Symmetry Breaking

and

Synchrony Breaking

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Why Study Patterns I

- **Patterns are surprising and pretty**

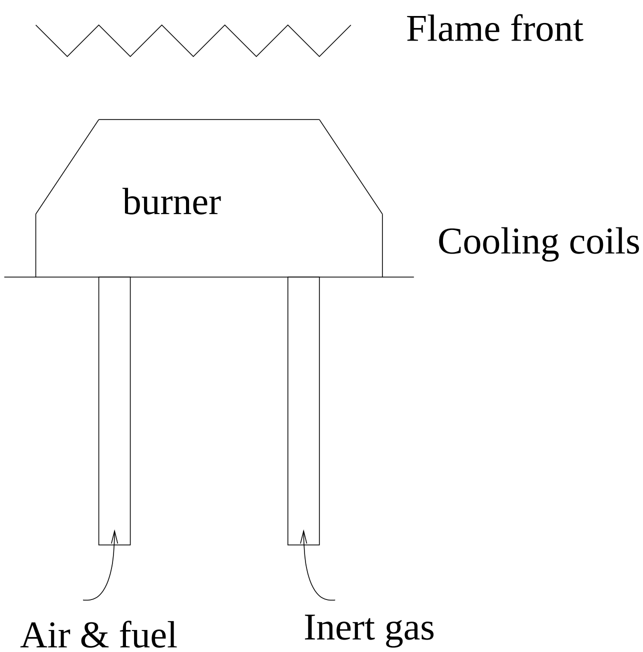
Mud Plains

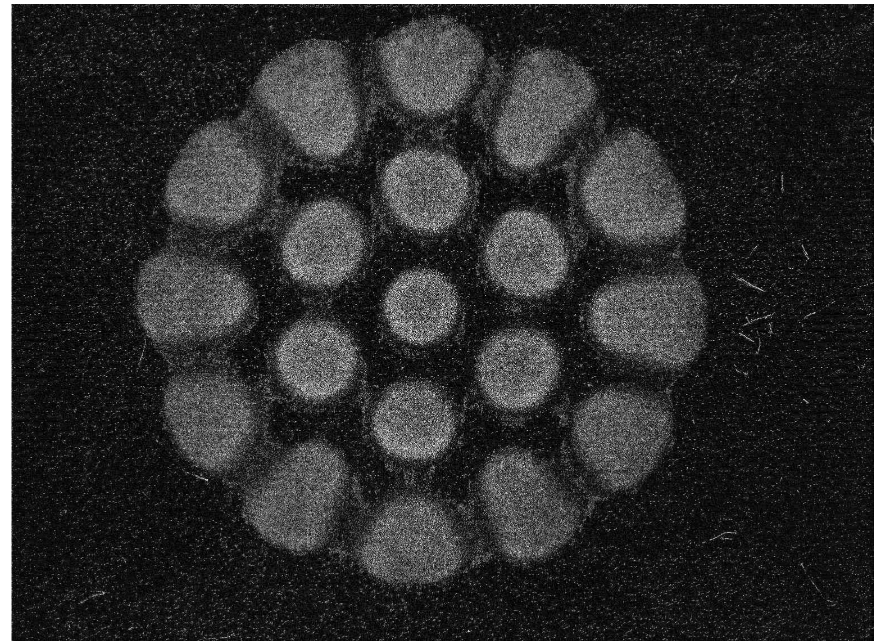
Leopard Spots

Sand Dunes in Namibian Desert

Zebra Stripes

Porous Plug Burner Flames (Gorman)





- Dynamic patterns
- A film in two parts
  - rotating patterns
  - standing patterns

