Geometric generation of permutation sequences

Dennis Roseman

University of Iowa
roseman@math.uiowa.edu

March 26, 2009
Overture

**Music**

Original motivation: apply mathematics to the composition of music.

**Mathematics Focus:**

Some geometry of the $n$-dimensional permutahedron.

**Visualization Focus**

Higher dimensional visualization including braids used as a visualization tool.
Two very different musical examples:

- change ringing
- Nomos Alpha of Xenakis
From *Change Ringing* by Wilfrid G. Wilson

**Geometric generation of permutation sequences**

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**Permutahedron**

**Change Ringing**

**Bouncing**

**Problem List**

**Cell Structure**

coloring edges

coloring facets

**Braids**

**Beam Calculation**

**Edges in layers**

**Tiling**

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**CHANGE RINGING**

The Art and Science of Change Ringing on Church and Hand Bells

by

WILFRID G. WILSON

Master of the London County Association of Change Ringers 1965-65

Vice President of the Oxford Diocesan Guild of Church Bell Ringers

Member of the Central Council of Church Bell Ringers

OCTOBER HOUSE INC.
New York
From *Change Ringing* by Wilfrid G. Wilson

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CHAPTER FIVE

Plain Bob—The Method

The simplest of the even bell methods and the one best suited for use as an introduction to the inexhaustible complexities and problems of change ringing is Plain Bob. The basis of the method is the plain hunt (see Chapter Four) and the variations from it are the simplest possible by which to produce additional changes. A thorough knowledge of Plain Bob is essential to a study of other aspects of change ringing.

On Four Bells

Very little change ringing is practised on four bells, but the whole principle of Plain Bob can be seen clearly and concisely on this number, thus making it easier to understand the method on the higher numbers of bells.

Starting from rounds ($1 \ 2 \ 3 \ 4$) write out a plain hunt on four bells (as on p. 14) until the first lead of the treble (number 1) is reached. Then, while the treble leads twice, let the bell which is second strike twice in that position. This is called making second place. It is then impossible without clashing for the bells in the third and fourth positions to return to their original positions as they otherwise would, so they dodge with each other. Each of these two bells takes a step backward in its hunting course. This block of changes from the time the treble left the front until it got back to it again is called a lead, and the backstroke row when the treble is leading is called the lead end.*

We now have a new lead end $1 \ 3 \ 4 \ 2$—see row A p. 19.

From this new lead end write out a plain hunt again until the treble is once more leading. As these changes start from $1 \ 3 \ 4 \ 2$ instead of from $1 \ 2 \ 3 \ 4$ they will be different ones. Then at the second lead of the treble make similar variations—the bell that is second makes second place and the other two bells dodge with each other in 3–4 (i.e. the third and fourth positions). This completes another lead and produces another lead end, $1 \ 4 \ 2 \ 3$, marked B.

* For note on this see p. 27.
From *Change Ringing* by Wilfrid G. Wilson

**Plain Bob – The Method**

```
1 2 3 4
2 1 4 3
2 4 3 1
4 2 3 1
4 3 2 1
3 4 1 2
3 1 4 2
1 3 2 4
1 3 4 2
3 1 2 4
3 2 1 4
2 3 4 1
2 4 3 1
4 2 1 3
4 1 2 3
1 4 3 2
1 4 2 3
```

A

```
3 1 2 4
3 2 1 4
2 3 4 1
2 4 3 1
4 2 1 3
4 1 2 3
1 4 3 2
1 4 2 3
```

B

```
4 3 1 2
3 4 2 1
3 2 4 1
2 3 1 4
2 1 3 4
1 2 4 3
1 2 3 4
```

C

From this new lead end write out another plain hunt until the treble reaches the front again. Make similar variations to those at the other lead ends and we are back at rounds (row C). We have produced all the possible 24 changes on four bells, we have no repetitions and we have missed none out. This block of changes (three complete leads) is called a plain course of Plain Bob Minimus, sometimes Plain Bob Singles. The major factor in this plain course is the continued plain hunting of the treble and you will find that there is a whole large class of methods in which the treble plain hunts continually among the other bells – however many there may be.

