Introduction To Discrete Mathematics

Review

If you put \( n + 1 \) pigeons in \( n \) pigeonholes then at least one hole would have more than one pigeon.
If \( n(r - 1) + 1 \) objects are put into \( n \) boxes, then at least one of the boxes contains \( r \) or more of the objects.
If the average of \( n \) nonnegative integers \( a_1, a_2, \ldots, a_n \) is greater than \( r - 1 \), i.e.,
\[
\frac{a_1 + a_2 + \cdots + a_n}{n} > r - 1,
\]
then at least one of the integers is greater than or equal to \( r \).
The number of \( r \)-permutations of an \( n \)-set equals
\[
P(n, r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.\]
The number of permutations of an \( n \)-set is \( P(n, n) = n! \).
The number of circular \( r \)-permutations of an \( n \)-set equals
\[
P(n, r) = \frac{n!}{(n-r)!r}.\]
The number of circular permutations of an \( n \)-set is equal to \((n-1)!\).
The number of \( r \)-combinations of an \( n \)-set equals
\[
\binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}.\]
The number of \( r \)-permutations of the multiset \( \{\infty \cdot x_1, \infty \cdot x_2, \ldots, \infty \cdot x_k\} \) equals \( k^r \).
The number of permutations of the multiset \( \{n_1 \cdot x_1, n_2 \cdot x_2, \ldots, n_k \cdot x_k\} \) equals
\[
\frac{n!}{n_1! n_2! \cdots n_k!}, \quad \text{where} \ n = n_1 + n_2 + \cdots + n_k
\]
Then the number of \( r \)-combinations of the multiset \( \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\} \) (the number of \( r \)-combinations with repetition allowed) equals \( \binom{k+r-1}{r} = \binom{k+r-1}{k-1} \).
The number of nonnegative integer solutions for the equation \( x_1 + x_2 + \cdots + x_k = r \) equals \( \binom{k+r-1}{r} = \binom{k+r-1}{k-1} \).
The number of positive integer solutions for the equation \( x_1 + x_2 + \cdots + x_k = r \) equals \( \binom{r-1}{k-1} \).
The number of ways to place \( r \) identical balls into \( k \) distinct boxes equals \( \binom{k+r-1}{r} = \binom{k+r-1}{k-1} \).
The number of ways to place \( r \) identical balls into \( k \) distinct boxes such that no box remains empty equals \( \binom{r-1}{k-1} \).
Algorithm for generating the permutations of \(\{1, 2, \ldots, n-1, n\}\):
Begin with \(\overline{1 2 \cdots n}\).
While there exists a mobile integer, do
(1) Find the largest mobile integer \(m\)
(2) Switch \(m\) and the adjacent integer its arrow points to.
(3) Switch the direction of all the arrows above integers \(p\) with \(p > m\).

Algorithm 1 for construction of a permutation from its inversion sequence \((a_1, a_2, \ldots, a_n)\):
(n) Write down \(n\).

\(\ldots\)
(n-k) Insert \(n - k\) to the right of the \(a_{n-k}\)th existing number
\(\ldots\)

Algorithm 2 for construction of a permutation from its inversion sequence \((a_1, a_2, \ldots, a_n)\):
(0) Mark down \(n\) empty spaces.
For \(k = 1\) till \(n\)
Put \(k\) into the \(a_k + 1\)st empty space from the left.

Algorithm for generating combinations of \(\{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}\):
Begin with \(a_{n-1}a_{n-2}\cdots a_1a_0 = 00 \ldots 00\).
While \(a_{n-1}a_{n-2}\cdots a_1a_0 \neq 11 \ldots 11\), do
(1) Find the smallest integer \(j\) such that \(a_j = 0\).
(2) Replace \(a_j\) by 1 and each of \(a_{j-1}, \ldots, a_1, a_0\) by 0.
The algorithm stops when \(a_{n-1}a_{n-2}\cdots a_1a_0 = 11 \ldots 11\).

Algorithm for generating reflected Gray codes of order \(n\):
Begin with \(a_{n-1}a_{n-2}\cdots a_1a_0 = 00 \ldots 00\).
While \(a_{n-1}a_{n-2}\cdots a_1a_0 \neq 10 \ldots 00\), do
(1) If \(a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 = \text{even}\), change \(a_0\) (from 0 to 1 or 1 to 0).
(2) If \(a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 = \text{odd}\), find the smallest \(j\) such that \(a_j = 1\) and change \(a_{j+1}\)
(from 0 to 1 or 1 to 0).