Why Study Patterns II

1) Patterns are surprising and pretty

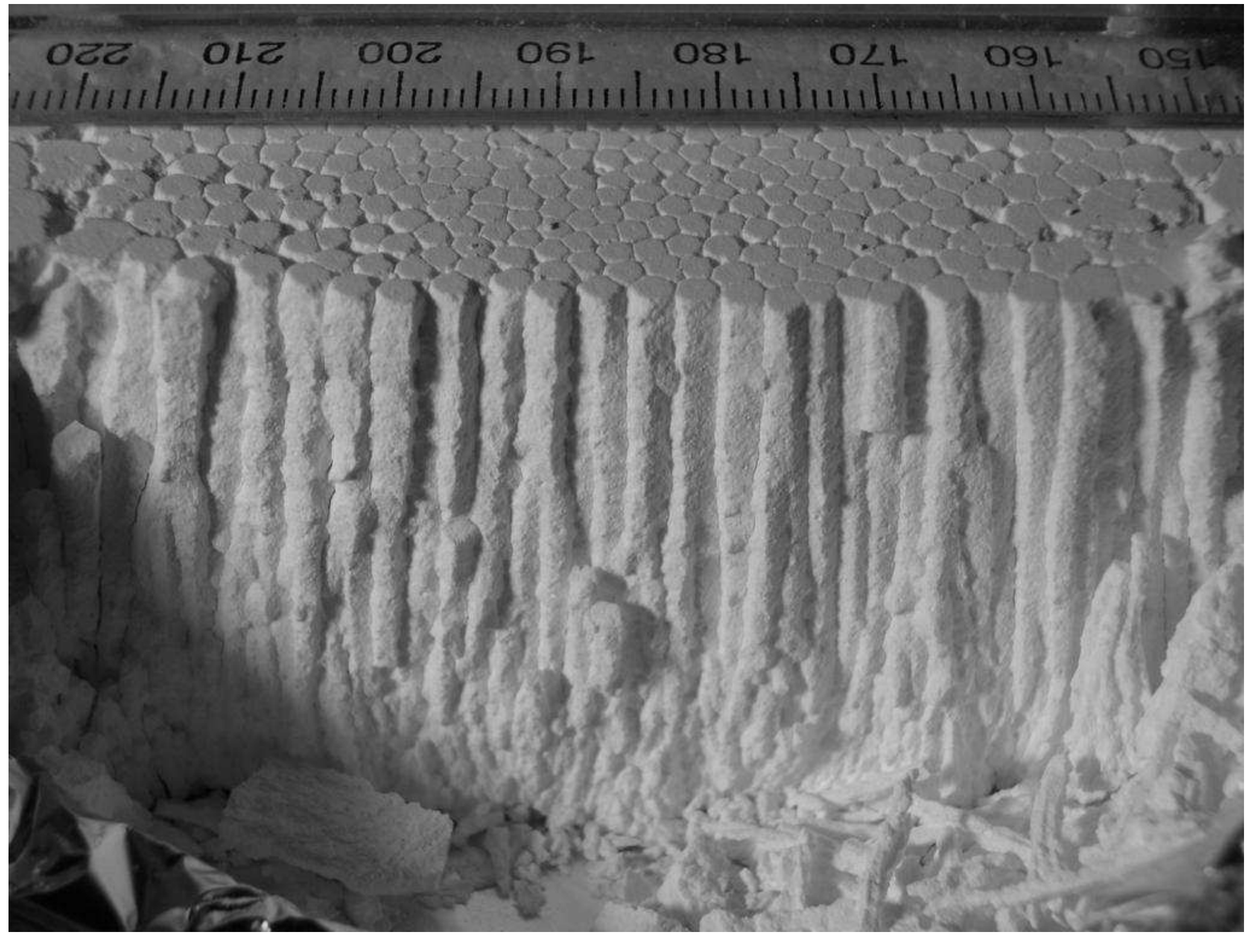
2) Science behind patterns

Columnar Joints on Staffa near Mull

Columns along Snake River

Irish Giants Causeway

Experiment on Corn Starch



Goehring and Morris, 2005

Why Study Patterns III

1) Patterns are surprising and pretty

2) Science behind patterns

3) **Change in patterns provide tests for models**

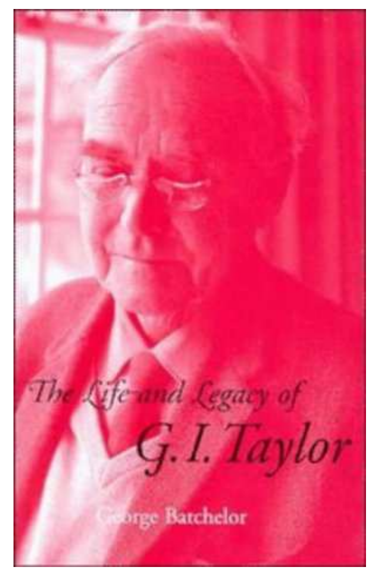
A Brief History of Navier-Stokes

Navier-Stokes equations for an incompressible fluid

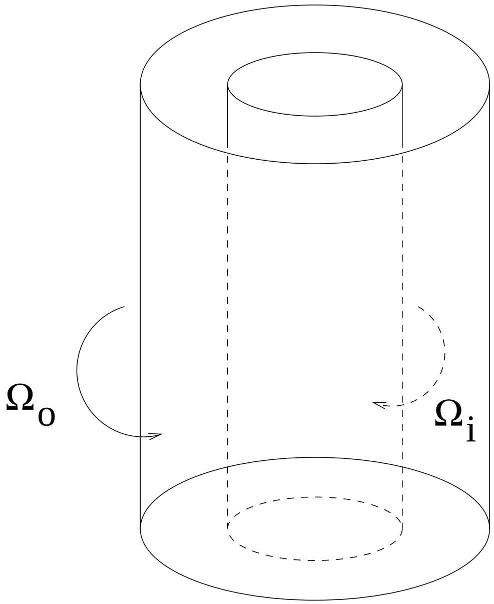
$$\begin{aligned} u_t &= \nu \nabla^2 u - (u \cdot \nabla) u - \frac{1}{\rho} \nabla p \\ 0 &= \nabla \cdot u \end{aligned}$$

u = velocity vector ρ = mass density
p = pressure ν = kinematic viscosity

- Navier (1821); Stokes (1856); Taylor (1923)

