Chapter 3
Discrete Random Variables and Probability Distributions

Part 1: Discrete Random Variables

Section 2.8 Random Variables
Section 3.1 Discrete Random Variables
Section 3.2 Probability Distributions and Probability Mass Functions
Section 3.3 Cumulative Distribution Functions
Consider tossing a coin two times. We can think of the following ordered sample space: \( S = \{(T, T), (T, H), (H, T), (H, H)\} \)

NOTE: for a fair coin, each of these are equally likely.

The outcome of a random experiment need not be a number, but we are often interested in some (numerical) measurement of the outcome.

For example, the **Number of Heads** obtained is numeric in nature can be 0, 1, or 2 and is a **random variable**.

**Definition (Random Variable)**

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.
Random Variables

Definition (Random Variable)

A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.

Example (Random Variable)

For a fair coin flipped twice, the probability of each of the possible values for Number of Heads can be tabulated as shown:

<table>
<thead>
<tr>
<th>SampleSpace</th>
<th>Number of Heads</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H,H)</td>
<td>2</td>
</tr>
<tr>
<td>(H,T)</td>
<td>1</td>
</tr>
<tr>
<td>(T,H)</td>
<td>0</td>
</tr>
<tr>
<td>(T,T)</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $X \equiv \#$ of heads observed. $X$ is a random variable.
Discrete Random Variables

Definition (Discrete Random Variable)

A **discrete** random variable is a variable which can only take-on a **countable** number of values (finite or countably infinite).

Example (Discrete Random Variable)

- Flipping a coin twice, the random variable **Number of Heads** \( \in \{0, 1, 2\} \) is a discrete random variable.
- Number of flaws found on a randomly chosen part \( \in \{0, 1, 2, \ldots\} \).
- Proportion of defects among 100 tested parts \( \in \{0/100, 1/100, \ldots, 100/100\}\).
- Weight measured to the nearest pound.*

*Because the possible values are discrete and countable, this random variable is discrete, but it might be a more convenient, simple approximation to assume that the measurements are values on a continuous random variable as ‘weight’ is theoretically continuous.
Definition (Continuous Random Variable)

A **continuous random variable** is a random variable with an interval (either finite or infinite) of real numbers for its range.

Example (Continuous Random Variable)

- Time of a reaction.
- Electrical current.
- Weight.
Discrete Random Variables

We often omit the discussion of the underlying sample space for a random experiment and directly describe the distribution of a particular random variable.

Example (Production of prosthetic legs)

Consider the experiment in which prosthetic legs are being assembled until a defect is produced. Stating the sample space...

\[ S = \{d, gd, ggd, gggd, \ldots\} \]

Let \( X \) be the trial number at which the experiment terminates (i.e. the sample at which the first defect is found).

The possible values for the random variable \( X \) are in the set \( \{1, 2, 3, \ldots\} \)

We may skip a formal description of the sample space \( S \) and move right into the random variable of interest \( X \).
Definition (Probability Distribution)

A **probability distribution** of a random variable $X$ is a description of the probabilities associated with the possible values of $X$.

Example (Number of heads)

Let $X \equiv \#$ of heads observed when a coin is flipped twice.

<table>
<thead>
<tr>
<th>Number of Heads</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>1/4</td>
<td>2/4</td>
<td>1/4</td>
</tr>
</tbody>
</table>

Probability distributions for discrete random variables are often given as a table or as a function of $X$...

Example (Probability defined by function $f(x)$)

Table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x) = f(x)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Function of $X$: $f(x) = \frac{1}{10}x$ for $x \in \{1, 2, 3, 4\}$
Example (Transmitted bits, example 3-4 p.68)

There is a chance that a bit transmitted through a digital transmission channel is received in error.

Let $X$ equal the number of bits in error in the next four bits transmitted. The possible values for $X$ are $\{0, 1, 2, 3, 4\}$.

Suppose that the probabilities are...