Algorithm for generating \(r\)-combinations of \(S = \{1, 2, \ldots, n-1, n\}\):
Begin with \(12 \cdots r\).
While \(a_1a_2\cdots a_r \neq (n-r+1)\cdots (n-1)n\), do
(1) Find the largest integer \(k\) such that \(a_k < n\) and \(a_k + 1\) is not in the \(a_1a_2\cdots a_r\).
(2) Replace \(a_1a_2\cdots a_r\) with
\[
\underbrace{a_1a_2\cdots a_{k-1}(a_k + 1)(a_k + 2)\cdots (a_k + r - k + 1)}.
\]

Algorithm for a linear extension of an \(n\)-poset:
Step 1. Choose a minimal element \(x_1\) from \(X\) (with respect to the ordering \(\leq\)).
Step 2. Delete \(x_1\) from \(X\); choose a minimal element \(x_2\) from \(X\).
Step 3. Delete \(x_2\) from \(X\) and choose a minimal element \(x_3\) from \(X\).
\(\ldots\)
Step \(n\). Delete \(x_{n-1}\) from \(X\) and choose the only element \(x_n\) in \(X\).
For a real $\alpha$ and an integer $k$, 
\[
\binom{\alpha}{k} = \begin{cases} 
\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k \geq 1 \\
1 & \text{if } k = 0 \\
0 & \text{if } k \leq -1.
\end{cases}
\]

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (1 \leq k \leq n-1)
\]

\[
\binom{n}{k} = \binom{n}{n-k} \quad (0 \leq k \leq n)
\]

\[
k \binom{n}{k} = n \binom{n-1}{k-1} \quad (1 \leq n)
\]

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \quad (n \geq 0)
\]

\[
\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0 \quad (n \geq 1)
\]

\[
\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots (= 2^{n-1}) \quad (n \geq 1)
\]

\[
l \binom{n}{1} + 2 \binom{n}{2} + \cdots + n \binom{n}{n} = n 2^{n-1} \quad (n \geq 1)
\]

\[
l^2 \binom{n}{1} + 2^2 \binom{n}{2} + \cdots + n^2 \binom{n}{n} = n(n+1)2^{n-2} \quad (n \geq 1)
\]

\[
\binom{n}{0} + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \quad (n \geq 0)
\]

\[
\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k} \quad (n \geq 1)
\]

\[
\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \quad (1 \leq k \leq n)
\]

**Binomial expansion.** For integer $n \geq 1$ and variables $x$ and $y$,

\[
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

**Newton's binomial expansion.** For a real $\alpha$ and variables $x$ and $y$ with $0 \leq |x| \leq |y|$,

\[
(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}.
\]

**Multinomial expansion.** For integer $n \geq 1$ and variables $x_1, x_2, \ldots, x_k$,

\[
(x_1 + x_2 + \cdots + x_t)^n = \sum_{n_1+n_2+\cdots+n_t=n; n_1,n_2,\ldots,n_t\geq0} \binom{n}{n_1,n_2,\ldots,n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.
\]
**Sperner’s theorem.** Any clutter of an $n$-set $S$ contains at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ subsets of $S$. The power set $P(S)$ can be partitioned into $m$ disjoint chains $C_1, C_2, \ldots, C_{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$.

Construction of a symmetric chain partition for the case $n - 1$: for each chain $A_1 \subset A_2 \subset \cdots \subset A_k$ for the case $n - 1$: if $k \geq 2$, do $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}$ and $A_1 \cup \{n\} \subset A_2 \cup \{n\} \subset \cdots \subset A_k \cup \{n\}$; if $k = 1$, do $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}$.

**Dilworth’s theorem.**

\[
\min \{ k : A_1 \cup \cdots \cup A_k \text{ is an antichain partition } \} = \max \{ |C| : C \text{ is a chain } \}.
\]

\[
\min \{ k : C_1 \cup \cdots \cup C_k \text{ is a chain partition } \} = \max \{ |A| : A \text{ is an antichain } \}.
\]

Let $P_1, P_2, \ldots, P_n$ be properties of the objects of a finite set $S$. Let $A_i$ be the set of all elements of $S$ that have the property $P_i$. The number of objects of $S$ that have none of the properties $P_1, P_2, \ldots, P_n$ is given by

\[
|A_1 \cap A_2 \cap \cdots \cap A_n| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|.
\]

The number of objects of $S$ that have at least one of the properties $P_1, P_2, \ldots, P_n$ is given by

\[
|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.
\]

A permutation $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ is called a **derangement** if $i_k \neq k$ for any $1 \leq k \leq n$ (no number remains in its position). The number $D_n$ of derangements of $\{1, 2, \ldots, n\}$ is given by

\[
D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}).
\]

The derangement sequence $D_n$ satisfies the following recurrence relations

\[
D_n = (n - 1)(D_{n-1} + D_{n-2}), \quad D_1 = 0, D_2 = 1, \quad \text{and} \quad D_n = nD_{n-1} + (-1)^n, \quad D_1 = 0.
\]

A permutation of $\{1, 2, \ldots, n\}$ is called **nonconsecutive** if none of $12, 23, \ldots, (n - 1)n$ occurs. The number $Q_n$ of nonconsecutive permutations of $\{1, 2, \ldots, n\}$ is given by

\[
Q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!
\]

For $n \geq 2$, $Q_n = D_n + D_{n-1}$.

