22m:033 Notes:
2.9 Dimension and Rank

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1 Coordinate systems

Because of linear independence provision of a basis, it follows that if $B = \{\overrightarrow{b_1}, \ldots, \overrightarrow{b_p}\}$ is a basis for a subspace H that every vector $\overrightarrow{x}$ in H can be written *uniquely* as a linear combination of the basis vectors

$$\overrightarrow{x} = c_1\overrightarrow{b_1} + \cdots + c_p\overrightarrow{b_p}.$$

Example 1.1 Suppose $B = \{\overrightarrow{b_1}, \overrightarrow{b_2}\}$ is a basis and we have two ways of writing

$$\begin{aligned} \overrightarrow{v} &= c_1\overrightarrow{b_1} + c_2\overrightarrow{b_2} \\ \overrightarrow{v} &= k_1\overrightarrow{b_1} + k_2\overrightarrow{b_2} \end{aligned}$$

Then

$$\begin{aligned} c_1\overrightarrow{b_1} + c_2\overrightarrow{b_2} &= k_1\overrightarrow{b_1} + k_2\overrightarrow{b_2} \text{ or} \\ (c_1 - k_1)\overrightarrow{b_1} + (c_2 - k_2)\overrightarrow{b_2} &= \overrightarrow{0} \end{aligned}$$

By definition of linear independence we must have $(c_1 - k_1) = 0$ and $(c_2 - k_2) = 0$, so $c_1 = k_1$ and $c_2 = k_2$ ∎

Definition 1.2 *The column matrix* $\begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}$ *is called* *the* ***coordinate vector of*** $\overrightarrow{x}$ ***with respect to basis*** *B and denoted by* $(\overrightarrow{x})_B$

2 Dimension of a subspace

Proposition 2.1 *If H is a subspace of* R^n *with one basis consisting of p vectors, then every basis for H also consists of p vectors.* ∎

This allows us to make the following important definitions:

Definition 2.2 *The* ***dimension*** *of any non-zero subspace H of* R^n *is the number of vectors in any basis for H. The dimension of a zero subspace is defined to be 0.*

Definition 2.3 *The* ***rank*** *of a matrix A is the dimension of the column space.*

Proposition 2.4 *(The Rank Theorem) If A is a matrix with n columns then*

$$rankA + dimNulA = n.$$

∎

Proposition 2.5 *(The Basis Theorem) If H is p-dimensional subspace of R^n, any linearly independent set of p vectors of H will be a basis for H.*

Also any set of p vectors of H that span H are a basis for H. ∎

3 Yet more on inverses

Proposition 3.1 *(The Invertible Matrix Theorem Continued) Let A be an $n \times n$ matrix then the following are equivalent to the statement that A is an invertible matrix:*

1. *the columns of A form a basis for R^n*
2. $ColA = R^n$
3. $dimColA = n$
4. $rankA = n$
5. $NulA = \{\overrightarrow{0}\}$
6. $dimNulA = 0$

4 The Rank Theorem and linear transformations

The contents of much of the last two sections have some important interpretations in terms of linear transformation.

Remark 4.1 Consider a linear transformation $T : R^n \rightarrow R^m$ given by $T(\overrightarrow{x}) = A\overrightarrow{x}$. The column space of A is the same as the range(image) of T. A basis for the column space is therefore a basis for the range of T.

Example 4.2 Suppose

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is already in reduced echelon form and we see that rank of A is 2.

We could describe the transformation T by $T(x, y, z) = (x, y, 0)$. Here $T : R^3 \rightarrow R^3$. It is clear that the range P of T is two dimensional. It is the xy-coordinate plane in R^3, namely the plane with equation $z = 0$.

The dimension of the column space of A is 2. It is easy to calculate the null space of A. It is the z-axis which has vector equation $t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Given any point $\overrightarrow{b} = (x, y, 0) \in P$, $T^{-1}(\overrightarrow{b})$ is the line $L_{\overrightarrow{b}}$ with vector equation

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

.

So we can say the the map T maps R^3 onto P by "collapsing each line $L_{\overrightarrow{b}}$ to a point—namely $\overrightarrow{b}$.

Example 4.3 Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Here $T : R^3 \to R^3$.

We get A into echelon form. First add -4 times row 1 to row 2, then add -7 times row 1 to row 3 and we get:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

Finally multiply row 2 by 2 and add to row 3 and we end up with:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

From this we conclude that the column space has basis

$$B = \{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \}$$

So the range of T is two dimensional (that is a plane); lets call it P. In particular T is not an onto map.

Now let us turn our attention to the null space of A. To get this we continue and get the row reduced form of A which is:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Writing the solutions of $A(\overrightarrow{x}) = \overrightarrow{0}$ in vector parametric form we see that the solutions are:

$$t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

. So the null space is a line N. This null space can be thought of as $T^{-1}(\overrightarrow{0})$.

We can say that what T does is "collapse this line N to the point $\overrightarrow{0}$.

What about any other point $\overrightarrow{b} \in P$. What can we say about $T^{-1}(\overrightarrow{b})$? Equivalently $T^{-1}(\overrightarrow{b})$ is the solution set of the non-homogeneous equations $A\overrightarrow{x} = \overrightarrow{b}$. As we have learned this will be a line in R^3 parallel to N. So we can say that T collapses lines parallel to N to points of P.