22m:033 Notes:
2.1 Matrix Operations

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1 Sums and scalar multiples

In a sense, a vector is a matrix with one column.

We can add vectors and multiply by a scalar.

We can extend these operations to matrices in the "obvious way".

NOTATION: For a matrix M, the entry in the i-th row and j-th column is denoted by m_{ij}

Example 1.1 If $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $m_{12} = 2, m_{21} = 4, m_{23} = 6$ but there is no such thing as m_{32}.

Definition 1.2 *If A and B are two $m \times n$ matrices, $M = A + B$ is the matrix so that $m_{ij} = a_{ij} + b_{ij}$*

If A is a matrix and c is a number then $M = cA$ the matrix so that $m_{ij} = ca_{ij}$. ∎

Example 1.3

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 5 & 5 & 8 \end{pmatrix}$$

$$2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ is not defined.} \quad \blacksquare$$

Definition 1.4 *An $m \times n$ matrix M with $m_{ij} = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$ is called the $m \times n$* ***zero matrix****.*

Definition 1.5 *An $n \times n$ matrix M with $m_{ii} = 1$ for all $0 \leq i \leq n$ and $m_{ij} = 0$ if $i \neq j$ is called the $n \times n$* ***identity matrix****. This is denoted by I_n.*

Example 1.6 So $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. $\blacksquare$

Definition 1.7 *For any matrix A, we define $-A = (-1)A$.*

The properties of scalar multiplication for matrices, and addition of matrices (See Theorem 1 page 108) satisfies all the rules we established for vectors which said:

Proposition 1.8 *Suppose $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ are vectors in R^n and c, d are numbers.*

1. $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$
2. $(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$
3. $\overrightarrow{u} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{u} = \overrightarrow{u}$
4. $\overrightarrow{u} + (-\overrightarrow{u}) = -\overrightarrow{u} + \overrightarrow{u} = \overrightarrow{0}$
5. $c(\overrightarrow{u} + \overrightarrow{v}) = c\overrightarrow{u} + c\overrightarrow{v}$
6. $(c+d)\overrightarrow{u} = c\overrightarrow{u} + d\overrightarrow{u}$
7. $c(d\overrightarrow{u}) = (cd)\overrightarrow{u}$
8. $1\overrightarrow{u} = \overrightarrow{u}$

We just replace this using matricides (of appropriate size) instead of these particular (column) matrices:

Proposition 1.9 *Suppose A, B, C are $n \times m$ matrices and c, d are numbers and 0 denotes the $n \times m$ zero matrix.*

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = 0 + A = A$
4. $A + (-A) = -A + A = 0$
5. $c(A + B) = cA + cB$
6. $(c + d)A = cA + dA$
7. $1A = A$

2 Matrix multiplication

Here is when things get interesting ....

Definition 2.1 *Suppose A is a $m \times n$ matrix and B is an $n \times p$ matrix. We can define a product $M = AB$ by*

$$m_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{in}b_{nj}. \quad \blacksquare$$

Remark 2.2 Note that this formula for the computation of m_{ij} involves the i-th row on the left and the j-th row on the right. ∎

Example 2.3

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 16 & 3 \end{pmatrix}$$

On the other hand note that

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 4 & 5 & 6 \\ 6 & 9 & 12 \end{pmatrix}$$

∎

Example 2.4 IN THE SPECIAL CASE OF SQUARE MATRICES WE CAN MULTIPLY IN ANY ORDER BUT GENERALLY THE ORDER MAKES A DIFFERENCE:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$$

but

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$$

In other words, for square $n \times n$ matrices A and B , we **generally** have

$$AB \neq BA \quad \blacksquare$$

3 Matrix multiplication and composition of linear transformations

In the section on linear transformations, we noted that if

$$T \colon R^n \to R^m$$

is a linear transformation and

$$S\colon R^m \to R^k$$

, is a linear transformation, then the composite

$$T \circ S\colon R^n \to R^k$$

is a linear transformation.

We also mentioned the important fact that every linear transformation corresponds to a matrix.

The statement and explanation of the following Proposition is found on page 109 and to top of page 110 of our text.

Proposition 3.1 *If A is the matrix of T and if B is the matrix for S then the matrix of $T \circ S$ is the matrix product AB.*

Remark 3.2 Recall some facts about compositions of ordinary real valued functions $f(x)$ and $g(x)$.

First of all the composition is not automatically defined—we need a match up between image of f and the domain of g.

For example if $f(x) = \sqrt{x}$ and $g(x) = \sin x$, $f \circ g(x) = \sqrt{\sin x}$ is not defined for many values of x such as $(\pi, 2\pi)$, $(3\pi, 4\pi)$ since the square root of a negative number is not a real number.

But more important, we do not in general have $f \circ g = g \circ f$. In our example we see that the functions $\sqrt{\sin x}$ and $\sin \sqrt{x}$ are not the same.

Similarly we do not generally have $T \circ S$ the same transformation as $S \circ T$ and consequently we should expect the matrix products AB and BA not to be equal. ∎

4 Properties of Matrix Multiplication

Proposition 4.1 *For matrices A, B, C for the proper size that which multiplication as shown is defined:*

$$
\begin{aligned}
A(BC) &= (AB)C \\
A(B+C) &= AB+AC \\
(B+C)A &= BA+CA \\
r(AB) &= (rA)B \text{ for any scalar } r \\
r(AB) &= A(rB) \text{ for any scalar } r \\
I_m A &= A I_n
\end{aligned}
$$

5 Powers of a Matrix

Definition 5.1 *If A is $n \times n$ matrix then*

$A^0 = I_n$, $A^2 = AA$, and generally A^k is obtained by multiplying together k copies of A. ∎

6 Transpose of a matrix

Definition 6.1 *If M is an $m \times n$ matrix then the transpose $X = M^T$ of M is the matrix produced by switching rows and columns. Specifically, $x_{ij} = m_{ji}$*

Example 6.2 The transpose of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

The basic properties of the transpose for matrices A, B, C of appropriate size and scalar r. Pay close attention to the order of multiplication in the last equation.

Proposition 6.3

$$\begin{aligned}
(A^T)^T &= A \\
(A+B)^T &= A^T + B^T \\
(rA)^T &= r(A^T) \\
(AB)^T &= B^T A^T
\end{aligned}$$

7 Problems for homework

Question 7.1 Let

$$M = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$$

Calculate:

$$MN, NM, M - N, M - I_3, M - N^T$$

Question 7.2 An important example of a linear transformation of the plane is rotation about the origin. This is explained on page 84 of text, a section we skip over.

IMPORTANT FACT: For a fixed θ consider the matrix

$$M_\theta = \begin{pmatrix} \cos\theta & \sin(-\theta) \\ \sin\theta & \cos\theta \end{pmatrix}$$

The linear transformation corresponding to M_θ is counterclockwise rotation of angle θ

Verify the following for any angles α and β:

1. $M_\alpha M_\beta = M_\beta M_\alpha$
2. $M_\alpha M_\beta = M_{\alpha+\beta}$
3. Explain these two equations in terms of the corresponding transformations.

Hint: I see trigonometric identities in your future.