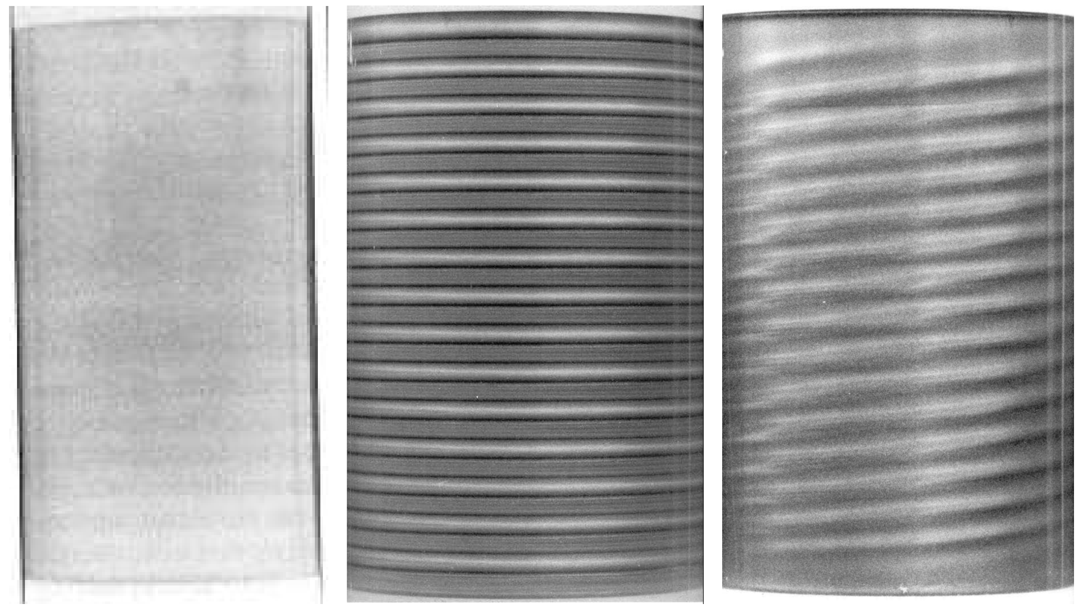


The Couette Taylor Experiment



- Ω_i = speed of inner cylinder
- Ω_o = speed of outer cylinder

Andereck, Liu, and Swinney (1986)



Couette Taylor Spiral

time independent time periodic

G.I. Taylor: Theory & Experiment (1923)

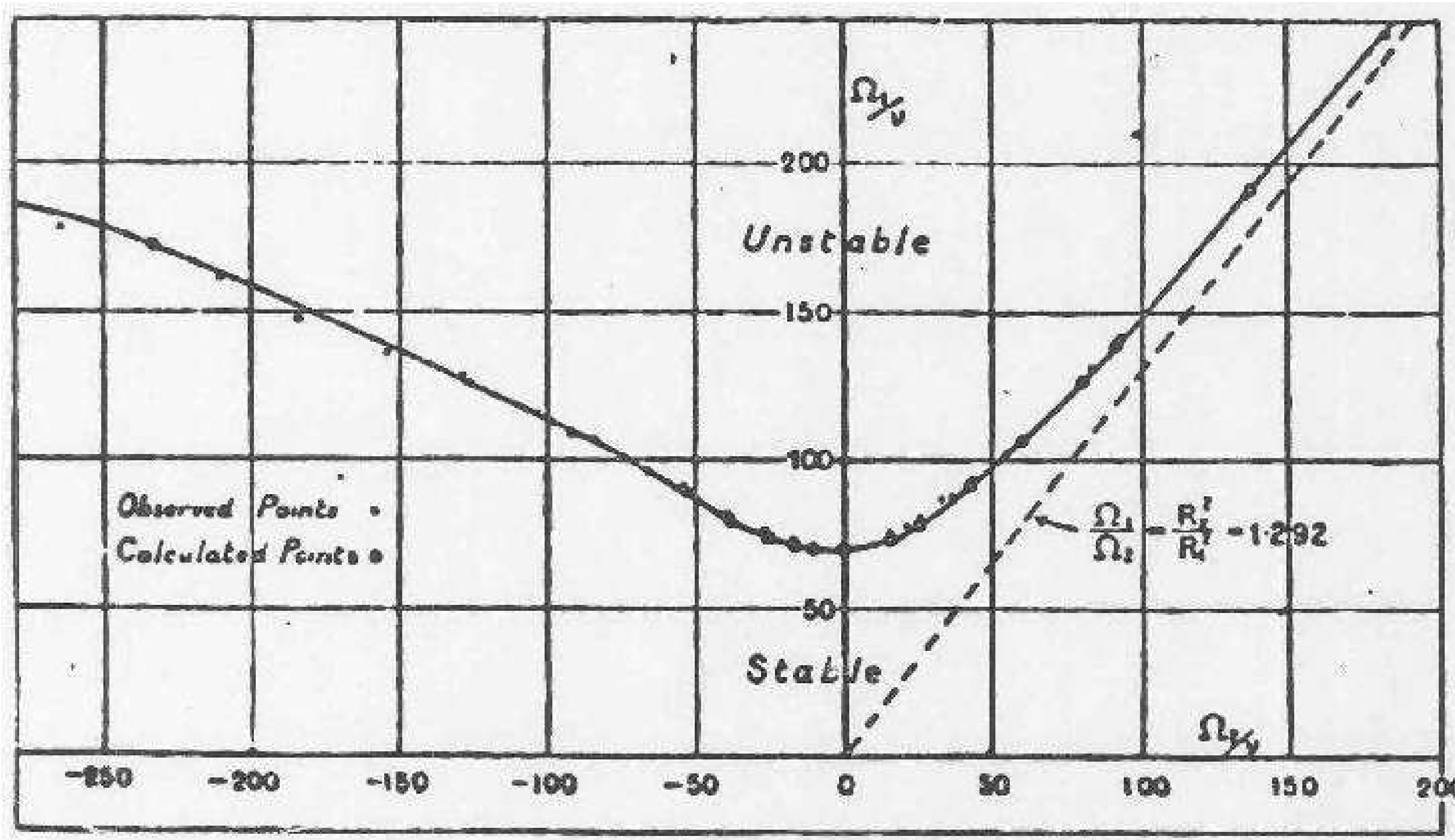


Fig. 18. Comparison between observed and calculated speeds at which instability first appears;

Why Study Patterns IV

1) Patterns are surprising and pretty

2) Science behind patterns

3) Change in patterns provide tests for models

4) **Model independence**

Mathematics provides menu of patterns

Planar Symmetry-Breaking

- Euclidean symmetry: translations, rotations, reflections
- Symmetry-breaking from translation invariant state in planar systems with Euclidean symmetry leads to
  - Stripes: invariant under translation in one direction

    Sand dunes, zebra
  - Spots: states centered at lattice points

    mud plains, leopard

Circle Symmetry-Breaking Oscillation

- There exist two types of time-periodic solutions

  near a circularly symmetric equilibrium

  - Rotating waves:

