Review II

For a real $\alpha$ and an integer $k$,

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1-\cdots-(\alpha-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k \leq -1. \end{cases}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (1 \leq k \leq n-1)$$

$$\binom{n}{k} = \binom{n}{n-k} \quad (0 \leq k \leq n)$$

$$k\binom{n}{k} = n\binom{n-1}{k-1} \quad (1 \leq n)$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \quad (n \geq 0)$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0 \quad (n \geq 1)$$

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots (= 2^{n-1}) \quad (n \geq 1)$$

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1} \quad (n \geq 1)$$

$$1^2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + n^2\binom{n}{n} = n(n+1)2^{n-2} \quad (n \geq 1)$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \quad (n \geq 0)$$

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k} \quad (1 \leq k \leq n)$$

$$\binom{n}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \quad (1 \leq k \leq n)$$

**Binomial expansion.** For integer $n \geq 1$ and variables $x$ and $y$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

**Newton’s binomial expansion.** For a real $\alpha$ and variables $x$ and $y$ with $0 \leq |x| \leq |y|$, 

$$(x+y)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{k} y^{\alpha-k}.$$

**Multinomial expansion.** For integer $n \geq 1$ and variables $x_1, x_2, \ldots, x_k$,

$$(x_1 + x_2 + \cdots + x_t)^n = \sum_{n_1+n_2+\cdots+n_t=nn_1,n_2,\ldots,n_t\geq0} \binom{n}{n_1,n_2,\ldots,n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.$$
Sperner’s theorem. Any clutter of an $n$-set $S$ contains at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ subsets of $S$.

The power set $P(S)$ can be partitioned into $m$ disjoint chains $C_1, C_2, \ldots, C_m$, where $m = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Construction of a symmetric chain partition for the case $n$ given a symmetric chain partition for the case $n-1$: for each chain $A_1 \subset A_2 \subset \cdots \subset A_k$ for the case $n-1$: if $k \geq 2$, do $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}$ and $A_1 \cup \{n\} \subset A_2 \cup \{n\} \subset \cdots \subset A_{k-1} \cup \{n\}$; if $k = 1$, do $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}$.

Dilworth’s theorem.

$$\min\{k : A_1 \cup \cdots \cup A_k \text{ is an antichain partition}\} = \max\{|C| : C \text{ is a chain}\}.$$  

$$\min\{k : C_1 \cup \cdots \cup C_k \text{ is a chain partition}\} = \max\{|A| : A \text{ is an antichain}\}.$$  

Let $P_1, P_2, \ldots, P_n$ be properties of the objects of a finite set $S$. Let $A_i$ be the set of all elements of $S$ that have the property $P_i$. The number of objects of $S$ that have none of the properties $P_1, P_2, \ldots, P_n$ is given by

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|.$$  

The number of objects of $S$ that have at least one of the properties $P_1, P_2, \ldots, P_n$ is given by

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$  

A permutation $i_1i_2\ldots i_n$ of $\{1, 2, \ldots, n\}$ is called a derangement if $i_k \neq k$ for any $1 \leq k \leq n$ (no number remains in its position). The number $D_n$ of derangements of $\{1, 2, \ldots, n\}$ is given by

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}).$$

$$D_1 = 0, \quad D_2 = 1, \quad D_3 = 2, \quad D_4 = 9, \quad D_5 = 44, \quad D_6 = 265.$$  

$$\lim_{n \to \infty} \frac{D_n}{n!} = e^{-1}.$$  

The derangement sequence $D_n$ satisfies the following recurrence relations

$$D_n = (n-1)(D_{n-1} + D_{n-2}), \quad D_1 = 0, D_2 = 1,$$

and

$$D_n = nD_{n-1} + (-1)^n, \quad D_1 = 0.$$  

A permutation of $\{1, 2, \ldots, n\}$ is called nonconsecutive if none of $12, 23, \ldots, (n-1)n$ occurs. The number $Q_n$ of nonconsecutive permutations of $\{1, 2, \ldots, n\}$ is given by

$$Q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!.$$  

For $n \geq 2$, $Q_n = D_n + D_{n-1}$.

