6.3 Probabilities with Large Numbers

In general, we can't perfectly predict any single outcome when there are numerous things that *could* happen.

But, when we <u>repeatedly observe many</u> <u>observations</u>, we expect the distribution of the observed outcomes to show *some* type of pattern or regularity.

Tossing a coin

One flip

 \Box It's a 50-50 chance whether it's a head or tail.

100 flips

I can't predict perfectly, but I'm not going to predict 0 tails, that's just not likely to happen.

I'm going to predict something close to 50 tails and 50 heads. That's much more likely than 0 tails or 0 heads.

Tossing a coin many times

Let \hat{p} represent the proportion of heads that I get when I toss a coin many times.

□ If I toss 45 heads on 100 flips, then $\hat{p} = \frac{45}{100} = 0.45$

 $\square \hat{p}$ is pronounced "*p*-hat". It is the relative frequency of heads in this example.

□ If I toss 48 heads on 100 flips, then $\hat{p} = \frac{48}{100} = 0.48$

Tossing a coin many times

I expect \hat{p} (the proportion of heads) to be somewhere near 50% or 0.50.

• What if I only toss a coin two times? • The only possible values for \hat{p} are...

1)
$$\hat{p} = 0/2 = 0.00$$

2) $\hat{p} = 1/2 = 0.50$
3) $\hat{p} = 2/2 = 1.00$

Pretty far from the true probability of flipping a head on a fair coin (0.5).

Tossing a coin many MANY timesIt turns out...

□ If I toss it 100 times I expect to be near 0.50

□ If I toss it 1000 times I expect to be even nearer to 0.50

□ If I toss it 10,000 times I expect to be even nearer to 0.50 than in the 1000 coin toss.

Tossing a coin many MANY times

- This shows a very possible observed situation...
 - □ Toss it 100 times, 45 heads.
 - \Box Toss it 1000 times, 485 heads. \hat{p}
 - Toss it 10,000 times, 4955 heads.

$$\hat{p} = \frac{45}{100} = 0.4500$$
$$\hat{p} = \frac{485}{1000} = 0.4850$$

$$\hat{p} = \frac{4955}{10000} = 0.4955$$

• With more tosses, the closer \hat{p} gets to the truth of 0.50 (it's "zero-ing in" on the truth).

Law of Large Numbers

For repeated *independent* trials, the long run (i.e. after many many trials) relative frequency of an outcome gets closer and closer to the true probability of the outcome.

If you're using your trials to estimate a probability (i.e. empirical probability), you'll do a better job at estimating by using a larger number of trials.

Computer simulation of rolling a die. We'll keep track of the proportion of 1's.



Number of rolls getting larger \rightarrow

Law of Large Numbers

- Suppose you don't know if a coin is fair.
 - \Box Let p represent the true probability of a head.
- \hat{p} from n=10 trials is an OK estimate for p.
- \hat{p} from n=1000 trials is a *better* estimate for p.

- $\hat{p} = \frac{6}{10} = 0.60$
- $\hat{p} = \frac{513}{1000} = 0.513$

 \hat{p} from n=10,000 trials is an even better estimate for p.

$$\hat{p} = \frac{5014}{10000} = 0.5014$$

Law of Large Numbers (LLN)

In the coin flip example, your estimate with n=10 may hit the truth right on the nose, but you have a better chance of \hat{p} being very close to p = 0.5 when you have a much larger *n*.

Law of Large Numbers (LLN)

- LLN let's insurance companies do a pretty good job of estimating costs in the coming year for large groups (i.e. when they have LOTS and LOTS of insurees).
 - Hard to predict for one person, but we can do a pretty good job of predicting total costs or total proportion of people who will have an accident for a group.

Example:

- The Binary Computer Company manufactures computer chips used in DVD players. Those chips are made with 0.73 defective rate.
 - □ A) When one chip is drawn, list the possible outcomes.
 - □ B) If one chip is randomly selected, find the probability that it is *good*.
 - □ C) If you select 100,000 chips, how many defects should you expect?

Answers:

 \Box A) Two possible outcomes: defective or good.

□ B) P(good)=1-P(defective) = 1-0.73=0.27

- □ C) If you select 100,000 chips, how many defects should you expect?
 - As the number of chips sampled gets larger, the proportion of defects in the sample approaches the true proportion of defects (which is 0.73). So, with this large of a sample, I would expect about 73% of the sample to be defects, or 0.73 x 100,000= 73,000.
 - I used the Law of Large numbers in the above.

Game based on the roll a die:

If a 1 or 2 is thrown, the player gets \$3. If a 3, 4, 5, or 6 is thrown, the house wins (you get nothing).

□ Would you play if it cost **\$5** to join the game?

- □ Would you play if it cost **\$1** to join the game?
- What do you EXPECT to gain (or lose) from playing?

We'll return to this example later... 14

 There are alternative ways to compute expected values (all with the same result).
 I will focus on those where the probabilities of the possible events sum to 1.

- Life insurance companies depend on the law of large numbers to stay solvent (i.e. be able to pay their debts).
- I pay \$1000 annually for life insurance for a \$500,000 policy (in the event of a death).
- Suppose they only insured me.
 If I live (high probability), they make \$1000.
 If I die (low probability), they lose \$499,000!!
- Should they gamble that I'm not going to die? Not sound business practice.

It's hard to predict the outcome for one person, but much easier to predict the 'average' of outcomes among a group of people.

