22m:033 Notes: Chapter 3 section 3
Cramer’s Rule, Volume and Linear Transformations

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/~roseman

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1 Solving 3 equations in 3 unknowns

Suppose \( A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \) and \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \).

We want to solve \( A \vec{x} = \vec{b} \).

Well we make the augmented matrix:

\[
\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}
\]

row reduce and get:

\[
\begin{pmatrix} 1 & 0 & 0 & -a_{23}a_{32}b_1 + a_{22}a_{33}b_1 + a_{13}a_{32}b_2 - a_{12}a_{33}b_2 - a_{13}a_{22}b_3 + a_{12}a_{23}b_3 \\ 0 & 1 & 0 & -a_{13}a_{23}b_1 + a_{12}a_{33}b_1 + a_{13}a_{32}b_2 - a_{12}a_{32}b_2 - a_{13}a_{21}b_3 + a_{12}a_{21}b_3 \\ 0 & 0 & 1 & a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} \\ 0 & 0 & 0 & -a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} \end{pmatrix}
\]

If we look at the denominators or the right hand column we see they are all the same and that in fact they are the determinant of \( A \). The numerators are more mysterious, but they do have the look of "the determinant of something". If we think about this for a while—perhaps days not hours—we would likely discover Cramer’s rule for this case.
2 Cramer’s Rule

To make a long story short, here is a final result of the above line of calculation.

**Important:** Cramer’s Rule is only for solving \( n \) equations with \( n \) unknowns.

Notation: If \( A \overrightarrow{x} = \overrightarrow{b} \) is an \( n \times n \) matrix, let \( A_i(\overrightarrow{b}) \) denote the matrix obtained by removing the \( i \)-th column of \( A \) and replacing it with \( \overrightarrow{b} \).

**Example 2.1** Suppose \( A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix} \) and \( \overrightarrow{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) then \( A_2(\overrightarrow{b}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix} \).

**Proposition 2.2** (Cramer’s Rule) If \( A \) is an \( n \times n \) invertible matrix, then for any \( \overrightarrow{b} \) the unique solution
of $A \vec{x} = \vec{b}$ is the the vector 
\[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_i \\
\vdots \\
x_n
\end{pmatrix}
\]
where
\[
x_i = \frac{\det A_i(\vec{b})}{\det A}.
\]

**Example 2.3** So using the matrix from the above
\[
A = \begin{pmatrix}
1 & 2 & 1 \\
0 & -1 & 1 \\
2 & 3 & -2
\end{pmatrix}
\]
and $\vec{b} = \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}$ and writing $\vec{x} = \begin{pmatrix}
x \\
y \\
z
\end{pmatrix}$.

We first calculate $\det A = 5$ thus $A$ is is invertible, so Cramer’s Rule applies. Next we calculate
\[
\det A_1(\vec{b}) = \det \begin{pmatrix}
1 & 2 & 1 \\
2 & -1 & 1 \\
3 & 3 & -2
\end{pmatrix} = 22
\]
\[
\det A_2(\vec{b}) = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix} = -9
\]

\[
\det A_3(\vec{b}) = \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 2 & 3 & 3 \end{pmatrix} = 1
\]

And so the unique solution to our equation is
\[
x = \frac{22}{5}, \ y = -\frac{9}{5} \text{ and } z = \frac{1}{5}
\]

**Remark 2.4** There is a faster way to do this that we already know:

If we row reduce the augmented matrix

\[
\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 2 & 3 & -2 & 3 \end{pmatrix}
\]

we get:

\[
\begin{pmatrix} 1 & 0 & 0 & \frac{22}{5} \\ 0 & 1 & 0 & -\frac{9}{5} \\ 0 & 0 & 1 & \frac{1}{5} \end{pmatrix}
\]
However Cramer’s Rule is useful for other things since it gives an explicit formula for the solution of a set of equations (if it has one).