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.6561</td>
</tr>
<tr>
<td>1</td>
<td>0.2916</td>
</tr>
<tr>
<td>2</td>
<td>0.0486</td>
</tr>
<tr>
<td>3</td>
<td>0.0036</td>
</tr>
<tr>
<td>4</td>
<td>0.0001</td>
</tr>
</tbody>
</table>
Example (Transmitted bits, example 3-4 p.68, cont.)

The probability distribution shown graphically:

Notice that the sum of the probabilities of the possible random variable values is equal to 1.
Probability Mass Function (PMF)

**Definition (Probability Mass Function (PMF))**

For a **discrete** random variable $X$ with possible values $x_1, x_2, x_3, \ldots, x_n$, a **probability mass function** $f(x_i)$ is a function such that

1. $f(x_i) \geq 0$
2. $\sum_{i=1}^{n} f(x_i) = 1$
3. $f(x_i) = P(X = x_i)$

**Example (Probability Mass Function (PMF))**

For the transmitted bit example,

$f(0) = 0.6561$, $f(1) = 0.2916$, ..., $f(4) = 0.0001$

$\sum_{i=1}^{n} f(x_i) = 0.6561 + 0.2916 + \cdots + 0.0001 = 1$

The probability distribution for a **discrete random variable** is described with a **probability mass function** (probability distributions for continuous random variables will use different terminology).
Example (Probability Mass Function (PMF))

Toss a coin 3 times.

- Let $X$ be the number of heads tossed.
  Write down the probability mass function (PMF) for $X$:
  
  {Use a table...}

- Show the PMF graphically:
A box contains 7 balls numbered 1,2,3,4,5,6,7. Three balls are drawn at random and *without replacement*.

Let $X$ be the number of 2’s drawn in the experiment.

Write down the probability mass function (PMF) for $X$:

{Use your counting techniques}
Cumulative Distribution Function (CDF)

Sometimes it’s useful to quickly calculate a cumulative probability, or $P(X \leq x)$, denoted as $F(x)$, which is the probability that $X$ is less than or equal to some specific $x$.

Example (Widgets, PMF and CDF)

Let $X$ equal the number of widgets that are defective when 3 widgets are randomly chosen and observed. The possible values for $X$ are $\{0, 1, 2, 3\}$.

The probability mass function for $X$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$ or $f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.550</td>
</tr>
<tr>
<td>1</td>
<td>0.250</td>
</tr>
<tr>
<td>2</td>
<td>0.175</td>
</tr>
<tr>
<td>3</td>
<td>0.025</td>
</tr>
</tbody>
</table>

Suppose we’re interested in the probability of getting 2 or less errors (i.e. either 0, or 1, or 2). We wish to calculate $P(X \leq 2)$. 

Cumulative Distribution Function (CDF)

Example (Widgets, PMF and CDF, cont.)

\[ P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) \]
\[ = 0.550 + 0.250 + 0.175 = 0.975 \]

Below we see a table showing \( P(X \leq x) \) for each possible \( x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P(X = x) )</th>
<th>( P(X \leq x) = F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.550</td>
<td>0.550</td>
</tr>
<tr>
<td>1</td>
<td>0.250</td>
<td>0.800</td>
</tr>
<tr>
<td>2</td>
<td>0.175</td>
<td>0.975</td>
</tr>
<tr>
<td>3</td>
<td>0.025</td>
<td>1.000</td>
</tr>
</tbody>
</table>

\( P(X \leq 0) = F(0) \)
\( P(X \leq 1) = F(1) \)
\( P(X \leq 2) = F(2) \)
\( P(X \leq 3) = F(3) \)

As \( x \) increases across the possible values for \( x \), the cumulative probability increases, eventually getting 1, as you accumulate all the probability.
The cumulative probabilities are shown below as a function of $x$ or $F(x) = P(X \leq x)$.

