Generating $r$-Combinations

September 26, 2008

Let $S$ be an $n$-set

$$S = \{1, 2, \ldots, n-1, n\}$$

with a natural order

$$1 < 2 < 3 < \cdots < n.$$  

For simplicity, we write an $r$-combination $\{a_1, a_2, \ldots a_r\}$ as an $r$-permutation

$$a_1a_2 \cdots a_r \text{ with } a_1 < a_2 < \cdots < a_r.$$  

For two $r$-combinations $A = a_1a_2 \cdots a_r$ and $B = b_1b_2 \cdots b_r$ of $S$, we say that $A$ precedes $B$ in the lexicographic order, written $A < B$, if there is an integer $k$ ($1 \leq k \leq r$) such that

$$a_1 = b_1, a_2 = b_2, \ldots, a_{k-1} = b_{k-1}, a_k < b_k.$$  

The first $r$-combination of $S$ in lexicographic order is $12 \cdots r$, and the last $r$-combination in lexicographic order is $(n-r+1) \cdots (n-1)n$.

**Theorem.** Let $a_1a_2 \cdots a_r$ be an $r$-combination of $S = \{1, 2, \ldots, n-1, n\}$. If $a_1a_2 \cdots a_r \neq (n-r+1) \cdots (n-1)n$ and $k$ is the largest integer such that $a_k < n$ and $a_k + 1$ is not in the $a_1a_2 \cdots a_r$, then the successor of $a_1a_2 \cdots a_r$ is

$$a_1a_2 \cdots a_{k-1}(a_k + 1)(a_k + 2) \cdots (a_k + r - k + 1).$$  

**Algorithm** for generating $r$-combinations of $S = \{1, 2, \ldots, n-1, n\}$:

1. Begin with $12 \cdots r$.
2. While $a_1a_2 \cdots a_r \neq (n-r+1) \cdots (n-1)n$, do
   1. Find the largest integer $k$ such that $a_k < n$ and $a_k + 1$ is not in the $a_1a_2 \cdots a_r$.
   2. Replace $a_1a_2 \cdots a_r$ with

$$a_1a_2 \cdots a_{k-1}(a_k + 1)(a_k + 2) \cdots (a_k + r - k + 1).$$

The collection of all 4-combinations of $\{1, 2, 3, 4, 5, 6\}$ are listed by the algorithm:

<table>
<thead>
<tr>
<th>1234</th>
<th>1245</th>
<th>1345</th>
<th>1456</th>
<th>2356</th>
</tr>
</thead>
<tbody>
<tr>
<td>1235</td>
<td>1246</td>
<td>1346</td>
<td>2345</td>
<td>2456</td>
</tr>
<tr>
<td>1236</td>
<td>1256</td>
<td>1356</td>
<td>2346</td>
<td>3456</td>
</tr>
</tbody>
</table>
Theorem. Let \( r \)-combination \( a_1a_2 \cdots a_r \) of \( S = \{1, 2, \ldots, n-1, n\} \) occurs in place number
\[
\binom{n}{r} - \binom{n-a_1}{r} - \binom{n-a_2}{r-1} - \cdots - \binom{n-a_{r-1}}{2} - \binom{n-a_r}{1}
\]
in the lexicographic order.

1. Find the position of the 4-combination 1258 in the list of all 4-combinations of \( \{1, 2, 3, 4, 5, 6, 7, 8\} \).

2. Generate all 3-combinations of \( \{1, 2, 3, 4, 5\} \).

3. Generate all 3-permutations of \( \{1, 2, 3, 4, 5\} \).