22C:44 Homework 2 Solution

1. (a) Use the Master Theorem. So $a = 2$, $b = 5$, $n^{\log_b a} = n^{\log_5 2} = n^{0.430677}$. Also $f(n) = n^{0.5}$. Therefore, there exists $\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a+\epsilon})$. Also $af(n/b) = 2f(n/5) = 2\sqrt{n/5} = 0.894427\sqrt{n}$. Hence the regularity condition is also satisfied. Applying Part (3) of the Master Theorem we get $T(n) = \Theta(\sqrt{n})$.

   (b) Use the Iteration Method. Then we see that

$$T(n) = 1/n + 2/n + 4/n + \cdots + 2^k/n + \Theta(1),$$

   where $2^k < n$ and $2^{k+1} \geq n$. Hence, $T(n) = 1/n(2^{k+1} - 1) + \Theta(1)$. Since $2^{k+1} \leq 2n$, we get

$$\frac{n-1}{n} + \Theta(1) < T(n) \leq \frac{2n-1}{n} + \Theta(1).$$

   Hence $T(n) = \Theta(1)$.
   Alternately, use the Master Theorem. So $a = 1$, $b = 2$, $n^{\log_b a} = n^0 = 1$, $f(n) = 1/n = n^{-1}$. Therefore, there exists $\epsilon > 0$ such that $f(n) = O(n^{\log_b a-\epsilon})$ and by using Part (1) of the Master Theorem we get $T(n) = \Theta(1)$.

   (c) Guess that $c3^n \leq T(n) \leq c'3^n$ for all $n \geq 1$. Choose $c = \min\{T(1)/3, T(2)/9\}$ and $c' = \max\{T(1)/3, T(2)/9\}$. This implies that $c3^n \leq T(n) \leq c'3^n$ for $n = 1, 2$. Suppose that $c3^k \leq T(k) \leq c'3^k$ for all k, $1 \leq k < n$. Then,

$$2c3^{n-1} + 3c3^{n-2} \leq T(n) \leq 2c3^{n-1} + 3c3^{n-2}.$$

   Simplifying, we get $c3^n \leq T(n) \leq c'3^n$.

   (d) Using the Iteration Method we get

$$T(n) = \sum_{i=1}^{k} 1 + \Theta(1).$$

   where $n^{1/2^k} > 2$ and $n^{1/2^{k+1}} \leq 2$. This implies that $\lg\lg n - 1 \leq k \leq \lg\lg n$. Therefore $T(n) = \Theta(\lg\lg n)$.

2. (a) $T(n) = \sum_{i=1}^{n}\sum_{j=i}^{n}\Theta(1)$. This can be simplified as

$$T(n) = \sum_{i=1}^{n}\Theta(n-i+1) = \Theta(n^2 - n(n+1)/2 + n) = \Theta(n^2/2 + n/2) = \Theta(n^2).$$

   (b) $T(1) = \Theta(1)$ and $T(n) = T(n-1) + \Theta(1)$ for all $n > 1$. Using the Iteration Method, we get $T(n) = \Theta(n)$.

   (c) $T(1) = \Theta(1)$ and $T(n) = 2T(n/2) + 1$ for all $n > 1$. Using the Master Method, this recurrence solves to $T(n) = \Theta(n)$.

3. (a) Let a *block* of $A[1 \ldots n]$ be a contiguous subsequence $A[i], A[i+1], \ldots, A[j]$, for any $1 \leq i \leq j \leq n$. Define the *weight* of a block to be the sum of the elements in the block. `Mystery` returns the the weight of the heaviest block in A.

(b) $T(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} \Theta(1)$. This can be simplified as follows:

$$\begin{aligned}
T(n) &= \sum_{i=1}^{n} \sum_{j=i}^{n} \Theta(j-i+1) \\
&= \sum_{i=1}^{n} 1 - i + n - i(1-i+n) - \frac{1}{2}(-1+i-n)(i+n) \\
&= \sum_{i=1}^{n} \frac{1}{2}\left(2 + i^2 + 3n + n^2 - i(3+2n)\right) \\
&= \frac{1}{2}\Big(2n + 3n^2 + n^3 + \frac{1}{6n}(1+n)(1+2n) - \frac{1}{2n}(1+n)(3+2n)\Big) \\
&= \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3} \\
&= \Theta(n^3).
\end{aligned}$$

(c) The idea for a $\Theta(n)$ algorithm is this. For the subarray `A[1...i]` maintain two pieces of information called `maxSum[i]` and `rightMaxSum[i]`. `maxSum[i]` equals the weight of the heaviest block in `A[1...i]` while `rightMaxSum[i]` equals the weight of the heaviest block in `A[1...i]` that contains `A[i]`. Given this information, the two new pieces of information `maxSum[i+1]` and `rightMaxSum[i+1]` corresponding to the subarray `A[1...i+1]` can be computed in $\Theta(1)$ time as follows.

```
if (A[i+1] + rightMaxSum[i] <= A[i+1]) then
        rightMaxSum[i+1] ⟵ A[i+1]
else
        rightMaxSum[i+1] ⟵ rightMaxSum[i] + A[i+1]

if (rightMaxSum[i+1] >= maxSum[i]) then
        maxSum[i+1] ⟵ rightMaxSum[i+1]
else
        maxSum[i+1] ⟵ maxSum[i]
```

Note that `maxSum[1] = rightMaxSum = A[1]` and the answer we want is `maxSum[n]`.

4. (a) Let C_i denote the outcome of the i th coin toss. Probability that Alice wins is

$$Prob[C_1 = H] + Prob[C_1 = T \wedge C_2 = H \wedge C_3 = H] = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}.$$

(b) Let H_A, T_A, H_B, and T_B denote the number of heads and tails obtained by Alice and Bob respectively. There are two possibilities for the relative sizes of H_A and H_B: (i) $H_B > H_A$ (ii) $H_B \leq H_A$. Note that $H_B \leq H_A$ is equivalnt to the possibility $T_B > T_A$. The two possibilities $H_B > H_A$ and $T_B > T_A$ are disjoint, cover all possibilities, and are equally likely due to symmetry. Therefore, $Prob[H_A > H_B] = 1/2$.

(c) Let F and B denote the events that the coin is fair and biased respectively. Let T_1 and T_2 denote the two coin tosses. Then

$$\begin{aligned}
Prob[F \mid T_1 = H \wedge T_2 = T] &= \frac{Prob[T_1 = H \wedge T_2 = T \mid F] \cdot Prob[F]}{Prob[T_1 = H \wedge T_2 = T]} \\
&= \frac{1/2 \cdot 1/2 \cdot 1/2}{Prob[T_1 = H \wedge T_2 = T]} \\
&= \frac{1}{4} \cdot \frac{1}{2Prob[T_1 = H \wedge T_2 = T]}
\end{aligned}$$

Similarly, we get

$$Prob[B \mid T_1 = H \wedge T_2 = T] = p(1-p) \cdot \frac{1}{2Prob[T_1 = H \wedge T_2 = T]}.$$

For $p \neq 0.5$, $p(1-p) < 0.25$ and hence it is more likely that the coin is fair.