The above cumulative distribution function $F(x)$ is associated with the probability mass function $f(x)$ below:
Connecting the PMF and the CDF

- We can get the PMF (i.e. the probabilities for $P(X = x_i)$) from the CDF by determining the height of the jumps.

- Specifically, because a CDF for a discrete random variable is a step-function with left-closed and right-open intervals, we have

$$P(X = x_i) = F(x_i) - \lim_{x \uparrow x_i} F(x_i)$$

and this expression calculates the difference between $F(x_i)$ and the limit as $x$ increases to $x_i$. 
Cumulative Distribution Function (CDF)

Definition (CDF for a discrete random variable)

The cumulative distribution function of a discrete random variable $X$, denoted as $F(x)$, is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

Definition (CDF for a discrete random variable)

For a discrete random variable $X$, $F(x)$ satisfies the following properties:

1. $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$
2. $0 \leq F(x) \leq 1$
3. If $x \leq y$, then $F(x) \leq F(y)$

- The CDF is defined on the real number line.
- The CDF is a non-decreasing function of $X$ (i.e. increases or stays constant as $x \to \infty$).
Cumulative Distribution Function (CDF)

- For each probability mass function (PMF), there is an associated CDF.
- If you’re given a CDF, you can come-up with the PMF and vice versa (know how to do this).
- Even if the random variable is discrete, the CDF is defined between the discrete values (i.e. you can state $P(X \leq x)$ for any $x \in \mathbb{R}$).
- The CDF ‘step function’ for a discrete random variable is composed of left-closed and right-open intervals with steps occurring at the values which have positive probability (or ‘mass’).
The cumulative distribution function $F(x)$ for a discrete random variable is a step-function.

Example (Widgets, PMF and CDF, cont.)

In the widget example, the range of $X$ is $\{0, 1, 2, 3\}$. There is no chance of a getting value outside of this set, e.g. $f(1.8) = P(X = 1.8) = 0$.

But $F(1.8) = P(X \leq 1.8) \neq 0$. Specifically...

$$F(1.8) = P(X \leq 1.8) = P(X \leq 1) = P(X = 0) + P(X = 1) = 0.800.$$  

So, if $f(x) = 0$, it does not necessarily mean $F(x) = 0$.

Here is $F(x)$ for the widget example:

$$F(x) = \begin{cases} 
0 & \text{if } x < 0 \\
0.550 & \text{if } 0 \leq x < 1 \\
0.800 & \text{if } 1 \leq x < 2 \\
0.975 & \text{if } 2 \leq x < 3 \\
1.0000 & \text{if } x \geq 3 
\end{cases}$$
Cumulative Distribution Function (CDF)

Example (Monitoring a chemical process)

The output of a chemical process is continually monitored to ensure that the concentration remains within acceptable limits. Whenever the concentration drifts outside the limits, the process is shut down and recalibrated. Let $X$ be the number of times in a given week that the process is recalibrated. The following table presents values of the cumulative distribution function $F(x)$ of $X$.

$F(x) = \begin{cases} 
0 & \text{if } x < 0 \\
0.17 & \text{if } 0 \leq x < 1 \\
0.53 & \text{if } 1 \leq x < 2 \\
0.84 & \text{if } 2 \leq x < 3 \\
0.97 & \text{if } 3 \leq x < 4 \\
1.0000 & \text{if } x \geq 4 
\end{cases}$

From the values in the far right column, I know that $X \in \{0, 1, 2, 3, 4\}$. 
Example (Monitoring a chemical process, cont.)

(1) Graph the cumulative distribution function.
Example (Monitoring a chemical process, cont.)

(2) What is the probability that the process is recalibrated fewer than 2 times during a week?

(3) What is the probability that the process is recalibrated more than three times during a week?
Example (Monitoring a chemical process, cont.)

(4) What is the probability mass function (PMF) for $X$?

(5) What is the most probable number of recalibrations in a week? (I’m not asking for an expected value here, just the one most likely).