Now on the table of 24 changes draw a line through all the 2’s. It will be this shape, though less squashed up.
From *Formal Music* by I. Xenakis—Nomos Alpha
From *Formal Music* by I. Xenakis—Nomos Alpha
Geometric generation of permutation sequences

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Permutahedron

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Ringing

Bouncing

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Towards a Philosophy of Music

Organization In-Time

IV. The products $K_1 \times C_4$ and $K_2 \times C_4$ are the result of the product of two graphs of closed transformations of the cube in itself. The mapping of the graphs is one-to-one and sounded successively; for example:

$$K_1 \rightarrow \text{graph} \quad (D_{4\sigma})$$

(See Figs. VIII-9, 10.)

V. Each $C_i$ is mapped onto one of the cells of $H \times X$ according to two principles: maximum expansion (minimum repetition), and maximum contrast or maximum resemblance. (See Fig. VIII-11.)
Definition

A **musical composition** is a family of sequences of related musical events.

Time and voices

Progression in time is related to succession in a sequence; each sequence represents a “voice”.

The mathematical objects we chose are permutations.

Construct families of sequences length $k$ of permutations of order $n$, where $k$ and $n$ are independent.
Permutations Geometrically: the Permutahedron

The Permutahedron

1. Take the $n!$ permutations $S_n$ to be all permutations of $(1, 2, \ldots, n)$
Permutations Geometrically: the Permutahedron

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2. They are \( n \)-tuples—plot them as points in \( \mathbb{R}^n \).
The Permutahedron

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3. Take the convex hull.
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1. Take the $n!$ permutations $S_n$ to be all permutations of $(1, 2, \ldots, n)$
2. They are $n$-tuples—plot them as points in $\mathbb{R}^n$.
3. Take the convex hull.
4. The resulting polytope is the **permutahedron** $\mathcal{P}(n)$
Low Dimensional Cases

Some Examples:

1. \( \mathcal{P}(2) \) is the line segment in \( \mathbb{R}^2 \) with endpoints (1, 2) and (2, 1).
Low Dimensional Cases

Some Examples:

1. $\mathcal{P}(2)$ is the line segment in $\mathbb{R}^2$ with endpoints $(1, 2)$ and $(2, 1)$, a subset of the line $x + y = 1 + 2$. 
Low Dimensional Cases

Some Examples:

1. $\mathcal{P}(2)$ is the line segment in $\mathbb{R}^2$ with endpoints $(1,2)$ and $(2,1)$, a subset of the line $x + y = 3$. 
Low Dimensional Cases

Some Examples:

1. $\mathcal{P}(2)$ is the line segment in $R^2$ with endpoints $(1, 2)$ and $(2, 1)$.
2. $\mathcal{P}(3)$ is a hexagon in $R^3$ in the plane.
Low Dimensional Cases

Some Examples:

1. $\mathcal{P}(2)$ is the line segment in $\mathbb{R}^2$ with endpoints $(1, 2)$ and $(2, 1)$.

2. $\mathcal{P}(3)$ is a hexagon in $\mathbb{R}^3$ in the plane $x + y + z = 6$. 
Low Dimensional Cases

Some Examples:

1. $\mathcal{P}(2)$ is the line segment in $R^2$ with endpoints (1, 2) and (2, 1).
2. $\mathcal{P}(3)$ is a hexagon in $R^3$ in the plane.
3. $\mathcal{P}(4)$ is a truncated octahedron in $R^3$. 
Low Dimensional Cases

Some Examples:

1. $\mathcal{P}(2)$ is the line segment in $R^2$ with endpoints $(1, 2)$ and $(2, 1)$.
2. $\mathcal{P}(3)$ is a hexagon in $R^3$ in the plane.
3. $\mathcal{P}(4)$ is a truncated octahedron in $R^3$ subset of the hyperplane $x + y + z + w = 10$. 
The Permutahedron of order 2