Insurance is based on the probability of events (death, accidents, etc.), some of which are very unlikely.

For the upcoming year, suppose: P(die) = 1/500 = 0.002 P(live) = 499/500 = 0.998 A life insurance policy costs \$100 and

A life insurance policy costs \$100 and pays out \$10,000 in the event of a death

If the company insures a million people, what do they expect to gain (or lose)?

- For each policy, one of two things can happen... a payout (die) or no payout (live).
 If the person dies and they pay out, they lose \$9,900 on the policy. (they did collect the \$100 regardless of death or not)
 - □ If the person lives and there's no payout, they gain **\$100** on the policy.

Expected Value = $(1/500)^{(-9,900)} + (499/500)^{(100)} = 80$

This amounts to a profit of \$80,000,000 on sales of 1 million policies



This amounts to a profit of \$80,000,000 on sales of 1 million policies (overall expected value of \$80,000,000). 20

For the upcoming year, suppose: $\Box P(die) = 1/500 = 0.002$

□ P(live) = 499/500 = 0.998

□ A life insurance policy costs **\$1000** and pays out **\$500,000** in the event of a death

Expected Value = $(1/500)^{*}(-$499,000) + (499/500)^{*}($1000) = 0

They **expect** to just 'break even' (if more people die than expected, they're in trouble). Charging any less would result in an expected loss.

The law of large numbers comes into play for insurance companies because their <u>actual</u> payouts will depend on the observed number of deaths (some uncertainty).

As they observe (or insure) more people, the <u>relative frequency</u> of a death will get closer and closer to the 1/500 probability on which they based their calculations.

Calculating Expected Value

Consider two events, each with its own value and probability. The expected value is

```
expected value =
```

(value of event 1) * (probability of event 1)
+ (value of event 2) * (probability of event 2)

This formula can be extended to any number of events by including more terms in the sum.

Game based on the roll a die:

If a 1 or 2 is thrown, the player gets \$3. If a 3, 4, 5, or 6 is thrown, the house wins (you get nothing).

□ If the game costs **\$5** to play, what is the expected value of a game?

Expected Value = $(2/6)^{*}(-$2)+(4/6)^{*}(-$5) = -$4$ (per game)

No matter what is rolled, you're losing money.

Game based on the roll a die:

If a 1 or 2 is thrown, the player gets \$3. If a 3, 4, 5, or 6 is thrown, the house wins (you get nothing).

□ If the game costs **\$1** to play, what is the expected value of a game?

Expected Value = $(2/6)^{*}(\$2) + (4/6)^{*}(-\$1) = \$0$ (per game)

This is a 'fair game'.

Definition

The **expected value** of a variable is the *weighted average* of all its possible events. Because it is an average, we should expect to find the "expected value" only when there are a large number of events, so that the law of large numbers comes into play.

For the life insurance company, the observed relative frequency of deaths approaches the truth (which is 1/500) as they get more and more insurees.

 Game based on the roll of two dice:
 If a sum of 12 is thrown, the player gets \$40. If anything else is thrown, the house wins (you get nothing).

□ If the game costs **\$4** to play, what is the expected value of a game?

Expected Value = $(1/36)^{*}(\$36) + (35/36)^{*}(-\$4)$ (per game) = - \$2.89

This is a 'unfair game' (for the gambler).

Everyone who bets any part of his fortune, however small, on a mathematically unfair game of chance acts **irrationally**...

-- Daniel Bernoulli, 18th century mathematician

In the long run, you know you'll lose. But perhaps you think you can 'beat the house' in the short run.

The Gambler's Fallacy

Definition

The **gambler's fallacy** is the mistaken belief that a streak of bad luck makes a person "due" for a streak of good luck.

 Game based on the toss of a fair coin:
 Win \$1 for heads and lose \$1 for tails (no cost to play).

Expected Value =
$$(1/2)^{*}(\$1) + (1/2)^{*}(-\$1)$$

(per game) = $\$0$

- □ After playing 100 times, you have 45 heads and 55 tails (you're down \$10).
- Thinking things will 'balance out in the end' you keep playing until you've played 1000 times. Unfortunately, you now have 480 heads and 520 tails (you're down \$40).

Do these results go against what we know about the law of large numbers? Nope.

$$\hat{p}_{n=100} = \frac{45}{100} = 0.450$$

$$\hat{p}_{n=1000} = \frac{480}{1000} = 0.480$$

□ The proportion of heads DID get closer to 0.50

But the difference between the number of heads and tails got larger, which is reasonable as the number of tosses gets larger.

Outcomes of coin tossing trials

				Difference
Number	Number	Number	Percentage	between number of
of tosses	of tails	of heads	of heads	heads and tails
100	55	45	45%	10
1000	520	480	48%	40
10,000	5,050	4,950	49.50%	100
100,000	50,100	49,900	49.90%	200

- Though the percentage of heads gets closer to 50%, the difference in the number of heads and tails doesn't get closer to zero
 - (and the difference is what relates to the money in your pocket).

Streaks

- When a sequence of events gives rise to a streak, this can also lend itself to the gambler's fallacy.
 - □ Flip a coin 7 times: HHHHHHH
 - But we know that the above streak is just as likely as HTTHTTH because these are equally likely outcomes.
 - Perhaps you think a tail is 'due' next in the first streak, but P(tail)=P(head)=0.50 no matter what you tossed in the past.