22C:131 Homework 1

Due: Tuesday, 9/18

Notes: (a) Undergraduate students are required to solve the first 5 problems in the homework. Graduate students are required to solve all problems. The problem numbers refer to problems in the textbook, by Arora and Barak. (b) It is possible that solutions to some of these problems are available to you via other theory of computation books or on-line lecture notes, etc. If you use any such sources, please acknowledge these in your homework *and* present your solutions in your own words. You will benefit most from the homework, if you sincerely attempt each problem on your own first, before seeking other sources. (c) It is okay to discuss these problems with your classmates. Just make sure that you take no written material away from these discussions *and* (as in (b)) you present your solutions in your own words. When discussing homework with classmates please be aware of guidelines on "Academic Dishonesty" as mentioned in the course syllabus.

1. For any binary string $x \in \{0,1\}^+$, let $dec(x)$ denote the non-negative decimal integer equivalent of x. Thus $dec(\texttt{000101}) = 5$. For $x, y \in \{0,1\}^+$ $x \le y$ iff $dec(x) \le dec(y)$.

   Pick an appropriate k and design a k-tape Turing machine M that computes the function $f : \{0,1\}^+ \times \{0,1\}^+ \to \{0,1\}$ where $f(x,y) = 1$ if $x \le y$ and $f(x,y) = 0$ otherwise. M should run in $O(|x| + |y|)$ time on input (x, y).

   While it is tedious to specify "low level" details of Turing machines, it is even more tedious to read a "low level" description of a Turing machine. So in the interests of clarity and readability, you should use Example 1.1 from the textbook to model your solution. If the text description on Page 14 seems too verbose, use the "state diagram" approach I used in class to describe a Turing machine. Annotate this "state diagram" with commentary on what each transition means in the grand scheme of things. Make sure you clearly specify the number of tapes being used and how the input appears on the input tape.

2. Problem 1.5 (Chapter 1, Page 34). As the hint in the textbook suggests, the solution to this problem is obtained by modifying the "simulation" in the proof of Claim 1.6. You do not have to provide as much detail as I did in class; you can mimic the level of detail in the "proof sketch" for the claim in the textbook (Pages 17-18).

3. An instance of the *Post Correspondence Problem* (PCP) is a collection P of "dominos:"

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}.$$

   Each t_i and each b_i, $1 \le i \le k$ is some string from a finite alphabet Γ. The collection P is said to contain a *match* if there is a sequence $i_1, i_2, \ldots, i_\ell$ where

$$t_{i_1} t_{i_2} \cdots t_{i_\ell} = b_{i_1} b_{i_2} \cdots b_{i_\ell}.$$

   The problem is to determine if the given instance of PCP contains a match. One can think of PCP also as a function that maps instances of PCP that contain a match into 1 and the rest of the instances into 0. In 1946 Emil Post proved that PCP is not computable by any Turing machine. While you don't have to prove this here, below are a couple of problems that might help you appreciate PCP.

(a) Find a match in the following instance of PCP:

$$\left\{ \left[\frac{ab}{abab}\right], \left[\frac{b}{a}\right], \left[\frac{aba}{b}\right], \left[\frac{aa}{a}\right] \right\}.$$

Here you can think of the alphabet Γ as $\{a, b\}$.

(b) Show that if the alphabet Γ is restricted to be $\{1\}$, then PCP is computable. You do not have to describe a Turing machine for this proof; a clearly stated algorithm in pseudocode with comments will suffice.

4. We say that a Turing machine M *accepts* a string $w \in \{0,1\}^*$ if on input w, M halts and outputs 1. A Turing machine M is said to have property R if whenever M accepts w it accepts w^R. (**Note:** w^R denotes the string obtained by reversing string w; e.g., $(\texttt{011})^R$ is `110`.) Define a function $R : \{0,1\}^* \to \{0,1\}$ as follows: $R(\alpha) = 1$ if M_α has property R and $R(\alpha) = 0$ otherwise. Prove that the function R is uncomputable.

5. Define a function $B : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ as follows: $B(\alpha, x) = 1$ if M_α writes a non-blank symbol onto its output tape at some point over the course of its computation with input x; $B(\alpha, x) = 0$ otherwise. Prove that the function B is uncomputable.

6. Problem 1.9 (Chapter 1, Page 35). A "proof sketch" of the type you see in the textbook will suffice for this problem.

7. Problem 1.12 (Chapter 1, Page 35).