**Figure:** The two permutations (1, 2) and (2, 1) : a line segment in $R^2$
The Permutahedron of order 3

Figure: Hexagon in $R^3$ of the six permutations of order 3: (1, 2, 3), (2, 1, 3), (3, 1, 2), (3, 2, 1), (2, 3, 1), (1, 3, 2), (1, 2, 3)
The Permutahedron of order 4

Figure: The 24 permutations of order 4 determine a truncated octahedron in $R^4$ which we show in $R^3$
Definition

A sequence $\Sigma$ of permutations is a \textit{change ringing} composition if

- $\Sigma$ begins and ends with the identity permutation of $S_n$
- Otherwise each of the $n!$ order $n$ permutations occurs exactly one time
- Two consecutive permutations of $\Sigma$ differ by switching two consecutive integers.
Ringing Changes on Three Bells

Table: One way to ring changes on 3 bells; the second reverses the order.
Ringing Changes Geometrically

**Figure:** The change *Double Canterbury Pleasure Minimus* corresponds to a Hamiltonian path in the edge set of $\mathcal{P}(4)$.
Critique of Change Ringing

- Change ringing is very limited—hard to get non-trivial examples.
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- There is no relationship between one change and another
Change Ringing

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Critique of Change Ringing

- Change ringing is very limited—hard to get non-trivial examples.
- There is no relationship between one change and another.
- Each permutation is treated equally. Musically one expects to make choices.
- There is a fixed length to a ring of changes.
- The difficulty of calculation increases rapidly with $n$. 
A new way to get a sequences of permutations

Bouncing a light in a mirrored $P(4)$
A new way to get a sequence of permutations

Bouncing a light in a mirrored $\mathcal{P}(4)$

- Build room in the shape of $\mathcal{P}(4)$ with all walls made of mirror.
A new way to get a sequences of permutations

Bouncing a light in a mirrored $\mathcal{P}(4)$

- Build room in the shape of $\mathcal{P}(4)$ with all walls made of mirror.
- From inside the room shine a “generic” laser beam from point $x_0$ in direction $\lambda_0$. 
A new way to get a sequences of permutations

**Bouncing a light in a mirrored \( P(4) \)**

- Build room in the shape of \( P(4) \) with all walls made of mirror.
- From inside the room shine a “generic” laser beam from point \( x_0 \) in direction \( \lambda_0 \).
- The beam as it reflects will hit successive walls giving a sequence of points \( x_1, x_2, \ldots \).
A new way to get a sequences of permutations

Bouncing a light in a mirrored $\mathcal{P}(4)$

- Build room in the shape of $\mathcal{P}(4)$ with all walls made of mirror.
- From inside the room shine a “generic” laser beam from point $x_0$ in direction $\lambda_0$.
- The beam as it reflects will hit successive walls giving a sequence of points $x_1, x_2, \ldots$.
- Since the beam is generic there will be a unique vertex (permutation) $\pi_i$ of $\mathcal{P}(4)$ nearest to $x_i$. 
Bouncing a light in a mirrored $\mathcal{P}(4)$

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- From inside the room shine a “generic” laser beam from point $x_0$ in direction $\lambda_0$.
- The beam as it reflects will hit successive walls giving a sequence of points $x_1, x_2, \ldots$.
- Since the beam is generic there will be a unique vertex (permutation) $\pi_i$ of $\mathcal{P}(4)$ nearest to $x_i$.
- Thus we generate our sequence of permutations $S(x_0, \lambda_0) = (\pi_1, \pi_2, \ldots)$
Bounce points

Definition
The points $x_1, x_2, \ldots$ are called either intersection points (they are calculated as an intersection of a ray and $\partial \mathcal{P}(n)$) or bounce points (since our beam bounces there).

