22m:033 Notes:
6.3 Orthogonal Projections

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/∼roseman

April 26, 2010

1 A more general view of orthogonal projection

In the previous section we discussed projection of a vector $\overrightarrow{y}$ onto a line L. We could then write $\overrightarrow{y} = proj_L \overrightarrow{y} + \overrightarrow{z}$ where $proj_L \overrightarrow{y}$ is orthogonal to $\overrightarrow{z}$.

In this section we generalize this—instead of only considering a one dimensional subspace L we consider a general subspace W and obtain a similar result.

Definition 1.1 *Let W be a subspace of R^n, $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_p}\}$ be an orthogonal basis of W, and $\overrightarrow{y} \in R^n$. The orthogonal projection of y onto W is defined to be*

$$proj_W \overrightarrow{y} = \frac{\overrightarrow{y} \cdot \overrightarrow{u_1}}{\overrightarrow{u_1} \cdot \overrightarrow{u_1}} \overrightarrow{u_1} + \frac{\overrightarrow{y} \cdot \overrightarrow{u_2}}{\overrightarrow{u_2} \cdot \overrightarrow{u_2}} \overrightarrow{u_2} + \cdots + \frac{\overrightarrow{y} \cdot \overrightarrow{u_p}}{\overrightarrow{u_p} \cdot \overrightarrow{u_p}} \overrightarrow{u_p}$$

Remark 1.2 Although our definition seems to rely on a choice of basis, it can be shown that it does not.

Proposition 1.3 *Let W be a subspace of R^n. Then each $\overrightarrow{y} \in R^n$ can be written uniquely*

$$\overrightarrow{y} = proj_W \overrightarrow{y} + \overrightarrow{z}$$

where $proj_W \overrightarrow{y}$ *is orthogonal to* $\overrightarrow{z}$.

Remark 1.4 The text uses notation $\widehat{\mathbf{y}}$. In the class notes and tests we use $\overrightarrow{y}$ to represent vectors rather than our text style wich uses bold font, $\mathbf{y}$. But if we were to use the "hat" symbol for projection it would look awkward as $\widehat{\overrightarrow{y}}$ or $\overrightarrow{\widehat{y}}$. We will avoid this and use the commonly used notation $proj_W \overrightarrow{y}$.

Example 1.5 Suppose we consider

$$\overrightarrow{u_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \overrightarrow{u_2} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} \text{ and } \overrightarrow{y} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

We note that u_1 and u_2 are orthogonal. Let W be the space spanned by u_1 and u_2. Then we calculate:

$$\overrightarrow{y} \cdot \overrightarrow{u_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 5$$

$$\overrightarrow{y} \cdot \overrightarrow{u_2} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = 11$$

$$\overrightarrow{u_1} \cdot \overrightarrow{u_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 14$$

$$\overrightarrow{u_2} \cdot \overrightarrow{u_2} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = 42$$

So the projection of y onto W is given by

$$proj_W \overrightarrow{y} = \frac{5}{14}\overrightarrow{u_1} + \frac{11}{42}\overrightarrow{u_2} = \frac{5}{14}\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \frac{11}{42}\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

Example 1.6 Next we consider the same basis u_1, u_2 with a different y

$$\overrightarrow{u_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \overrightarrow{u_2} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} \text{ and } \overrightarrow{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Let W be the space spanned by u_1 and u_2. Then we calculate:

$$\overrightarrow{y} \cdot \overrightarrow{u_1} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 7$$

$$\overrightarrow{y} \cdot \overrightarrow{u_2} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = 21$$

$$\overrightarrow{u_1} \cdot \overrightarrow{u_1} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 14$$

$$\overrightarrow{u_2} \cdot \overrightarrow{u_2} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = 42$$

So the projection of y onto W is given by

$$proj_W \overrightarrow{y} = \frac{1}{2}\overrightarrow{u_1} + \frac{1}{2}\overrightarrow{u_2} = \frac{1}{2}\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

So the projection of the vector to W is the vector itself. How is this possible? Hint: Think about $\frac{1}{2}\overrightarrow{u_1} + \frac{1}{2}\overrightarrow{u_2}$ as linear combination and recall the definition of subspace.

2 The Best Approximation Theorem

Proposition 2.1 *Let W be a subspace of R^n, and $\overrightarrow{y} \in R^n$ any vector. Then $proj_W \overrightarrow{y}$ is* ***the closest***

point in W ***to*** $\overrightarrow{y}$ *in the sense that*

$$||\overrightarrow{y} - proj_W \overrightarrow{y}|| < ||\overrightarrow{y} - \overrightarrow{v}||$$

for any $\overrightarrow{v} \in W$ *with* $\overrightarrow{v} \neq proj_W \overrightarrow{y}$.

The text introduces the next definition in Example 4 page 399

Definition 2.2 *The* ***distance between a point*** $\overrightarrow{y}$ ***and subspace*** W *is* $||\overrightarrow{y} - proj_W \overrightarrow{y}||$

Example 2.3 Refering to Example 1.5 we calculate the distance from the vector to the subspace:

$$\overrightarrow{y} - proj_W \overrightarrow{y} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

So

$$||\overrightarrow{y} - proj_W \overrightarrow{y}|| = \frac{2}{\sqrt{3}}$$

In Example 1.6 we calculate this distance to be 0, as it should be since $\overrightarrow{y} \in W$.