22C:296 Seminar on Randomization

Scribe: Sriram Penumatcha

8th October

Edge Disjoint Cycles

Theorem 1 *Any k-regular digraph with no parallel edges has atlest $\Omega(k^2)$ edge disjoint cycles.*

Theorem 2 *Let G be an Eulerian digraph with no parallel edges and minimum degree k, then G has atlest $\frac{5k}{2} - 2$ edge disjoint cycles.*

Proof: Let x be a vertex with degree k in G.Let $c_1, c_2, ..., c_k$ be k edge disjoint cycles passing through x chosen to minimize total number of edges.Let S be the union of edges in $c_1, c_2, ..., c_k$ .Let D be the subgraph of G induced by S.Let H be D minus the edges incident on x, then H is acyclic.Also H induces a partial order $<$ on $V(G)$.

Claim: Let $x < y$ then there is no (x, y) edge outside of S.
Let y be a minimal element in $<$. Hence any edges of H incident on y are outgoing. The only way y could not have incoming edges is because $(x, y) \in E(G)$ and $(x, y) \in S$. Since there are no parallel edges in-degree of y in D is 1. So the number of outedges of x in D is also 1. Hence there is only one cycle from among $c_1, c_2, ..., c_k$ incident on y.Therefore after deleting $c_1, c_2, ..., c_k$ from G , the degree of y is atleast $k - 1$. And we can therefore choose $k - 1$ additional cycles to get a total of $(2k - 1)$ Edge-Disjoint cycles.

Rather than pick y to be an arbitrary minimal element , we pick a specific minimal element as follows. Let Z be elements of $V(G) - \{x\}$ that participate in more than $\frac{k}{2}$ cycles of $c_1, c_2, ..., c_k$.

Claim: Any pair of elements in Z are comparable.
Proof: For any $a, b \in Z, a \neq b$ a and b participate in a common cycle. This implies that $<$ restricted to Z is a total order.(Assuming $|Z| \geq 1$) . Let $a \in Z$ be the minimum element in Z. It cannot be a minimal element of $<$ because any minimal element participates in only one cycle.So let y be a minimal element of $<$ such that $y < a$.

We get $c_1, c_2, ..., c_k$ cycles incident on x. Then we pick $(k - 1)$ cycles incident on y again minimizing the total number of edges in those cycles.Again define a partial order as before with respect to the new set of cycles and let z be a minimal element in this partial order.So, for all $b \in Z$, $y < b$ .Hence if $(y, b) \in E(G)$ then $(y, b) \in S$ (by the claim we proved earlier). Now delete k cycles $c_1, c_2, ..., c_k$ to get the graph $G - D$. Note that this is also Eulerian and in this graph y has degree $\geq k-1$.So we can pick $(k-1)$ cycles passing through y, call these $c_1{}', c_2{}', ..., c_{k-1}{}'$ choosing them so as to minimize the total length.Let $S^{'}$ be the union of the edges in $c_1{}', c_2{}', ..., c_{k-1}{}'$. Let $D^{'}$ be the subgraph of G-D induced by $S^{'}$.Let $H^{'}$ be the graph obtained from $D^{'}$ by deleting edges of y. AS before , $H^{'}$ is acyclic and induces a partial order $<^{'}$ on V(G). Let z be an arbitrary minimal element of $<^{'}$. By same argument as before, there is exactly one cycle among $c_1{}', c2^{'}, ..., c_k{}'$ incident on Z.

We now claim that $z \in Z$. If this is true then atmost $\frac{k}{2}$ cycles among $c_1, c_2, ..., c_k$ are incident on z. Hence a total of $\frac{k}{2}+1$ cycles are from among $c_1, c_2, ..., c_k$ and $c_1{}', c_2{}', ..., c_{k-1}{}'$ are incident on z. Hence degree of z in $G-D-D^{'}$ is atlest $\frac{k}{2}-1$ and this allows a choice of $\frac{k}{2}-1$ more cycles.

□