22m:033 Notes:
2.3 Characterizations of invertible matrices

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1 Definition of Inverse

The following is a version of Theorem 8 of the text:

Proposition 1.1 *Suppose A is an $n \times n$ matrix. Then the following statements are equivalent:*

1. *A is an invertible matrix*
2. *A is row equivalent to I_n*
3. *The equation $A\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution*
4. *The equation $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution for any $\overrightarrow{b} \in R^n$*
5. *The columns of A form a set of linearly independent vectors in R^n*
6. *The columns of A span R^n*
7. *The linear transformation $T(\overrightarrow{x}) = A\overrightarrow{x}$ is one-to-one*
8. *The linear transformation $T(\overrightarrow{x}) = A\overrightarrow{x}$ maps R^n onto R^n*
9. *There is an $n \times n$ matrix C such that $CA = I_n$*
10. *There is an $n \times n$ matrix D such that $AD = I_n$*
11. *A^T is an invertible matrix*

Remark 1.2 Parts 5 and 6 of Proposition 1.1 say roughly that you really only need half of our definition for inverse, since one part can be deduced from the other.

Remark 1.3 Here is another way to format Proposition 1.1. Note that items in part 3 of Proposition 1.4 basically relate to sets of **homogeneous equations** while items in part 4 of Proposition 1.4 basically relate to sets of **non-homogeneous equations**:

Proposition 1.4 *Suppose A is an $n \times n$ matrix. Then the following statements are equivalent:*

1. *A is an invertible matrix*
2. *A is row equivalent to I_n*
3. (a) *The equation $A\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution*
   (b) *The columns of A form a set of linearly independent vectors in R^n*
   (c) *The linear transformation $T(\overrightarrow{x}) = A\overrightarrow{x}$ is one-to-one*
4. (a) *The equation $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution for any $\overrightarrow{b} \in R^n$*
   (b) *The columns of A span R^n*
   (c) *The linear transformation $T(\overrightarrow{x}) = A\overrightarrow{x}$ maps R^n onto R^n*
5. *There is an $n \times n$ matrix C such that $CA = I_n$*
6. *There is an $n \times n$ matrix D such that $AD = I_n$*
7. *A^T is an invertible matrix*

Remark 1.5 It is important to note that this theorem only tells us about square matrices, n equations with n unknowns etc.

It gives **absolutely no information** on other situations of non-square matrices, n equations with k unknowns etc.

2 Invertible linear transformations

Recall the notion of inverse of a function. We will see that linear functions have inverse functions if (and only if) the corresponding matrix is invertible.

Definition 2.1 *A linear transformation $T\colon R^n \to R^n$ is called **invertible** if there is a function $S\colon R^n \to R^n$ such that:*

$$S(T(\overrightarrow{x})) = \overrightarrow{x} \textit{ for all } \overrightarrow{x} \in R^n$$

and

$$T(S(\overrightarrow{x})) = \overrightarrow{x} \textit{ for all } \overrightarrow{x} \in R^n$$

*If there is such an S it is called the **inverse** of T.* ∎

This is Theorem 9 of text:

Proposition 2.2 *Suppose linear transformation* $T\colon R^n \to R^n$ *has matrix* A*. Then* T *is invertible if and only if* A *is invertible.*

Furthermore the inverse of T *is also a linear transformation with matrix* A^{-1}*.* ∎