    Time evolution is the same as spatial rotation

    Standing waves:

    Fixed lines of symmetry for all time

- Examples: Gorman's flame experiments
  - PDE systems on interval with periodic boundary conditions

Primer on Steady-State Bifurcation

- Solve $\dot{x} = f(x, \lambda) = 0$ where $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$
- Local theory: Assume $f(0, 0) = 0$ – find solns near $(0, 0)$
- If $J = (d_x f)_{0,0}$ nonsingular, IFT implies unique soln $x(\lambda)$
- Bifurcation of steady states $\Longleftrightarrow \ker J \neq \{0\}$

Equivariant Steady-State Bifurcation

Let $\gamma : \mathbf{R}^n \to \mathbf{R}^n$ be linear

- γ is a symmetry iff γ(soln)=soln iff $f(\gamma x, \lambda) = \gamma f(x, \lambda)$
- Chain rule $\Longrightarrow$ $J\gamma = \gamma J$ $\Longrightarrow$ $\ker J$ is γ-invariant
- Theorem: Fix Γ. Generically $\ker J$ is an absolutely irreducible representation of Γ

  i.e. only commuting matrices are multiples of identity
- Reduction implies that there is a unique steady-state bifurcation theory for each absolutely irreducible rep

Primer on Hopf Bifurcation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - (x^2 + y^2) \begin{bmatrix} x \\ y \end{bmatrix}$$

Origin is an equilibrium for all values of λ

Primer on Hopf Bifurcation

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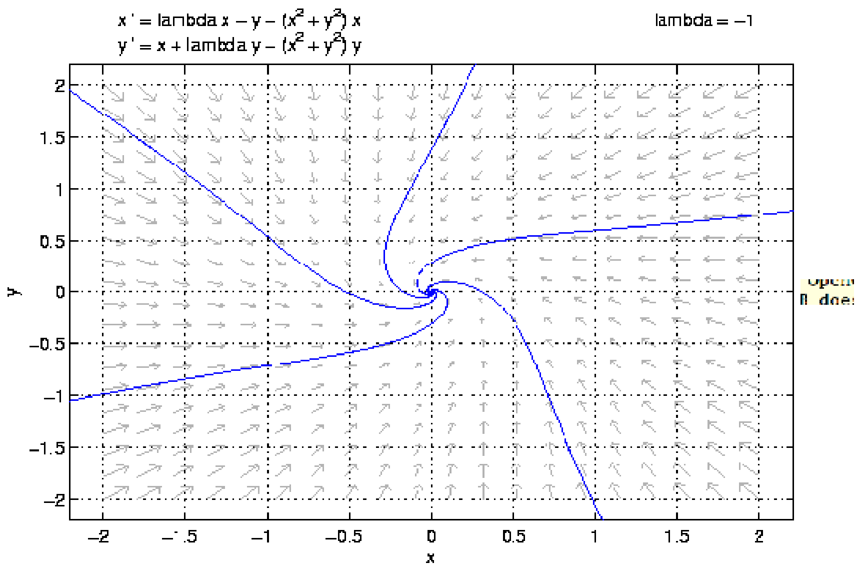
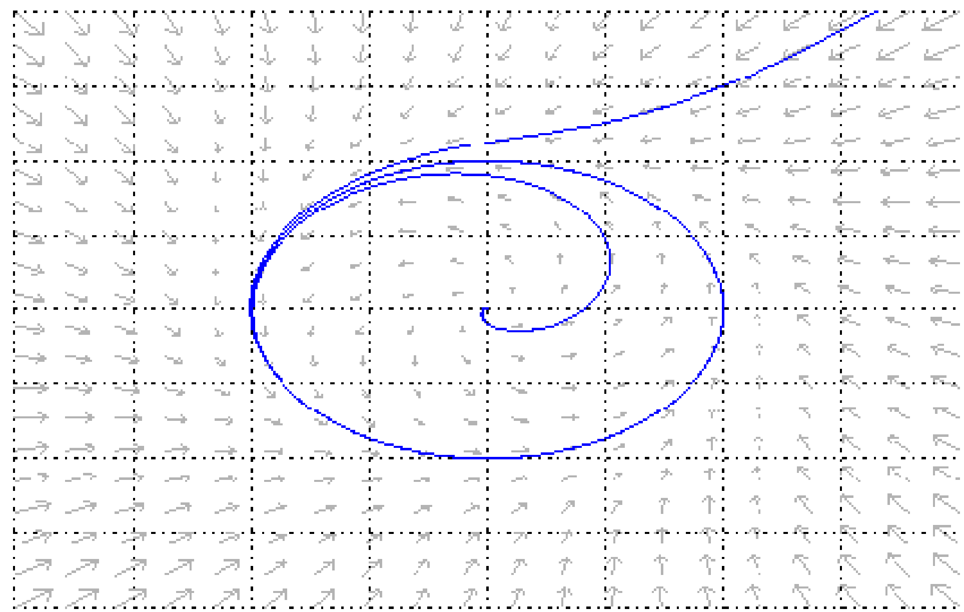


$\lambda = -1$



$\lambda = 1$

Primer on Hopf Bifurcation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - (x^2 + y^2) \begin{bmatrix} x \\ y \end{bmatrix}$$

- Origin goes from spiral sink to spiral source as $\lambda \nearrow 0$
- Let $r^2 = x^2 + y^2$. Then $\dot{r} = (\lambda - r^2)r$

  1) Unique branch of periodic trajectories (for $\lambda > 0$)

  2) Amplitude growth of periodic solution is $\lambda^{\frac{1}{2}}$

Primer on Hopf Bifurcation

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Hopf Theorem: Generically (1) and (2) hold when pair of eigenvalues of Jacobian on imaginary axis

Primer on Equivariant Hopf Bifurcation

- Hopf bifurcation $\Longleftrightarrow J$ has eigenvalues $\pm \omega i$
- Suppose

$$\mathbf{R}^n = V_1 \oplus \cdots \oplus V_\ell$$

where V_j are distinct absolutely irreducible

Then

  - $J : V_j \to V_j$ is a real multiple of I_{V_j}
  - all eigenvalues of J are real
  - Hopf bifurcation is not possible.

