22m:033 Notes: Chapter 3 section 3 Cramer's Rule, Volume and Linear Transformations

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1 Solving 3 equations in 3 unknowns

Suppose $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

We want to solve $A\overrightarrow{x} = \overrightarrow{b}$.

Well we make the augmented matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$$

row reduce and get:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-a_{23}a_{32}b_1+a_{22}a_{33}b_1+a_{13}a_{32}b_2-a_{12}a_{33}b_2-a_{13}a_{22}b_3+a_{12}a_{23}b_3}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}} \\ 0 & 1 & 0 & \frac{-a_{23}a_{31}b_1+a_{21}a_{33}b_1+a_{13}a_{31}b_2-a_{11}a_{33}b_2-a_{13}a_{21}b_3+a_{11}a_{23}b_3}{a_{13}a_{22}a_{31}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33}} \\ 0 & 0 & 1 & \frac{-a_{22}a_{31}b_1+a_{21}a_{32}b_1+a_{12}a_{31}b_2-a_{11}a_{32}b_2-a_{12}a_{21}b_3+a_{11}a_{22}b_3}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}} \end{pmatrix}$$

If we look at the denominators or the right hand column we see they are all the same and that in fact they are the determinant of A. The numerators are more mysterious, but they do have the look of "the determinant of something". If we think about this for a while—perhaps days not hours—we would likely discover Cramer's rule for this case.

2 Cramer's Rule

To make a long story short, here is a final result of the above line of calculation.

Important: Cramer's Rule is only for solving n equations with n unknowns.

Notation: If $A\overrightarrow{x} = \overrightarrow{b}$ is an $n \times n$ matrix, let $A_i(\overrightarrow{b})$ denote the matrix obtained by removing the i-th column of A and replacing it with $\overrightarrow{b}$.

Example 2.1 Suppose $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then $A_2(\overrightarrow{b}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix}$.

Proposition 2.2 *(Cramer's Rule) If A is an $n \times n$ invertible matrix, then for any $\overrightarrow{b}$ the unique solution* *of* $A\overrightarrow{x} = \overrightarrow{b}$ *is the the vector* $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix}$ *where*

$$x_i = \frac{\det A_i(\overrightarrow{b})}{\det A}.$$

Example 2.3 So using the matrix from the above $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and writing $\overrightarrow{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

We first calculate $\det A = 5$ thus A is is invertible, so Cramer's Rule applies. Next we calculate

$$\det A_1(\overrightarrow{b}) = \det \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 3 & -2 \end{pmatrix} = 22$$

$$\det A_2(\overrightarrow{b}) = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix} = -9$$

$$\det A_3(\overrightarrow{b}) = \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 2 & 3 & 3 \end{pmatrix} = 1$$

And so the unique solution to our equation is

$$x = \frac{22}{5}, y = \frac{-9}{5} \text{ and } z = \frac{1}{5}$$

.

Remark 2.4 There is a faster way to do this that we already know:

If we row reduce the augmented matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 2 & 3 & -2 & 3 \end{pmatrix}$$

we get:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{22}{5} \\ 0 & 1 & 0 & -\frac{9}{5} \\ 0 & 0 & 1 & \frac{1}{5} \end{pmatrix}$$

However Cramer's Rule is useful for other things since it gives an explicit formula for the solution of a set of equations (if it has one).

3 A formula for an inverse of a 3×3 matrix

So if we want to calculate an inverse for a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ we form the larger matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{pmatrix}$$

row reduce and get

$$\left(\begin{matrix} 1 & 0 & 0 & \frac{a_{22}a_{33}-a_{23}a_{32}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}} & \frac{a_{13}a_{32}-a_{12}a_{33}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}} & \frac{}{a_{13}a_{22}a_{31}-a_{12}a_{23}\cdots} \\ 0 & 1 & 0 & \frac{a_{23}a_{31}-a_{21}a_{33}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}} & \frac{a_{13}a_{31}-a_{11}a_{33}}{a_{13}a_{22}a_{31}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33}} & \frac{}{-a_{13}a_{22}a_{31}+a_{12}a_{23}\cdots} \\ 0 & 0 & 1 & \frac{a_{22}a_{31}-a_{21}a_{32}}{a_{13}a_{22}a_{31}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33}} & \frac{a_{12}a_{31}-a_{11}a_{32}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}} & \frac{}{a_{13}a_{22}a_{31}-a_{12}a_{23}\cdots} \end{matrix} \right.$$

