22m:033 Notes:
6.1 Inner Product, Length and Orthogonality

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1 The inner product

Arithmetic is based on addition and multiplication. Geometry is based on length and angle.

Using a dot product we obtain length and angle from addition and multiplication in a relatively simple way.

Definition 1.1 *Suppose* $\overrightarrow{u}$ *is an n-dimensional vector with coordinates* $\{u_i\}$ *and* $\overrightarrow{v}$ *is an n-dimensional vector with coordinates* $\{v_i\}$ *then the* ***inner product*** *or* ***dot product of*** $\overrightarrow{u}$ ***and*** $\overrightarrow{v}$ *is defined:*

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Remark 1.2 If we express a vector as a matrix with one column the inner product can be expressed in terms of matrix multiplication as: $\overrightarrow{u} \cdot \overrightarrow{v} = (\overrightarrow{u})^T \overrightarrow{v}$.

So the dot product of $\overrightarrow{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{v} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ can be calculated:

$$\overrightarrow{u} \cdot \overrightarrow{v} = (\overrightarrow{u})^T \overrightarrow{v} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = 2 - 6 + 15 = 11.$$

It is important to note that the dot product of two *vectors* is a *number*. This makes it very different from more familiar multiplications where we multiply two things of some type and get another thing of that same type.

Other than that, the formal properties for dot product have the familiar look of multiplication:

Proposition 1.3 *Suppose* $\overrightarrow{u}$ *and* $\overrightarrow{v}$ *are* n*-dimensional vectors and* c *a number. Then*

1. $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u}$
2. $(\overrightarrow{u} + \overrightarrow{v}) \cdot \overrightarrow{w} = \overrightarrow{u} \cdot \overrightarrow{w} + \overrightarrow{v} \cdot \overrightarrow{w}$
3. $(c\overrightarrow{u}) \cdot \overrightarrow{v} = \overrightarrow{u} \cdot (c\overrightarrow{v}) = c(\overrightarrow{u} \cdot \overrightarrow{v})$
4. $0 \leq \overrightarrow{u} \cdot \overrightarrow{u}$
5. $\overrightarrow{u} \cdot \overrightarrow{u} = 0$ *if and only if* $\overrightarrow{u} = \overrightarrow{0}$

Remark 1.4 Note in that last item, there are two *different* "zeros'—the number zero and the vector zero.

2 Length of a vector

Definition 2.1 *The length (or norm) of a vector* $\overrightarrow{v}$ *is defined to be* $\sqrt{\overrightarrow{v} \cdot \overrightarrow{v}}$*,*

A common notation for the length of $\overrightarrow{v}$ is $||\overrightarrow{v}||$.

Remark 2.2 For two and three dimensional vectors, the formula agrees with the familiar length formulas.

If $\overrightarrow{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $||\overrightarrow{v}|| = \sqrt{x^2 + y^2}$. This is the familiar Pythagorean Theorem in the plane.

If $\overrightarrow{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $||\overrightarrow{v}|| = \sqrt{x^2 + y^2 + z^2}$. This is the three dimensional version of the Pythagorean Theorem.

For higher dimensions we have If $\overrightarrow{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ then $||\overrightarrow{v}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. This is clearly an analogue version of the Pythagorean Theorem. This allows us to define length in any R^n even though we may have difficulty in visualizing things in high dimensions.

A vector $\overrightarrow{v} \in R^n$ lies on a line $L = t\overrightarrow{v}$, and our length is just length as measured in this line.

Example 2.3 So the length of $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ is

$$\sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{1 + 4 + 9 + 16} = \sqrt{30}.$$

Remark 2.4 It is easy to check from the definition that if c is a number, then $||c\overrightarrow{v}|| = |c| \ ||\overrightarrow{v}||$.

For example if $\overrightarrow{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $c\overrightarrow{v} = \begin{pmatrix} cx \\ cy \end{pmatrix}$

And so

$$||c\overrightarrow{v}|| = \sqrt{(cx)^2 + (cy)^2} = \sqrt{c^2(x^2 + y^2)} =$$
$$\sqrt{c^2} \ \sqrt{(x^2 + y^2)} = |c| \ \sqrt{(x^2 + y^2)}.$$

Definition 2.5 *A vector with length 1 is called a **unit vector**.*

Definition 2.6 *If $\overrightarrow{v}$ is a non zero vector, the vector $(\frac{1}{||\overrightarrow{v}||}\overrightarrow{v})$ is called the* ***normalization of*** $\overrightarrow{v}$.

Remark 2.7 If $\overrightarrow{v}$ is a non zero vector, then $(\frac{1}{||\overrightarrow{v}||})\overrightarrow{v}$ is a unit vector since, using Remark 2.4, we see:

$$\left|\left|\left(\frac{1}{||\overrightarrow{v}||}\right)\overrightarrow{v}\right|\right| = \left|\frac{1}{||\overrightarrow{v}||}\right| ||\overrightarrow{v}|| = \frac{||\overrightarrow{v}||}{||\overrightarrow{v}||} = 1$$

Example 2.8 The normalization of $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{3}{\sqrt{30}} \\ \frac{4}{\sqrt{30}} \end{pmatrix}$.

3 Distance in R^n

Definition 3.1 *Suppose $\overrightarrow{u}$ and $\overrightarrow{u}$ are n-dimensional vectors. We define the distance between $\overrightarrow{u}$ and $\overrightarrow{u}$ to be*

$$dist(\overrightarrow{u}, \overrightarrow{v}) = ||\overrightarrow{u} - \overrightarrow{v}||$$

Remark 3.2 In other words, the distance between two vectors is length of their difference.