A circular permutation of $\{1, 2, \ldots, n\}$ is called **nonconsecutive** if none of $12, 23, \ldots, n1$ occurs. The number $C_n$ of nonconsecutive circular permutations is given by

\[
C_n = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k-1)! + (-1)^n.
\]

Let $|X| = m$ and let $|Y| = n$. The number of all functions from $X$ to $Y$ equals $n^m$. The number of injective functions from $X$ to $Y$ equals $\binom{n}{m} m! = P(n, m)$. The number $S(m, n)$ of surjective functions from $X$ to $Y$ is given by

\[
S(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.
\]
Theorem 1. Let $q \neq 0$. The geometric sequence $h_n = q^n$ is a solution of the recurrence relation

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}; \quad a_k \neq 0, n \geq k$$

(1)

if and only if the number $q$ is a root of the characteristic equation

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1} x - a_0 = 0.$$  

(2)

Theorem 2. If the characteristic equation (2) has $k$ distinct roots $q_1, q_2, \ldots, q_k$, then the general solution of the recurrence relation (1) is

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n.$$  

Theorem 3. Let $q_1, q_2, \ldots, q_s$ be distinct roots with the multiplicities $m_1, m_2, \ldots, m_s$ respectively for the characteristic equation (2). Then the sequences

$$q_1^n, nq_1^n, n^2 q_1^n, \ldots, n^{m_1-1} q_1^n;$$

$$q_2^n, nq_2^n, n^2 q_2^n, \ldots, n^{m_2-1} q_2^n;$$

$$\vdots$$

$$q_s^n, nq_s^n, n^2 q_s^n, \ldots, n^{m_s-1} q_s^n;$$

are linearly independent solutions of the recurrence relation (1). Their linear combinations form the general solution of the recurrence relation (1).

Theorem 4. Let $h^*_n$ be any particular solution of the recurrence relation

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n; \quad a_k \neq 0, n \geq k,$$

(3)

and let $h^*_n$ be the general solution of its corresponding the homogeneous recurrence relation. Then $h_n = h^*_n + h^*$ is the general solution of the recurrence relation (3).

Consider a first-order linear nonhomogeneous recurrence relation

$$h_n = ah_{n-1} + b_n; \quad n \geq 1$$

(4)

Theorem 5. Let $b_n = cq^n$. Then (4) has a particular solution of the following form:

- If $q \neq a$, then $h^*_n = Aq^n$.
- If $q = a$, then $h^*_n = Anq^n$.

Theorem 6. Let $b_n = \sum_{i=0}^{k} c_i n^i$.

- If $a \neq 1$, then (4) has a particular solution of the form $h^*_n = A_0 + A_1 n + A_2 n^2 + \cdots + A_k n^k$.
- If $a = 1$, then the solution of (4) is $h_n = h_0 + \sum_{i=0}^{k} b_i$.

Theorem 7. Given a nonhomogeneous linear recurrence relation of the second order

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + cq^n; \quad n \geq 2$$

(5)

Let $q_1$ and $q_2$ be solutions of its characteristic equation $x^2 - a_1 x - a_2 = 0$. Then (6) has a particular solution of the following forms:

- If $q \neq q_1, q \neq q_2$, then $h^*_n = Aq^n$.
- If $q = q_1, q \neq q_2$, then $h^*_n = Anq^n$. 

- If $q = q_1, q = q_2$, then $h^*_n = Aq^n$. 

- If $q = q_1, q = q_2$, then $h^*_n = Anq^n$. 

Theorem 8. Given a nonhomogeneous linear recurrence relation of the second order

\[ h_n = a_1 h_{n-1} + a_2 h_{n-2} + b_n; \quad n \geq 2 \]

where \( b_n \) is a polynomial function of \( n \) with degree \( k \).

