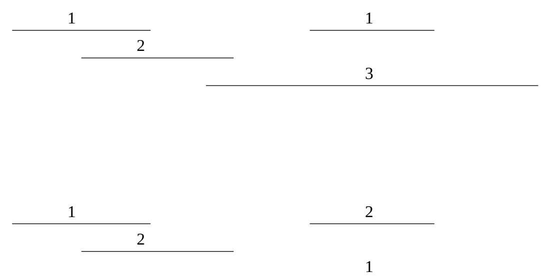
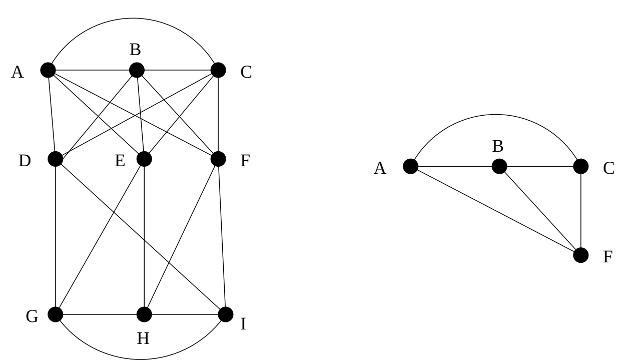
22C:44 Homework 7 Solution

1. A counterexample with 4 intervals is shown below. In the top portion of the figure I show how the greedy algorithm will produce a partition containing 3 sets, whereas in the bottom portion of the figure I show that a size 2 partition suffices.



2. Consider the graph on the left in the above figure. Each of vertices G, H, and I have degree 4 and the each of the remaining vertices have degree 5. Suppose the greedy algorithm picks vertex G first. G and all of its neighbors are deleted from the graph and we get the graph shown on the right. This graph is a clique with 4 vertices (K_4) and therefore only one vertex can be picked from it. So the greedy algorithm produces a solution with 2 vertices. On the other hand, it is clear that $\{D, E, F\}$ is a larger solution to the problem. Had the greedy algorithm chosen vertex H or vertex I first we would still have obtained a size-2 solution.



3. (a) Suppose the vertices of the graph are arbitrarily labeled 1 through n. Without loss of generality, suppose the algorithm process vertices in the order 1 through n. Also suppose that the $\Delta + 1$ colors the algorithm is allowed to use are $\{1, 2, \ldots, \Delta + 1\}$. We prove the claim by induction on the number of vertices processed. The inductive hypothesis is that after i vertices have been processed, each of the vertices 1 through i is assigned a color in the set $\{1, 2, \ldots, \Delta + 1\}$ such that for any edge $\{p, q\} \in E$, $p, q \leq i$, the color assigned to p is distinct from the color assigned to q. Note that after all n vertices have been processed this gives us a valid coloring of the graph.

   Base Case: After 1 vertex has been processed color 1 is used for vertex 1 and the induction hypothesis is trivially true.

   Inductive Case: Suppose that the inductive hypothesis is true after i vertices have been processed. We now show that it is also true after $(i + 1)$ vertices have been processed. Vertex $i + 1$ has at most Δ neighbors. This implies that it has at most Δ neighbors that have already been assigned a color. This in turn implies that there is at least one color in the set $\{1, 2, \ldots, \Delta + 1\}$ that has not been used for any neighbor of $i + 1$. The greedy algorithm chooses the smallest color in $\{1, 2, \ldots, \Delta + 1\}$ that has not been used for a neighbor, to color $i + 1$. Therefore the inductive hypothesis holds after processing $i + 1$ as well.

(b) Consider a path with 4 vertices, 1, 2, 3, and 4. Suppose the greedy algorithm colors the vertices in the order 1, 4, 2, and 3. Then vertices 1 and 4 will get assigned color 1, vertex 2 will get assigned color 2, and vertex 3 will get assigned color 3 because it has a neighbor colored 1 and a neighbor colored 2.

(c) The path 1, 2, 3, 4 can be colored with 2 colors by assigning color 1 to vertices 1 and 3 and color 2 to vertices 2 and 4.

4. The optimal Huffman tree is completely unbalanced and leads to the following assignment of codes: $code(a) = 0^7$, $code(b) = 0^6 1$, $code(c) = 0^5 1$, $code(d) = 0^4 1$, $code(e) = 0^3 1$, $code(f) = 0^2 1$, $code(g) = 01$, and $code(h) = 1$. Here I am using 0^i to denote a string with i 0's.

   Let $c_1, c_2, \ldots, c_n$ be the n characters with Fibonacci frequencies $f_1, f_2, \ldots, f_n$, where $f_1 = f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$ for all i, $3 \le i \le n$. Then the codes assigned to characters c_i are

$$code(c_1) = 0^{n-1} \qquad code(c_i) = 0^{n-i}1 \qquad \text{for all } i = 2, 3, \ldots, n.$$

5. See solution to Problem 2 in Homework 9 from Fall 2000. This problem set and its solution is posted in the practice problems section of the course page.