Bouncing for $\mathcal{P}(n)$
Cleary we can define this process for permutations of order $n$. 
Figure: An example of 16 bounces
### Corresponding Sequence of Permutations

<table>
<thead>
<tr>
<th>2 1 4 3</th>
<th>1 4 2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 1 4 3</td>
<td>3 4 2 1</td>
</tr>
<tr>
<td>1 3 2 4</td>
<td>4 2 1 3</td>
</tr>
<tr>
<td>2 4 3 1</td>
<td>1 2 4 3</td>
</tr>
<tr>
<td>3 1 2 4</td>
<td>1 3 4 2</td>
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<td>2 1 3 4</td>
</tr>
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<tr>
<td>2 4 3 1</td>
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</table>
Figure: Another example of 16 bounces
The numbers we use are not necessarily pitches

**Figure:** Any knob or input/output on the Control Panel of this Moog corresponds to a number. Photo by Kevin Lightner
Comparing Bouncing to Change Ringing

- **Change ringing**: finite number of possibilities
Comparing Bouncing to Change Ringing

- **Bouncing**: infinite number of possibilities
Comparing Bouncing to Change Ringing

- **Bouncing**: infinite number of possibilities
- **Change ringing** is hard is it to calculate.
Bouncing, Ringing

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Comparing Bouncing to Change Ringing

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- **Bouncing** is easy to calculate as we will see
Comparing Bouncing to Change Ringing

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Bouncing, Ringing

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- **Bouncing**: infinite number of possibilities
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Comparing Bouncing to Change Ringing

- **Bouncing**: infinite number of possibilities
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- **Bouncing**: if one varies $x$ and $\lambda$ one gets related permutation sequences $S(x, \lambda)$
- **Change Ringing**: each permutation is treated equally
Comparing Bouncing to Change Ringing

- **Bouncing**: infinite number of possibilities
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- **Bouncing**: distinct sequences have distinct characteristics
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- **Bouncing**: infinite number of possibilities
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- **Bouncing**: if one varies $x$ and $\lambda$ one gets related permutation sequences $S(x, \lambda)$
- **Bouncing**: distinct sequences have distinct characteristics
- **Change Ringing**: difficulty of calculation increases rapidly with $n$
Bouncing, Ringing

Comparing Bouncing to Change Ringing

- **Bouncing**: infinite number of possibilities
- **Bouncing**: is easy to calculate
- **Bouncing**: if one varies $x$ and $\lambda$ one gets related permutation sequences $S(x, \lambda)$
- **Bouncing**: distinct sequences have distinct characteristics
- **Bouncing**: calculation is quadratic with respect to $n$
The rest of the talk: mathematics and visualization

Questions we now address

1. How fast can we calculate the bouncing path for fairly high orders—8, 12, 16, 32?
2. How can we visualize the calculational process and the results?
3. What is the geometry of a high dimensional permutahedron?
4. How does the geometry of the permutahedron change with $n$?
Change Ringing

1. Construct a special polygonal path in the wireframe of a permutahedron.
2. The sequence of vertices on that path is the desired permutation sequence.

Bouncing, an alternative

1. Take a generically generated generic path in $\mathbb{R}^n$ that avoids the wireframe
2. Obtain the sequence of permutations by “digitizing” to permutations near the path.
Vertices of the Permutahedron

**Theorem**

There are $n!$ vertices in $\mathcal{P}(n)$.

All vertices of $\mathcal{P}(n)$ all lie on an $(n - 2)$-sphere with center $C_n$, the centroid of $\mathcal{P}(n)$, and radius $\rho_n$.

**Definition**

This sphere is the permutahedral sphere of order $n$ and $\rho_n$ the **permutahedral radius**. The distance between $C_n$ and the centroid of $Y_\alpha$ is the **inner permutahedral radius**.
Generators of the Symmetric Group

Definition

An elementary transposition is a permutation that interchanges consecutive integers.

Note:

This is interchange of consecutive integers (wherever they are) not interchange of integers in consecutive positions (whatever the integers are).
**Definition**

The union of edges of $\mathcal{P}(n)$ is called the **wireframe** of $\mathcal{P}(n)$.

