2.1 Matrix Operations

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/~roseman

February 10, 2010
1 Sums and scalar multiples

In a sense, a vector is a matrix with one column.

We can add vectors and multiply by a scalar.

We can extend these operations to matrices in the ”obvious way”.

NOTATION: For a matrix $M$, the entry in the $i$-th row and $j$-th column is denoted by $m_{ij}$

**Example 1.1** If $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $m_{12} = 2, m_{21} = 4, m_{23} = 6$ but there is no such thing as $m_{32}$.

**Definition 1.2** If $A$ and $B$ are two $m \times n$ matrices, $M = A + B$ is the matrix so that $m_{ij} = a_{ij} + b_{ij}$

If $A$ is a matrix and $c$ is a number then $M = cA$ the matrix so that $m_{ij} = ca_{ij}$.

**Example 1.3**

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 5 & 5 & 8 \end{pmatrix}$$
\[
2 \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{pmatrix} = \begin{pmatrix}
2 & 4 & 6 \\
8 & 10 & 12 \\
\end{pmatrix}
\]

and
\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{pmatrix} + \begin{pmatrix}
1 & 2 \\
3 & 4 \\
\end{pmatrix}
\]
is not defined.

**Definition 1.4** An \(m \times n\) matrix \(M\) with \(m_{ij} = 0\) for all \(0 \leq i \leq m\) and \(0 \leq j \leq n\) is called the \(m \times n\) **zero matrix**.

**Definition 1.5** An \(n \times n\) matrix \(M\) with \(m_{ii} = 1\) for all \(0 \leq i \leq n\) and \(m_{ij} = 0\) if \(i \neq j\) is called the \(n \times n\) **identity matrix**. This is denoted by \(I_n\).

**Example 1.6** So \(I_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}\).

**Definition 1.7** For any matrix \(A\), we define \(-A = (-1)A\).

The properties of scalar multiplication for matrices,
and addition of matrices (See Theorem 1 page 108) satisfies all the rules we established for vectors which said:

**Proposition 1.8** Suppose $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$ are vectors in $R^n$ and $c, d$ are numbers.

1. $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$
2. $(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$
3. $\overrightarrow{u} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{u} = \overrightarrow{u}$
4. $\overrightarrow{u} + (-\overrightarrow{u}) = -\overrightarrow{u} + \overrightarrow{u} = \overrightarrow{0}$
5. $c(\overrightarrow{u} + \overrightarrow{v}) = c\overrightarrow{u} + c\overrightarrow{v}$
6. $(c + d)\overrightarrow{u} = c\overrightarrow{u} + d\overrightarrow{u}$
7. $c(d\overrightarrow{u}) = (cd)\overrightarrow{u}$
8. $1\overrightarrow{u} = \overrightarrow{u}$

We just replace this using matricides (of appropriate size) instead of these particular (column) matrices:

**Proposition 1.9** Suppose $A, B, C$ are $n \times m$ matrices and $c, d$ are numbers and $0$ denotes the $n \times m$ zero matrix.
1. \( A + B = B + A \)

2. \( (A + B) + C = A + (B + C) \)

3. \( A + 0 = 0 + A = A \)

4. \( A + (-A) = -A + A = 0 \)

5. \( c(A + B) = cA + cB \)

6. \( (c + d)A = cA + dA \)

7. \( 1A = A \)

2 Matrix multiplication

Here is when things get interesting . . . .

Definition 2.1 Suppose \( A \) is a \( m \times n \) matrix and \( B \) is an \( n \times p \) matrix. We can define a product \( M = AB \) by

\[
m_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.
\]

Remark 2.2 Note that this formula for the computation of \( m_{ij} \) involves the \( i \)-th row on the left and the \( j \)-th row on the right.
Example 2.3

\[
\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 16 & 3 \end{pmatrix}
\]

On the other hand note that

\[
\begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 4 & 5 & 6 \\ 6 & 9 & 12 \end{pmatrix}
\]

\[\blacksquare\]
Example 2.4 In the special case of square matrices we can multiply in any order but generally the order makes a difference:

\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = 
\begin{pmatrix}
-2 & 1 \\
-4 & 3
\end{pmatrix}
\]

but

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} = 
\begin{pmatrix}
3 & 4 \\
-1 & -2
\end{pmatrix}
\]

In other words, for square \(n \times n\) matrices \(A\) and \(B\), we generally have

\[AB \neq BA\]

3 Matrix multiplication and composition of linear transformations

In the section on linear transformations, we noted that if \(T: \mathbb{R}^n \rightarrow \mathbb{R}^m\)
is a linear transformation and

\[ S: \mathbb{R}^m \rightarrow \mathbb{R}^k \]

is a linear transformation, then the composite

\[ T \circ S: \mathbb{R}^n \rightarrow \mathbb{R}^k \]

is a linear transformation.