Even with small type we cannot fit this onto our page so we look at the top row, columns of this matrix one at a time:

$$\frac{a_{22}a_{33}-a_{23}a_{32}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}}$$
$$\frac{a_{23}a_{31}-a_{21}a_{33}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}}$$
$$\frac{a_{22}a_{31}-a_{21}a_{32}}{a_{13}a_{22}a_{31}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33}}$$

$$\frac{a_{13}a_{32}-a_{12}a_{33}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}}$$
$$\frac{a_{13}a_{31}-a_{11}a_{33}}{a_{13}a_{22}a_{31}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33}}$$
$$\frac{a_{12}a_{31}-a_{11}a_{32}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}}$$

$$\frac{a_{13}a_{22}-a_{12}a_{23}}{a_{13}a_{22}a_{31}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33}}$$
$$\frac{a_{23}a_{31}-a_{21}a_{33}}{-a_{13}a_{22}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{11}a_{22}a_{33}}$$
$$\frac{a_{22}a_{31}-a_{21}a_{32}}{a_{13}a_{22}a_{31}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}-a_{11}a_{22}a_{33}}$$

We can see that the denominators are the determinant and the negative of the determinant. The numerators appear to be determinants of some 2×2 matrix.

It would take a while to puzzle the pattern out, so here is the solution—not just for this case but also the $n \times n$ case.

First we recall a definition from Section 1 of Chapter 3:

Given a square matrix A, the **(i, j)-cofactor of** A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Definition 3.1 *Given an $n \times n$ matrix A the **classical adjoint** or **adjugate** of A, denoted adj$A$ is the matrix whose entry in the ij position is C_{ji}.*

Remark 3.2 This definition is "tricky". Note that the adjugate is *not* simply the matrix of cofactors.

Read the definition of adjugate carefully and note that in the ij position is C_{ji} (which is *not* the same as C_{ij}). We will see this most clearly when we work an example.

Proposition 3.3 *If A is a square matrix then*

$$A^{-1} = \frac{1}{\det A} \; adj A.$$

Example 3.4 Lets use this formula to find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix}$. This is the matrix used in Example 2.3. We have calculated that $\det A = 5$.

We begin to calculate $adj A$.

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} = (-1)(-2) - (1)3 = -1$$

Be careful on the next one and watch your i and j.

$$C_{21} = +\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = (2)(-2) - (1)3 = -7$$

Continuing down the first row for $adj A$:

$$C_{31} = +\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = (2)(1) - (1)(-1) = 3$$

NOTE in this calculation the top entries of the top row are $C_{11}, C_{21},$ and C_{31} If we complete the calculation we get

$$adj A = \begin{pmatrix} -1 & 7 & 3 \\ 2 & -4 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

And so we get

$$A^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 7 & 3 \\ 2 & -4 & -1 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{7}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{4}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \end{pmatrix}.$$

4 What is the meaning of the value of the determinant if it is not zero?

We first consider a 2×2 matrix. It has two column vectors and these are vectors in the plane. We can view these as adjacent edges of a parallelogram P IF they are linearly independent.

Next consider a 3×3 matrix. It has three column vectors and these are vectors in space. We can view these as three edges of a parallelogram P which meet at a common corner of P IF they are linearly independent.

Proposition 4.1 *If A is a 2×2 matrix then $|\det A|$ is the area of the parallelogram determined by the column vectors of A.*

If A is a 3×3 matrix then $|\det A|$ is the area of the parallelepiped determined by the column vectors of A.

Proposition 4.2 *Let $T\colon R^2 \to R^2$ be a linear transformation determined by matrix A. If S is a parallelogram in R^2 then:*

$$\{area\ of\ T(S)\} = |\det A| \{area\ of\ S\}.$$

Let $T\colon R^3 \to R^3$ be a linear transformation determined by matrix A. If S is a parallelepiped in R^3 then:

$$\{volume\ of\ T(S)\} = |\det A| \{volume\ of\ S\}.$$

Remark 4.3 If A is not invertible then $\det A = 0$