Primer on Equivariant Hopf Bifurcation

- Hopf bifurcation $\Longleftrightarrow J$ has eigenvalues $\pm\omega i$
- Representation on E^c is Γ-simple iff either
  - $E^c = V \oplus V$ where V is absolutely irreducible, or
  - Γ acts nonabsolutely irreducibly on E^c
- Theorem: Fix Γ. At Hopf bifurcation, generically, Γ acts Γ-simply on center subspace E^c
- Reduction implies that there is a unique Hopf bifurcation theory for each irreducible rep

Spatiotemporal Symmetries

- What kind of symmetries do periodic solutions have?

- Let $x(t)$ be a time-periodic solution
  - $K = \{\gamma \in \Gamma : \gamma x(t) = x(t)\}$ **space symmetries**
  - $H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\}$ **spatiotemporal symm's**

- $\gamma \in H \Longrightarrow \theta \in \mathbf{S}^1$ such that $\gamma x(t) = x(t + \theta)$

Spatiotemporal Symmetries

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- **Example**: $\Gamma = \mathbf{O}(2)$; $E^c = \mathbf{R}^2 \oplus \mathbf{R}^2$

  Two periodic solutions types emanate from bifurcation
  - rotating waves: $H = \mathbf{SO}(2)$; $K = \mathbf{1}$
  - standing waves: $H = \mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2(R_\pi)$; $K = \mathbf{Z}_2(\kappa)$, where κ is a reflection

Spatiotemporal Symmetries

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- $\gamma \in H \Longrightarrow \theta \in \mathbf{S}^1$ such that $\gamma x(t) = x(t + \theta)$

- H/K is cyclic or $\mathbf{S}^1$ since

  $\gamma \mapsto \theta$ is a homomorphism with kernel K

Summary on Pattern Formation

- There is a codimension one steady-state bifurcation from a group invariant equilibrium corresponding to each absolutely irreducible subspace

  There is a codimension one Hopf bifurcation from a group invariant equilibrium corresponding to each irreducible subspace

Summary on Pattern Formation

- There is a codimension one steady-state bifurcation from a group invariant equilibrium corresponding to each absolutely irreducible subspace

  There is a codimension one Hopf bifurcation from a group invariant equilibrium corresponding to each irreducible subspace

