The Multinomial Theorem

Pascal’s Formula

Multinomial coefficient:
\[
\binom{n}{n_1, n_2, \ldots, n_t} = \frac{n!}{n_1! n_2! \cdots n_t!},
\]

where \( n_1 + n_2 + \cdots + n_t = n \).

Binomial coefficients are a particular case of multinomial coefficients:
\[
\binom{n}{k} = \binom{n}{n-k}.
\]

**Theorem 1 (Pascal’s Formula for multinomial coefficients.)** For integers \( n, n_1, n_2, \ldots, n_t \) such that \( n_1 + n_2 + \cdots + n_t = n \),
\[
\binom{n}{n_1, n_2, \ldots, n_t} = \binom{n-1}{n_1-1, n_2, \ldots, n_t} + \binom{n-1}{n_1, n_2-1, \ldots, n_t} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_t-1}.
\]

The Multinomial Theorem

**Theorem 2 (Multinomial Expansion.)** For integer \( n \geq 1 \) and variables \( x_1, x_2, \ldots, x_k \),
\[
(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1 + n_2 + \cdots + n_k = n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.
\]

1. Write down the expansion of \((x_1 + x_2 + x_3)^3\).
2. Find the coefficient of \(x_1^2 x_3 x_4^3 x_5\) in the expansion of \((x_1 + x_2 + x_3 + x_4 + x_5)^7\).
3. Find the coefficient of \(x_1^3 x_2 x_3^2\) in the expansion of \((2x_1 - 3x_2 + 5x_3)^6\).
4. How many different terms are there in the expansion of \((x_1 + x_2 + \cdots + x_t)^n\)?
5. Prove that
\[
\sum_{n_1 + n_2 + \cdots + n_t = n} \binom{n}{n_1, n_2, \ldots, n_t} = t^n.
\]