A circular permutation of $\{1, 2, \ldots, n\}$ is called nonconsecutive if none of $12, 23, \ldots, n1$ occurs. The number $C_n$ of nonconsecutive circular permutations is given by

$$C_n = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k-1)! + (-1)^n.$$
Let $|X| = m$ and let $|Y| = n$. The number of all functions from $X$ to $Y$ equals $n^m$. The number of injective functions from $X$ to $Y$ equals $(\binom{n}{m})m! = P(n, m)$. The number $S(m, n)$ of surjective functions from $X$ to $Y$ is given by

$$S(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$ 

### Practice Problems

1. Find the coefficient of $x^3$ in the expansion of $(2 + 5x)^6$.
2. Find the coefficient of $x_1^3 x_2 x_2^2$ in the expansion of $(2x_1 - 3x_2 + 5x_3)^6$.
3. Find symmetric chain partition for $P(S)$, where $S$ has one, two, three or four elements.
4. Find clutters of maximal size for $P(\{1, 2, 3, 4, 5\})$.
5. Prove that there is only one maximal clutter for $P(\{1, 2, 3, 4\})$.
6. Consider the poset $(\{1, 2, \ldots, 12\}, |)$:
   (a) determine a chain of the largest size and a a partition of $X$ into the smallest number of antichains;
   (b) determine an antichain of the largest size and a a partition of $X$ into the smallest number of chains.
7. Determine the number of $10$-combinations of the multiset $M = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.
8. Find the number of integer solutions for the equation

   $$x_1 + x_2 + x_3 + x_4 = 15,$$

   where $2 \leq x_1 \leq 6, \quad -2 \leq x_2 \leq 1, \quad 0 \leq x_3 \leq 6, \quad 3 \leq x_4 \leq 8$.
9. How many ways can a hatcheck girl hand back the 10 hats of 10 gentlemen, one to each gentleman, with no man getting his hat?
10. Determine the number of permutations of $\{1, 2, \ldots, n\}$ in which no odd integer is in its natural position.
11. How many ways are there to rearrange 10 camels in a caravan, so that every camel has a different camel in front of it (the position of the original first camel is arbitrary)?
12. How many ways are there for 8 children on a merry-go-round to change places so that somebody new is in front of each child? (The seats are indistinguishable and in a circle.)
13. Solve the recurrence relation $h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}$, with $h_0 = 1, h_1 = 2, h_2 = 0$.
14. Solve the recurrence relation $h_n = 4h_{n-1} - 4h_{n-2}$, with $h_0 = a, h_1 = b$.
15. Solve the recurrence relation $h_n = 3h_{n-1} - 4n$, with $h_0 = 2$. 
16. Solve the recurrence relation \( h_n = h_{n-1} - 3n^2 - 5n^3 \), with \( h_0 = 2 \).

17. Solve the recurrence relation \( h_n = 10h_{n-1} - 25h_{n-2} + 5^{n+1} \), with \( h_0 = 5, h_1 = 15 \).

18. Solve the recurrence relation \( h_n = 6h_{n-1} - 9h_{n-2} + 8n^2 - 24n \), with \( h_0 = 5, h_1 = 5 \).

19. Find the coefficient of \( x^6 \) in \( (x^2 + x^3 + x^4 + x^5 + \ldots)^2 \).

20. Use (ordinary) generating functions to find the number of ways distribute \( r \) jelly beans among 8 children if
   (a) each child gets at least one jelly bean;
   (b) each child gets at even number of beans.

21. Find the number of nonnegative integer solutions for the equation
   \( y_1 + 2y_2 = n \).

22. Find the closed form for the (ordinary) generating function of the sequence \( a_i = \frac{1}{i} \).

23. Solve the recurrence relation \( a_n = 2a_{n-1} + a_{n-2}, a_0 = 1, a_1 = 3 \).

24. Use \( \frac{(1-x^2)^m}{(1-2x)^n} \) to evaluate the sum \( \sum_{i=0}^{\frac{n}{2}} (-1)^i \binom{n}{i} \binom{n+m-2i-1}{n-1} \), if \( m \leq n \) and \( m \) is even.

25. Find the closed form for the exponential generating function of the sequence \( a_i = \frac{1}{i+1} \).

26. Use exponential generating functions to find the number of \( k \)-permutations of the multiset \( \{\infty \cdot x_1, \infty \cdot x_2, \ldots, \infty \cdot x_n\} \).

27. Determine the number of ways to color the squares of a 1-by-\( n \) chessboard using the colors, red, white, and blue, if an even number of squares are colored red and there is at least one blue square.