- Mathematics leads to a menu of patterns

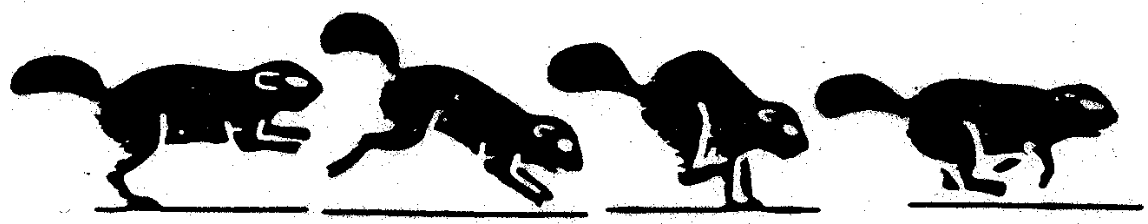
  This menu is model independent

  Physics & Biology choose from that menu

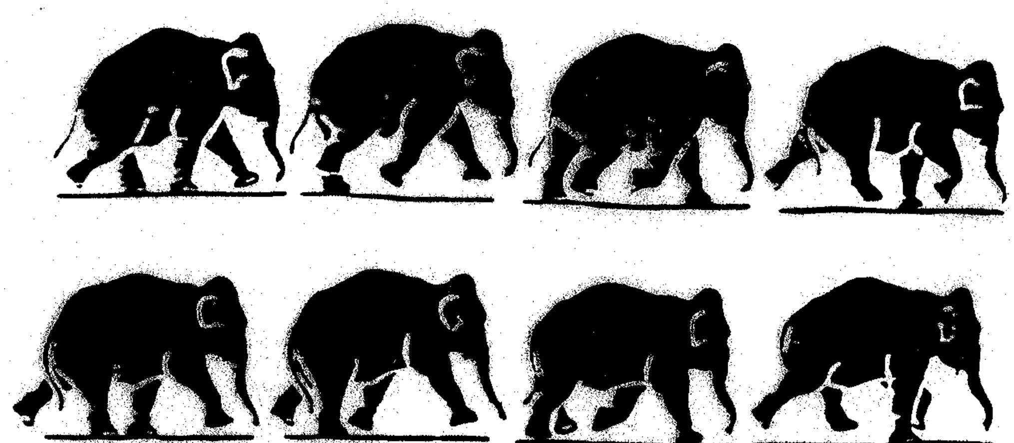
  This choice is model dependent

Quadruped Gaits

- Bound of the Siberian Souslik

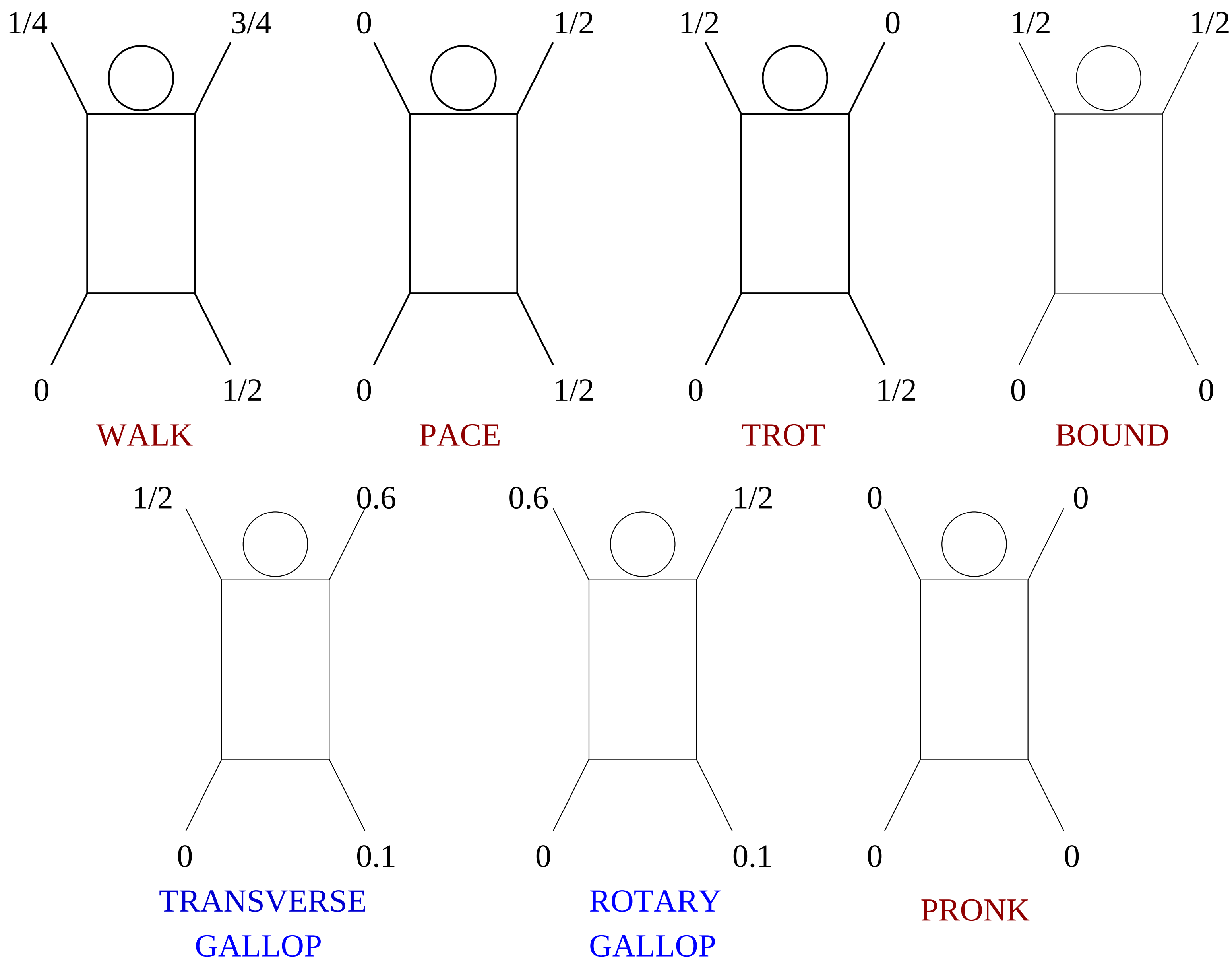


- Amble of the Elephant

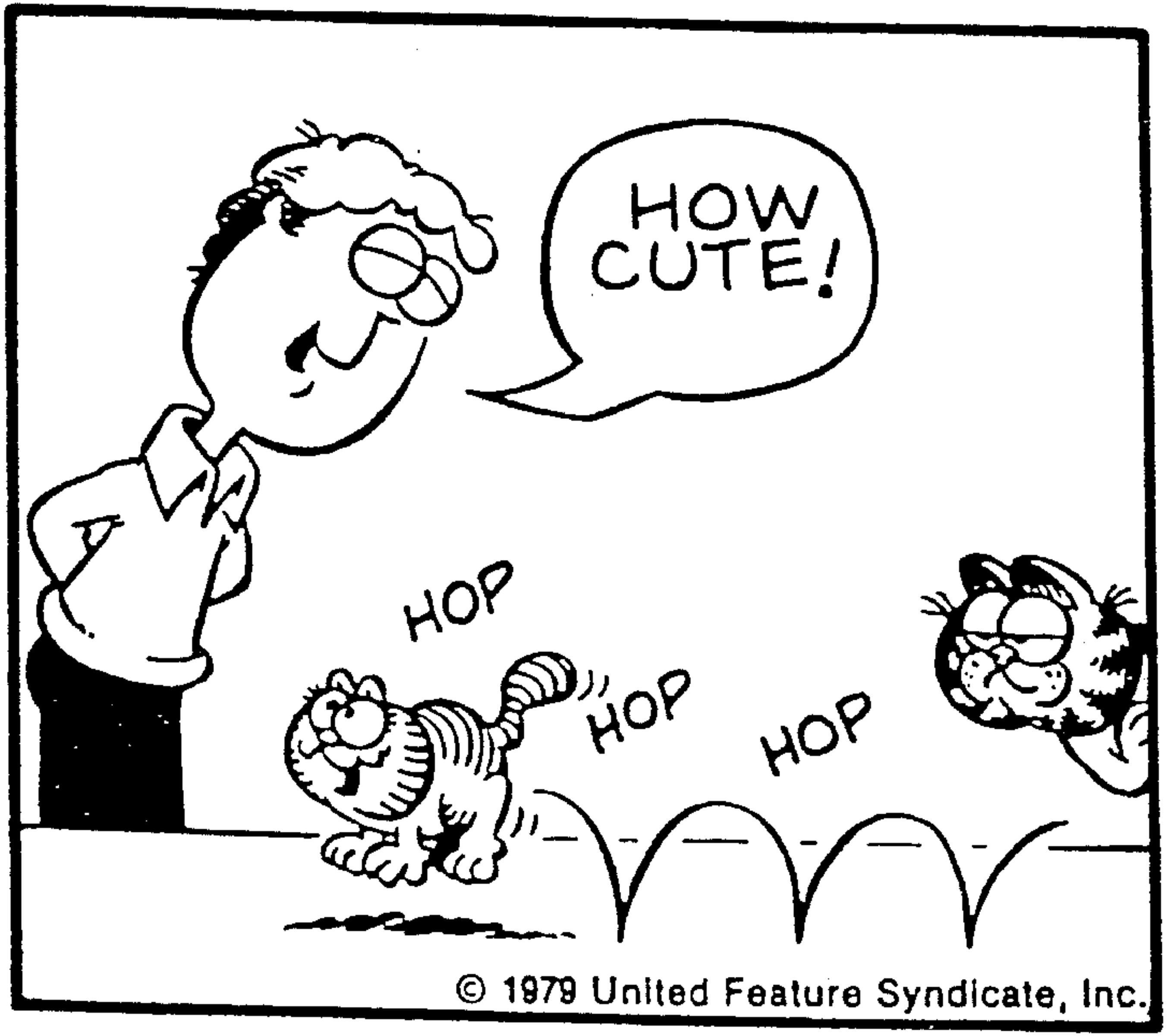


- Trot of the Horse

Standard Gait Phases



The Pronk



Gait Symmetries

Gait	Spatio-temporal symmetries
Trot	(Left/Right, $\frac{1}{2}$) and (Front/Back, $\frac{1}{2}$)
Pace	(Left/Right, $\frac{1}{2}$) and (Front/Back, 0)
Walk	(Figure Eight, $\frac{1}{4}$)

- **Three gaits are different**