**Example**

The four edges from $(1, 2, 3, 4, 5)$ go to $(2, 1, 3, 4, 5)$, $(1, 3, 2, 4, 5)$, $(1, 2, 4, 3, 5)$, and $(1, 2, 3, 5, 4)$.
Edges of the Permutahedron

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Basic general edge facts

- Two permutations are connected by an edge if and only if coordinates differ by a switch of two coordinates of consecutive value.
### Edges of the Permutahedron

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- Two permutations are connected by an edge if and only if coordinates differ by a switch of two coordinates of consecutive value.
- Thus any edge corresponds to an elementary transposition.
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The union of edges of $\mathcal{P}(n)$ is called the **wireframe** of $\mathcal{P}(n)$.

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The four edges from $(1, 2, 3, 4, 5)$ go to $(2, 1, 3, 4, 5)$, $(1, 3, 2, 4, 5)$, $(1, 2, 4, 3, 5)$, and $(1, 2, 3, 5, 4)$.

**Basic general edge facts**

- Two permutations are connected by an edge if and only if coordinates differ by a switch of two coordinates of consecutive value.
- Thus any edge corresponds to an elementary transposition.
- Thus all edges have length $\sqrt{2}$. 
## Edges of the Permutahedron

### Definition

The union of edges of $P(n)$ is called the **wireframe** of $P(n)$.

### Example

The four edges from $(1, 2, 3, 4, 5)$ go to $(2, 1, 3, 4, 5)$, $(1, 3, 2, 4, 5)$, $(1, 2, 4, 3, 5)$, and $(1, 2, 3, 5, 4)$.

### Basic general edge facts

- Two permutations are connected by an edge if and only if coordinates differ by a switch of two coordinates of consecutive value.
- Thus any edge corresponds to an elementary transposition.
- Thus all edges have length $\sqrt{2}$.
- The order of any vertex is $(n - 1)$. 
Visualizing the edges

Rotate and project to low dimensions

We can generically rotate a wireframe of any order permutahedron then project homemorphically into $R^3$.

We can generically rotate a wireframe of any order permutahedron then project non-homemorphically into $R^2$ and still get a meaningful image.

Color the edges

We can use $(n - 1)$ colors on the edges to code the corresponding transpositions.
Coloring Edges: Order 4
Geometric generation of permutation sequences

Dennis Roseman

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Coloring Edges: Order 5
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Coloring Edges: Order 6
Cells of the Permutahedron

Proposition

An $k$-cell of $P(n)$ is either subgroup which is a product of $k$ symmetric groups or a coset of one of such subgroup. Here $P(0) = \{1\}$.

Definition

Let $Y_\alpha = \{(x_1, \ldots, x_n) \in S_n : x_1 = 1\}$ and $Y_\omega = \{(x_1, \ldots, x_n) \in S_n : x_n = n\}$. We call $Y_\alpha$ the first Young subgroup of $S_n$, $Y_\omega$ the last Young subgroup of $S_n$.

Remark

$Y_\alpha$ and $Y_\omega$ are isomorphic to $S_{n-1}$.
Re-examining \( \mathcal{P}(4) \)

**Figure:** The edges of one color are all the cosets of a Young subgroup of order two. The hexagons are all cosets of the two Young subgroups isomorphic to \( \mathcal{P}(3) \). The squares are cosets of \( \mathcal{P}(2) \times \mathcal{P}(2) \).
Re-examining $\mathcal{P}(4)$

**Figure:** The edges of one color are all the cosets of a Young subgroup of order two. The hexagons are all cosets of the two Young subgroups isomorphic to $\mathcal{P}(3)$. The squares are cosets of $\mathcal{P}(2) \times \mathcal{P}(2)$. 
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Re-examining $\mathcal{P}(4)$
## Facets of the Permutahedron

### Definition
The **facets** are the $(n - 2)$-cells of $\mathcal{P}(n)$.

### Proposition
$\mathcal{P}(n)$ has $2^n - 2$ facets

### Example
So $\mathcal{P}(8)$ has 254 facets and $\mathcal{P}(12)$ has 4094.

### Implication
The number of facets is exponential in $n$. Our light beam calculation should not be based on examination of all facets.
A duality of facets and edges

Transposition colors for facets

At a vertex $v$ of facet $F$ you see $(n - 1)$ edges all of distinct colors.