If we write out the formulas we find:

The distance between two vectors is the square root of the sum of the squares of the differences of the coordinates:

$$dist(\overrightarrow{u}, \overrightarrow{v}) = \sqrt{\sum_{i=1}^{n}(u_i^2 - v_i^2)}$$

Also note we can write:

$$dist(\overrightarrow{u}, \overrightarrow{v}) = \sqrt{(\overrightarrow{u} - \overrightarrow{v}) \cdot (\overrightarrow{u} - \overrightarrow{v})}.$$

4 Angles in R^n

The famed Pythagorean Theorem: $a^2 + b^2 = c^2$ only holds for right triangles where the angle θ opposite the side of length c (that is, the hypotenuse) is $\frac{\pi}{2}$.

The theorem covering the case when $\theta \neq \frac{\pi}{2}$ is the Law of Cosines:

$$a^2 + b^2 = c^2 - 2bc\cos\theta.$$

Two vectors in the plane R^2, $\overrightarrow{u}$ and $\overrightarrow{v}$ (for the moment assume they are linearly independent) determine a triangle whose vertices are: the origin and two "end-points" of the vectors. Let θ be the angle of this triangle at $\overrightarrow{0}$; we refer to this as the angle between the vectors.

Applying the Law of Cosines to this triangle we obtain:

$$\overrightarrow{u} \cdot \overrightarrow{v} = ||\overrightarrow{u}||\ ||\overrightarrow{v}||\cos\theta$$

It is easy to check that this formula works if $\overrightarrow{u}$ and $\overrightarrow{v}$ are not linearly independent.

In the general case suppose we have two vectors in R^n, $\overrightarrow{u}$ and $\overrightarrow{v}$ (for the moment assume they are linearly independent). Then they lie in a well defined two-dimensional subset which contains the origin and is geometrically is just like R^2. In this plane we can measure our angle θ. And we will find that our formula

$$\overrightarrow{u} \cdot \overrightarrow{v} = ||\overrightarrow{u}||\ ||\overrightarrow{v}||\cos\theta$$

still holds.

5 Orthogonal vectors

Two line segments joined at a point are called orthogonal if the meet at a right angle. We can think of these line segments as giving two vectors. Since $\cos\frac{\pi}{2} = 0$ we see from the formula of Section 4 that the dot product of these two vectors must be zero.

Definition 5.1 *Two vectors in R^n, $\overrightarrow{u}$ and $\overrightarrow{v}$ are orthogonal if and only if $\overrightarrow{u} \cdot \overrightarrow{v} = 0$.*

Example 5.2 So we can see that

$$\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

are not orthogonal since their dot product is $2+4+2 \neq 0$.

Also

$$\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

are orthogonal since their dot product is $2 - 4 + 2 = 0$.

Example 5.3 Find a vector of the form $\begin{pmatrix} a \\ -1 \end{pmatrix}$ which is orthogonal to $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. The dot product of these vectors is $-a-3$, and we need $-a-3=0$, so we conclude that $a=3$.

6 Orthogonal complements

Definition 6.1 *If $\overrightarrow{z}$ is a vector in R^n and W a subspace we say $\overrightarrow{z}$* ***is orthogonal to*** *W if $\overrightarrow{z}$ is orthogonal to every vector in W.*

Remark 6.2 If $\overrightarrow{z}$ is a vector in R^n the set of *all* vectors orthogonal to $\overrightarrow{z}$ is a subspace.

If W is a subspace of R^n the set of *all* vectors orthogonal to some vector in W is a also subspace of R^n. We denote this subspace as $W^{\perp}$.

Definition 6.3 *If W is a subspace in R^n, the subspace $W^{\perp}$ of all vectors orthogonal to W is called* ***the orthogonal complement of*** *W .*

Definition 6.4 *Let A be an $m \times n$ matrix. The **row space,** $RowA$**, of** A is the subspace of R^m spanned by the row vectors of A. The **column space,** $ColA$**, of** A is the subspace of R^n spanned by the column vectors of A*

When we do matrix multiplication $AB = C$ we calculate c_{ij} from the i-th row of A and the j-th column of B. That calculation can be viewed as a dot product of the i-th row vector of A and the j-th column vector of B.

Now think about an equation $A\overrightarrow{x} = \overrightarrow{0}$. We can think of this as saying that the vector $\overrightarrow{x}$ to every row vector of A. This leads us to the following theorem.

Proposition 6.5 *Let A be an $m \times n$ matrix. Then*

1. $(RowA)^{\perp} = NulA$
2. $(ColA)^{\perp} = NulA^T$

Example 6.6 Find a basis for the orthogonal complement of the subspace W spanned by

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

By the Proposition 6.5 we need only find the null space of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

We do this by row reducing A to get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Letting $x_4 = s$ and $x_3 = t$ we will have $x_2 + t + 2s = 0$ and $x_1 + t = 0$, So the null space is all vectors

$$\begin{pmatrix} -t \\ -t-2s \\ t \\ s \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

We conclude that

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $W^{\perp}$.

7 Problems

1. Let $\overrightarrow{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find a basis for the subspace of R^3 orthogonal to the subspace spanned by $\overrightarrow{v}$.
2. Find a basis for the orthogonal complement of the

subspace W spanned by

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$