- Assumption: **There is a network in the nervous system that produces the characteristic rhythms of each gait**
- Design simplest network to produce walk, trot, and pace

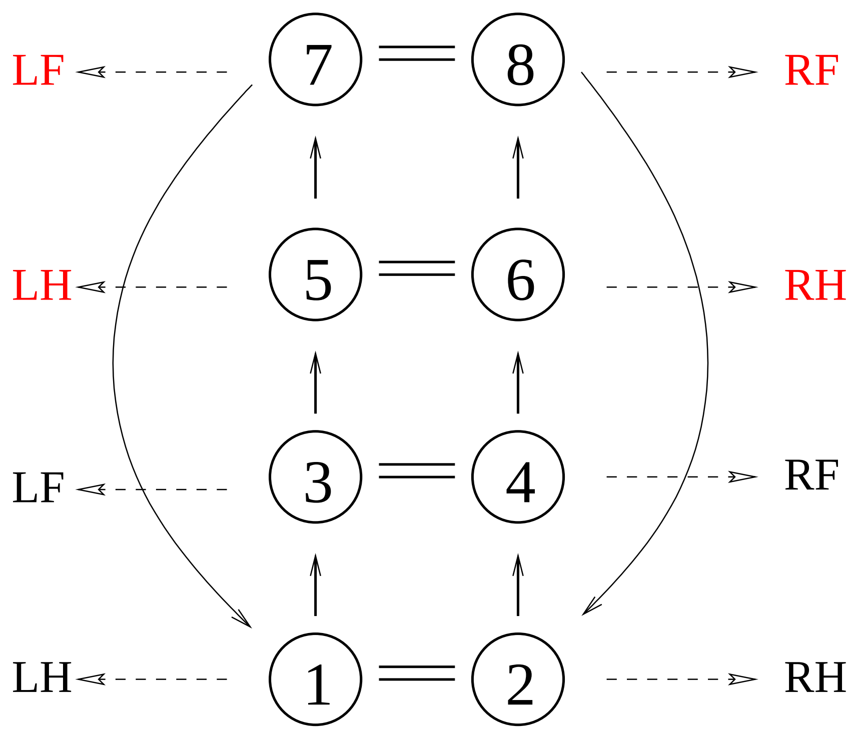
Central Pattern Generators (CPG)

- **Use gait symmetries to construct coupled network**

  1) walk $\Longrightarrow$ four-cycle ω in symmetry group

  2) pace or trot $\Longrightarrow$ transposition κ in symmetry group

- Simplest network has $\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$ symmetry

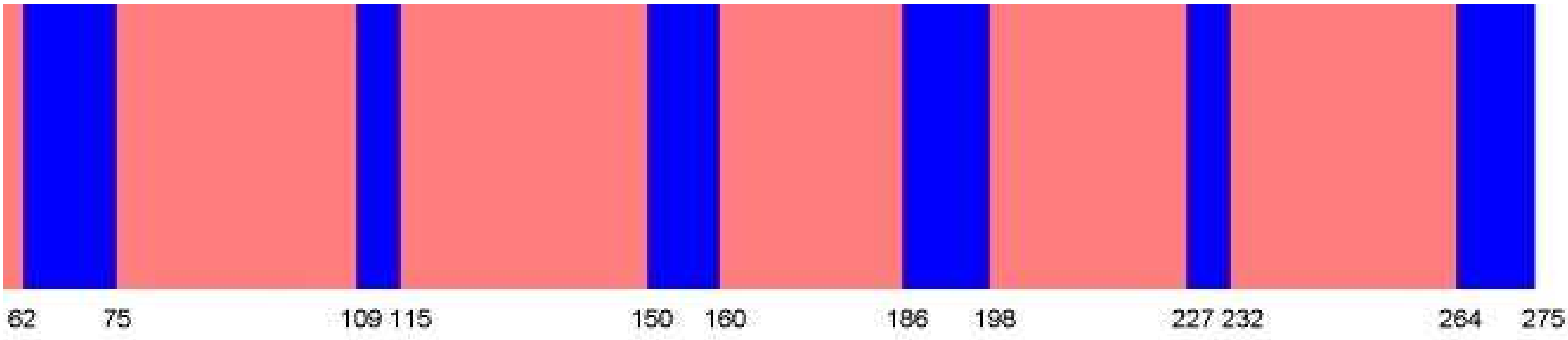


Primary Gaits: Hopf from Stand

Six Irreducible Representations of $\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$