One of these colors is not an edge of $F$.

This color will identify our corresponding elementary transposition.
Figure: Here we color the three generators: $\sigma_1 \sigma_2 \sigma_3$
Coloring the facets

Figure: Here we color the three generators: red green blue
Figure: Here we color the three generators: $\sigma_1 \sigma_2 \sigma_3$
Figure: Here we color the three generators: $\sigma_1 \sigma_2 \sigma_3$ The facet color is the unique color *not* an edge color of the facet.
Two group presentations:

**Presentation of Order $n$ Braid Group**

Generators: $\sigma_1, \ldots, \sigma_{n-1}$

Relations:
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $j \neq i \pm 1$
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

**Presentation of Order $n$ Symmetric Group**

Generators: $\sigma_1, \ldots, \sigma_{n-1}$

Relations:
- $\sigma_i = \sigma_i^{-1}$
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $j \neq i \pm 1$
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
Inverses and elementary transpositions

From the symmetric group to the braid group

A finite sequence of elementary transpositions \( \tau_1, \tau_2, \ldots, \tau_n \) corresponds to a word in the symmetric group:

\[
\tau_1 \tau_2 \cdots \tau_n.
\]

But if (somehow) we can distinguish elementary transpositions from their inverses we would obtain a word in the braid group:

\[
\tau_1^{\epsilon_1} \tau_2^{\epsilon_2} \cdots \tau_n^{\epsilon_n}.
\]
Signs for transpositions: one of many methods

**Definition**

A **braid sign convention** is a function that associates to any bounce point $x_i$ of any bouncing path $\epsilon(x_i) = \pm 1$.

**Example**

Let $\vec{N}$ be the vector from the identity permutation to the reverse of the identity. Define the **sign at** $x_i$ to be the sign of the dot product $\vec{x}_{i-1} \vec{x}_i \cdot \vec{N}$. Think of the identity as the “south pole”. **Positive** means we were heading north before we “bounced”; **negative** means heading “south”.

A bouncing path braid

Definition

Given a bouncing path $x_0, x_1, x_2, x_3 \ldots$ and a braid sign convention we obtain a bounce path braid—the braid given by the word in the braid group:

$$\sigma(x_1)^{\epsilon_1} \sigma(x_2)^{\epsilon_2} \sigma(x_3)^{\epsilon_3} \ldots$$
Example of Bounce Braid for Order 4
Example of Bounce Braid for Order 8
Example of Bounce Braid for Order 12
Example of Bounce Braid for Order 16
Braid as an Aid

In general we need to look at the bounce permutations \textit{together} with the bounce braid.

The bounce path indicates \textit{where} the bounce occurs, the braid tells us something about \textit{how} the “type” of bounce.
The nearest permutation to a point

**Definition**

A generic point \( z = (z_1, \ldots, z_n) \) of \( \mathbb{R}^n \) will have \( n \) distinct coordinate values.

The rank of \( z_i \), denoted \( r(z_i) \), is one plus the number of coordinates of \( z \) smaller than \( z_i \).

**Definition**

The rank vector \( \rho(z) = (r(z_1), \ldots, r(z_n)) \)

**In other words:**

Simply Put: The rank of \( z \) is the closest permutation to \( z \).

Or not: A generic point is mapped to a chamber of the real \( n \)-braid arrangement.
Finding the intersection of a ray and $\partial P(n)$

The key

Focus on the plane $P$ that contains the three points

1. the centroid $C$ of $P(n)$,
2. the initial point $x_0$
3. the tip of our vector $x_0 + \lambda$.