We also mentioned the important fact that every linear transformation corresponds to a matrix.

The statement and explanation of the following Proposition is found on page 109 and to top of page 110 of our text.

**Proposition 3.1** If \( A \) is the matrix of \( T \) and if \( B \) is the matrix for \( S \) then the matrix of \( T \circ S \) is the matrix product \( AB \).

**Remark 3.2** Recall some facts about compositions of ordinary real valued functions \( f(x) \) and \( g(x) \).

First of all the composition is not automatically defined—we need a match up between image of \( f \) and the domain of \( g \).
For example if \( f(x) = \sqrt{x} \) and \( g(x) = \sin x \), \( f \circ g(x) = \sqrt{\sin x} \) is not defined for many values of \( x \) such as \((\pi, 2\pi), (3\pi, 4\pi)\) since the square root of a negative number is not a real number.

But more important, we do not in general have \( f \circ g = g \circ f \). In our example we see that the functions \( \sqrt{\sin x} \) and \( \sin \sqrt{x} \) are not the same.

Similarly we do not generally have \( T \circ S \) the same transformation as \( S \circ T \) and consequently we should expect the matrix products \( AB \) and \( BA \) not to be equal.

4 Properties of Matrix Multiplication

**Proposition 4.1** For matrices \( A, B, C \) for the proper size that which multiplication as shown is defined:
\[ A(BC) = (AB)C \]
\[ A(B + C) = AB + AC \]
\[ (B + C)A = BA + CA \]
\[ r(AB) = (rA)B \text{ for any scalar } r \]
\[ r(AB) = A(rB) \text{ for any scalar } r \]
\[ I_mA = AI_n \]

5 Powers of a Matrix

**Definition 5.1** If \( A \) is \( n \times n \) matrix then

\[ A^0 = I_n, \ A^2 = AA, \text{ and generally } A^k \text{ is obtained by multiplying together } k \text{ copies of } A. \]

6 Transpose of a matrix

**Definition 6.1** If \( M \) is an \( m \times n \) matrix then the transpose \( X = M^T \) of \( M \) is the matrix produced by switching rows and columns. Specifically, \( x_{ij} = m_{ji} \)
Example 6.2 The transpose of \(
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\) is \(
\begin{pmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{pmatrix}
\)

The basic properties of the transpose for matrices \(A, B, C\) of appropriate size and scalar \(r\). Pay close attention to the order of multiplication in the last equation.

Proposition 6.3

\[
(A^T)^T = A \\
(A + B)^T = A^T + B^T \\
(rA)^T = r(A^T) \\
(AB)^T = B^T A^T
\]

7 Problems for homework

Question 7.1 Let

\[
M = \begin{pmatrix}
1 & 2 & -1 \\
-3 & 0 & 1 \\
0 & 1 & 3
\end{pmatrix}
\]
and

\[ N = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \]

Calculate:

\[ MN, NM, M - N, M - I_3, M - N^T \]

**Question 7.2** An important example of a linear transformation of the plane is rotation about the origin. This is explained on page 84 of text, a section we skip over.

**IMPORTANT FACT:** For a fixed \( \theta \) consider the matrix

\[ M_\theta = \begin{pmatrix} \cos \theta & \sin(-\theta) \\ \sin \theta & \cos \theta \end{pmatrix} \]

The linear transformation corresponding to \( M_\theta \) is counterclockwise rotation of angle \( \theta \)

Verify the following for any angles \( \alpha \) and \( \beta \):

1. \( M_\alpha M_\beta = M_\beta M_\alpha \)
2. \( M_\alpha M_\beta = M_{\alpha + \beta} \)
3. Explain these two equations in terms of the corresponding transformations.
Hint: I see trigonometric identities in your future.