- If \( a_1 + a_2 \neq 1 \), then (6) has a particular solution of the form:

\[ h^*_n = A_0 + A_1 n + A_2 n^2 + \cdots + A_k n^k, \]

where the coefficients \( A_0, A_1, \ldots, A_k \) are to be determined. If \( k \leq 2 \), then a particular solution has the form

\[ h^*_n = A_0 + A_1 n + A_2 n^2. \]

- If \( a_1 + a_2 = 1 \), then (6) can be reduced to a first order recurrence relation

\[ g_n = (a_1 - 1) g_{n-1} + b_n, \quad \text{where} \quad g_n = h_n - h_{n-1} \quad \text{for} \quad n \geq 1. \]

For the sequence \( a_0, a_1, a_2, \ldots, a_k, \ldots \), its ordinary and exponential generating functions are given by

\[
A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots \\
A^{(e)}(x) = a_0 + b_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \cdots + a_k \frac{x^k}{k!} + \ldots
\]

\[
A(x) B(x) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i b_{k-i} \right) x^k \\
A^{(e)}(x) B^{(e)}(x) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} \binom{k}{i} a_i b_{k-i} \right) \frac{x^k}{k!}
\]

Some ordinary generating functions:

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<th>( a_i )</th>
<th>1</th>
<th>( e^x )</th>
<th>( x )</th>
<th>( e^{i x} )</th>
<th>( i )</th>
<th>( e^{i} )</th>
<th>( i^2 )</th>
<th>( e^{i} )</th>
<th>( i )</th>
<th>( e^{i} )</th>
<th>( i^2 )</th>
<th>( e^{i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(x) )</td>
<td>( \frac{1}{1-x} )</td>
<td>( \frac{1}{1-e x} )</td>
<td>( \frac{x}{(1-x)^2} )</td>
<td>( \frac{e^{(1+i)x}}{(1-e x)^2} )</td>
<td>( (1+x)^n )</td>
<td>( (1-x)^n )</td>
<td>( \ln \frac{1}{1-x} )</td>
<td>( \frac{1}{(1-x)^n} )</td>
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Some exponential generating functions:

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<th>( e^{i} )</th>
<th>( x e^x )</th>
<th>( i e^x )</th>
<th>( i! e^x )</th>
<th>( i^2 e^x )</th>
<th>( i! e^{i} )</th>
<th>( i^2 e^{i} )</th>
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<th>( i! e^{i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A^{(e)}(x) )</td>
<td>( e^x )</td>
<td>( e^{i x} )</td>
<td>( x e^x )</td>
<td>( (1+x)^n e^x )</td>
<td>( (1-x)^{n-1} )</td>
<td>( (1+x)^n )</td>
<td>( (1-x)^{n-1} )</td>
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</tr>
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</table>
Given a coloring \( c \in C \), the stabilizer of \( c \) is the set \( G(c) = \{ f \in G \mid f \ast c = c \} \).

Given a permutation \( f \in G \), the invariant set of \( f \) is the set \( C(f) = \{ c \in C \mid f \ast c = c \} \).

Given a coloring \( c \in C \), the orbit of \( c \) is the set \( \bar{c} = \{ f(c) \mid f \in G \} \).

Let \( C \) be the set of all \( k^n \) colorings of \( X \) into \( k \) colors. Then \( |C(f)| = k^{\#(f)} \), where \( \#(f) \) is the number of cycles in the disjoint cycle factorization of \( f \).

**Burnside’s Lemma** Suppose a group \( G \) of permutations of \( X \) acts on a set \( C \) of colorings of \( X \). Then the number \( N(G, C) \) of nonequivalent colorings in \( C \) is given by

\[
N(G, C) = \frac{1}{|G|} \sum_{f \in G} |C(f)|.
\]

Given a permutation \( f \in G \), the type of \( f \) is an \( n \)-tuple \( \text{type}(f) = (e_1, e_2, \ldots, e_n) \), where \( e_i \) is the number of \( i \)-cycles in a disjoint cycle factorization of \( f \).

\[
e_1 + e_2 + \cdots + e_n = \#(f), \quad 1e_1 + 2e_2 + \cdots + ne_n = n.
\]

To each permutation \( f \in G \) with type \( \text{type}(f) = (e_1, e_2, \ldots, e_n) \) we associate a monomial

\[
\text{mon}(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}.
\]

The cycle index of \( G \) is

\[
P_G(z_1, z_2, \ldots, z_n) = \frac{1}{|G|} \sum_{f \in G} \text{mon}(f) = \frac{1}{|G|} \sum_{f \in G} z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}.
\]

**Theorem** Suppose there are \( k \) colors. Let \( C \) be a set of all \( k^n \) colorings of \( X \). Then the number \( N(G, C) \) of nonequivalent colorings in \( C \) is given by

\[
N(G, C) = P_G(k, k, \ldots, k).
\]

**Theorem (Polya)** Let \( \{u_1, u_2, \ldots, u_k\} \) be a set of \( k \) colors. Let \( C \) be a set of any colorings of \( X \) such that the group \( G \) of permutations of \( X \) acts on the set \( C \). Then the generating function for the number of nonequivalent colorings in \( C \) according to the number of colors of each kind is given by

\[
P_G(u_1 + \cdots u_k, u_1^2 + \cdots u_k^2, \ldots, u_1^n + \cdots u_k^n).
\]