22m:033 Notes:

2.2 Inverse of a Matrix

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March 3, 2010
1 Definition of Inverse

Definition 1.1 An $n \times n$ matrix $A$ is invertible if there exists an $n \times n$ matrix $C$ such that

$$AC = I_n \text{ and } CA = I_n.$$ 

If $C$ exists it is called the inverse of $A$ and denoted by $A^{-1}$.

Remark 1.2 If an inverse exists, it is unique.

With our notation we can express the equations in Definition 1.1 as:

$$AA^{-1} = I_n \text{ and } A^{-1}A = I_n.$$ 

A matrix that is not invertible is called a singular matrix.

A matrix that is invertible is called a nonsingular matrix.

Example 1.3 We can check that the inverse of

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$ 

is

$$\begin{pmatrix} -2 & 1 \\ 3 & -\frac{1}{2} \end{pmatrix}$$
by multiplying the two matrices.

For a $2 \times 2$ matrix we can easily explain invertibility (Theorem 4 of text):

**Proposition 1.4** If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A$ is invertible if and only if $ad - bc \neq 0$. In fact if $ad - bc \neq 0$

\[
A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

The number $ad - bc$ is called the **determinant of $A$**.

Note that we could have calculated the inverse in Example 1.3 using these formulas. Note that the determinant of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is -2.

**Remark 1.5** **WARNING:** The simple formula for the inverse of a $2 \times 2$ matrix only works for a $2 \times 2$ matrix. We will find an extension of this formula later in the course but it is not simple. In fact find better ways of calculating inverses in this section.
2 Using inverse to solve equations

This is Theorem 5 in text:

**Proposition 2.1** If $A$ is an $n \times n$ invertible matrix then for any $\vec{b} \in \mathbb{R}^n$ the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

3 Properties of inverse

These are listed in Theorem 6 in text

**Proposition 3.1**  
- If $A$ is invertible then so is $A^{-1}$ and $(A^{-1})^{-1} = A$
- If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is also invertible and
  $$(AB)^{-1} = B^{-1}A^{-1}$$
- If $A$ is invertible then so is $A^T$ and
  $$(A^T)^{-1} = (A^{-1})^T$$
4 Elementary Matrices

An elementary matrix allows us to express the process of a row operation by matrix multiplication.

**Definition 4.1** An elementary matrix is one obtained from an identity matrix by performing a single row operation.

**Proposition 4.2** If we obtain $B$ from $A$ by a single row operation, and if $E$ is the elementary matrix corresponding to this row operation. Then $B = EA$

**Example 4.3** Suppose we take $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and add twice the first row to the second row. We then get $B = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \end{pmatrix}$.

The corresponding elementary operation is $E = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and we can check that $B = EA$.

This is Theorem 7 of the text
Proposition 4.4 An $n \times n$ matrix $A$ is invertible if and only if it is row equivalent to $I_n$. Also any sequence of row operations that reduces $A$ to $I_n$ also transforms $I_n$ to $A^{-1}$.

Remark 4.5 For the simple case that one row operation reduces $A$ to $I_n$ then it also transforms $I_n$ to $A^{-1}$. Let $E$ the the corresponding elementary matrix. Then

$$EA = I$$

Take the inverse of both sides and use the fact that $I^{-1} = I$ we get

$$A^{-1}E^{-1} = (EA)^{-1} = I^{-1} = I$$

So

$$(A^{-1}E^{-1})E = IE$$

or

$$A^{-1} = A^{-1}I = (A^{-1}(E^{-1})E) = IE = E \blacksquare$$

5 Finding inverses, if they exist

Remark 5.1 Take a square matrix $A$ and make an augmented matrix $M = (AI)$. Row reduce $M$. 

IF $M$ has the form $M = (IB)$, then $A$ does have an inverse and $A^{-1} = B$. IF $M$ ends up with a row of zeros then $A$ does not have an inverse.

**Example 5.2** We can calculate that the inverse of
\[
\begin{pmatrix}
1 & -1 & 1 \\
1 & 2 & 3 \\
3 & -2 & 1
\end{pmatrix}
\]
is
\[
\begin{pmatrix}
-1 & \frac{1}{8} & \frac{5}{8} \\
-1 & \frac{1}{4} & \frac{1}{4} \\
1 & \frac{1}{8} & -\frac{3}{8}
\end{pmatrix}
\] since when we row reduce
\[
\begin{pmatrix}
1 & -1 & 1 & 1 & 0 & 0 \\
1 & 2 & 3 & 0 & 1 & 0 \\
3 & -2 & 1 & 0 & 0 & 1
\end{pmatrix}
\]
we get
\[
\begin{pmatrix}
1 & 0 & 0 & -1 & \frac{1}{8} & \frac{5}{8} \\
0 & 1 & 0 & -1 & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 1 & 1 & \frac{1}{8} & -\frac{3}{8}
\end{pmatrix}
\] .

**Example 5.3** On the other hand we deduce that
\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]
has no inverse since when we row reduce
\[
\begin{pmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
4 & 5 & 6 & 0 & 1 & 0 \\
7 & 8 & 9 & 0 & 0 & 1
\end{pmatrix}
\]
we get

$$\begin{pmatrix}
1 & 0 & -1 & 0 & -\frac{8}{3} & \frac{5}{3} \\
0 & 1 & 2 & 0 & \frac{7}{3} & -\frac{4}{3} \\
0 & 0 & 0 & 1 & -2 & 1
\end{pmatrix}$$

6 Problems

**Question 6.1** Let $M_\theta = \begin{pmatrix} \cos \theta & \sin(-\theta) \\
\sin \theta & \cos \theta \end{pmatrix}$. Show, using Definition 1.1 that $M_\theta^{-1} = M_{-\theta}$.

**Question 6.2** Following the argument of Remark 4.5 show that For the simple case that two row operations reduces $A$ to $I_n$ then they also transform $I_n$ to $A^{-1}$.

**Question 6.3** Use the method of Remark 5.1 to derive the formula for the inverse given in Proposition 1.4.