3 A formula for an inverse of a $3 \times 3$ matrix

So if we want to calculate an inverse for a $3 \times 3$ matrix

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

we form the larger matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 1
\end{pmatrix}
\]

row reduce and get
Even with small type we cannot fit this onto our page so we look at the top row, columns of this matrix one at a time:

\[
\begin{array}{c}
a_{22}a_{33} - a_{23}a_{32} \\
-a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} \\
\frac{-a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33}}{a_{23}a_{31} - a_{21}a_{33}} \\
a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}
\end{array}
\]

\[
\begin{array}{c}
a_{13}a_{32} - a_{12}a_{33} \\
-a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} \\
\frac{-a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33}}{a_{13}a_{31} - a_{11}a_{33}} \\
a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}
\end{array}
\]

\[
\begin{array}{c}
a_{13}a_{22} - a_{12}a_{23} \\
-a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} \\
\frac{-a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33}}{a_{23}a_{31} - a_{21}a_{33}} \\
a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}
\end{array}
\]

We can see that the denominators are the determinant and the negative of the determinant. The numerators appear to be determinants of some $2 \times 2$ matrix.

It would take a while to puzzle the pattern out, so here is the solution—not just for this case but also the $n \times n$ case.
First we recall a definition from Section 1 of Chapter 3:

Given a square matrix $A$, the $(i, j)$-cofactor of $A$ is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

**Definition 3.1** Given an $n \times n$ matrix $A$ the classical adjoint or adjugate of $A$, denoted $\text{adj} A$ is the matrix whose entry in the $ij$ position is $C_{ji}$.

**Remark 3.2** This definition is “tricky”. Note that the adjugate is not simply the matrix of cofactors.

Read the definition of adjugate carefully and note that in the $ij$ position is $C_{ji}$ (which is not the same as $C_{ij}$). We will see this most clearly when we work an example.

**Proposition 3.3** If $A$ is a square matrix then

$$A^{-1} = \frac{1}{\det A} \text{adj} A.$$  

**Example 3.4** Lets use this formula to find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix}$. This is the matrix used in Example 2.3. We have calculated that $\det A = 5$.  

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We begin to calculate $\text{adj} A$.

$$C_{11} = + \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} = (-1)(-2) - (1)3 = -1$$

Be careful on the next one and watch your $i$ and $j$.

$$C_{21} = + \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = (2)(-2) - (1)3 = -7$$

Continuing down the first row for $\text{adj} A$:

$$C_{31} = + \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = (2)(1) - (1)(-1) = 3$$

NOTE in this calculation the top entries of the top row are $C_{11}, C_{21},$ and $C_{31}$ If we complete the calculation we get

$$\text{adj} A = \begin{pmatrix} -1 & 7 & 3 \\ 2 & -4 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

And so we get

$$A^{-1} = \frac{1}{5} \begin{pmatrix} -1 & 7 & 3 \\ 2 & -4 & -1 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{7}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{4}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \end{pmatrix}.$$
4 What is the meaning of the value of the determinant if it is not zero?

We first consider a $2 \times 2$ matrix. It has two column vectors and these are vectors in the plane. We can view these as adjacent edges of a parallelogram $P$ IF they are linearly independent.

Next consider a $3 \times 3$ matrix. It has three column vectors and these are vectors in space. We can view these as three edges of a parallelogram $P$ which meet at a common corner of $P$ IF they are linearly independent.

**Proposition 4.1** If $A$ is a $2 \times 2$ matrix then $| \det A |$ is the area of the parallelogram determined by the column vectors of $A$.

*If $A$ is a $3 \times 3$ matrix then $| \det A |$ is the area of the parallelepiped determined by the column vectors of $A$.*

**Proposition 4.2** Let $T : R^2 \rightarrow R^2$ be a linear transformation determined by matrix $A$. If $S$ is a parallelogram in $R^2$ then:

$$\{ \text{area of } T(S) \} = | \det A | \{ \text{area of } S \}.$$
Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation determined by matrix $A$. If $S$ is a parallelepiped in $\mathbb{R}^3$ then:

$$\{\text{volume of } T(S)\} = |\det A|\{\text{volume of } S\}.$$

**Remark 4.3** If $A$ is not invertible then $\det A = 0$