We then project the wireframe of $P(n)$ onto this plane.
Projection wireframe $\mathcal{P}(5)$
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Projection wireframe $\mathcal{P}(5)$
Projection wireframe $\mathcal{P}(6)$
Geometric generation of permutation sequences

Dennis Roseman

Permutahedron

Change Ringing

Bouncing

Problem List

Cell Structure
coloring edges

coloring facets

Braids

Beam Calculation

Edges in layers

Tiling

Projection wireframe $\mathcal{P}(6)$
Projection wireframe $\mathcal{P}(7)$
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- The “central” portion of figures is hard to understand but the area around the edge is much clearer.
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- The higher permutahedra are not round. (This is important since if we do our bouncing inside a round ball, the path will be planar)
- There seems to be some structure there that is evident from the projections. (Some will be clearer with a colored wireframe)
- The “central” portion of figures is hard to understand but the area around the edge is much clearer. (In fact the bounding polygonal path of the projection is the projection of a simple closed polygonal path of $\mathcal{P}(n)$ edges)
Creating great path: projection to the plane $P$
Same as previous figure after rotation in $R^3$
Let $P$ be the plane through points: $x_0$, $x_0 + \lambda$ and the centroid of $P(n)$.
A Great Path Method

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At each $p_i$ find the intersection point of the ray with hyperplanes determined by facets at $p_i$.

By convexity of $P(n)$ the closest such intersection point is our bounce point.
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4. Each edge of $\partial D$ is a projection of a single edge of $W$. 
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6. From $\pi_0$, follow this great path in the general direction of $\lambda$ obtaining vertex sequence $p_0, p_1, \ldots$. 
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8. By convexity of $P(n)$ the closest such intersection point is our bounce point.
A braid from edges, not facets

1. Put together the great paths used in calculating a bounce sequence.
2. This sequence of edges of $\partial \mathcal{P}(n)$ gives a sequence of elementary transpositions.
3. There are ways to define signs to this sequence giving yet more braids.
Very briefly—one way to get signs

Example

1. Consider the polygonal path $\beta = P \cap \partial P(n)$
2. Orient $\beta$ using $\lambda$
3. Orient the $(n - 3)$-cells of $\partial P(n)$
4. Use intersection numbers of the $\beta$ with those cells to get our sign

Note:

There is a quick indirect way to calculate this from the construction of the great path.
A connection to topic in computer science

There is a relationship between a colored great path which joins two antipodal permutations and the concept of a sorting network.
A different bouncing braid

A braid based on edges near bounce

1. Take bounce points $x_1, x_2, \ldots, x_n$ where $x_i$ lies in facet $F_i$
2. Let $e_i$ be the edge of $F_i$ closest to $x_i$ with associated an elementary transposition $\Sigma(x_i)$
3. There are a number of ways to assign a braid sign convention $\epsilon_i$;
4. This gives a braid word

$$\Sigma(x_1)^{\epsilon_1} \Sigma(x_2)^{\epsilon_2} \ldots \Sigma(x_n)^{\epsilon_n}$$
There is not time to go into detail . . .

To give a sense that the color of an edge tells something about the nature of the bounce consider the following graphics.
A projection of labeled wireframe of $\mathcal{P}(5)$
A projection of labeled wireframe of $\mathcal{P}(5)$
A projection of labeled wireframe of $\mathcal{P}(6)$
A projection of labeled wireframe of $\mathcal{P}(7)$
An alternative to bouncing: permutahedral tiles

**Theorem**

\[ R^n \text{ can be tiled with translated copies } \mathcal{P}(n) \]
A black and white tiling of the plane by hexagons
Fitting two adjacent $\mathcal{P}(4)$s
Fitting three adjacent $\mathcal{P}(4)$s
Fitting four adjacent $\mathcal{P}(4)s$
A “black and white tiling of $R^3$ by $\mathcal{P}(4)s$
Our Questions:

1. How fast can we calculate the bouncing path for fairly high orders—8, 12, 16, 32?
   - Quadratic, not exponential.

2. How can we visualize the calculational process and the results?
   - Projections, color code, braids.

3. What is the geometry of a high dimensional permutahedron?
   - Related to Young subgroups and cosets.

4. How does the geometry of the permutahedron change with $n$?
   - It does not become rounder. In fact in some directions it "flattens out".
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Review
Thank you