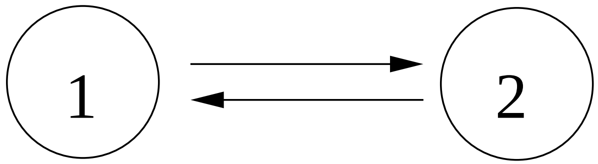
Phase Diagram	Gait
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	pronk
$\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$	pace
$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	trot
$\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	bound
$\begin{pmatrix} \pm\frac{1}{4} & \pm\frac{3}{4} \\ 0 & \frac{1}{2} \end{pmatrix}$	walk$^{\pm}$
$\begin{pmatrix} 0 & 0 \\ \pm\frac{1}{4} & \pm\frac{1}{4} \end{pmatrix}$	jump$^{\pm}$

The Jump



- Average Right Rear to Right Front = 31.2 frames
- Average Right Front to Right Rear = 11.4 frames
- $\frac{31.2}{11.4} = 2.74$

Two Identical Cells



$$\dot{x}_1 = f(x_1, x_2)$$
$$\dot{x}_2 = f(x_2, x_1)$$

where $x_1, x_2 \in \mathbf{R}^k$

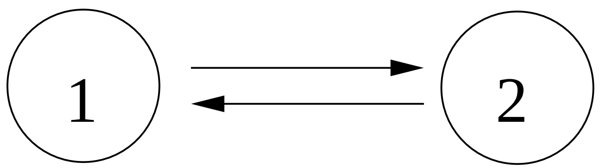
- Time-periodic solutions exist robustly where two cells oscillate a in phase

$$x_2(t) = x_1(t)$$

- Time-periodic solutions exist robustly where two cells oscillate a half-period out of phase

$$x_2(t) = x_1(t + \frac{1}{2})$$

Two Identical Cells

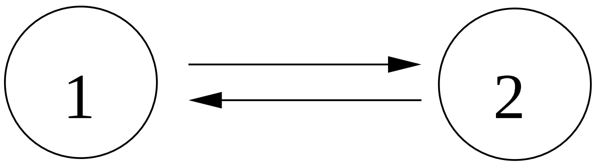


$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, \lambda) \\ \dot{x}_2 &= f(x_2, x_1, \lambda) \qquad x_1, x_2 \in \mathbf{R}^k \\ 0 &= f(0, 0, \lambda) \end{aligned}$$

- $J(\lambda) = \begin{bmatrix} \alpha(\lambda) & \beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{bmatrix}$; $\begin{bmatrix} x \\ x \end{bmatrix}$, $\begin{bmatrix} x \\ -x \end{bmatrix}$ invariant subsp's

eigenvalues of J are eigenvalues of $\alpha + \beta$ and $\alpha - \beta$

Two Identical Cells



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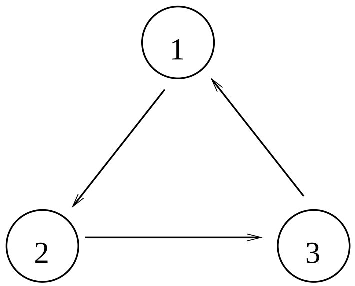
  eigenvalues of J are eigenvalues of $\alpha + \beta$ and $\alpha - \beta$

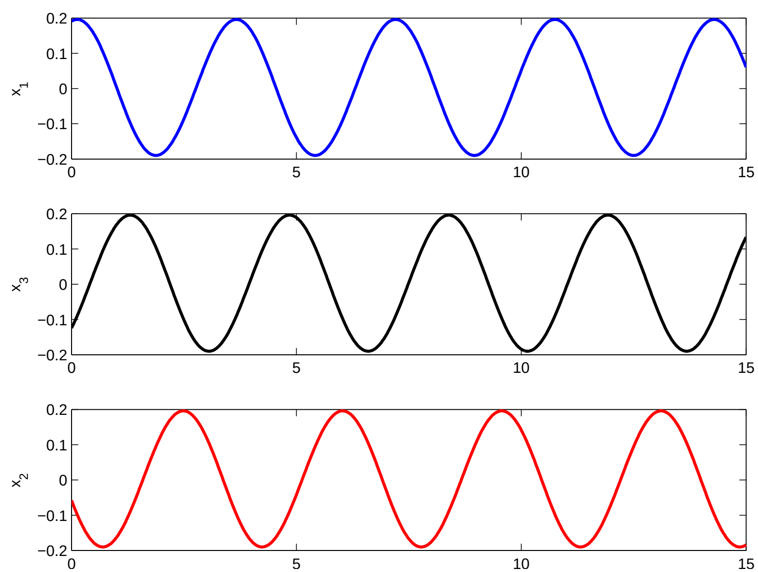
- $\alpha + \beta$ critical: synchronous periodic solutions
- $\alpha - \beta$ critical: periodic solutions where two cells are half-period out of phase $\quad x_2(t) = x_1(t + \frac{T}{2})$

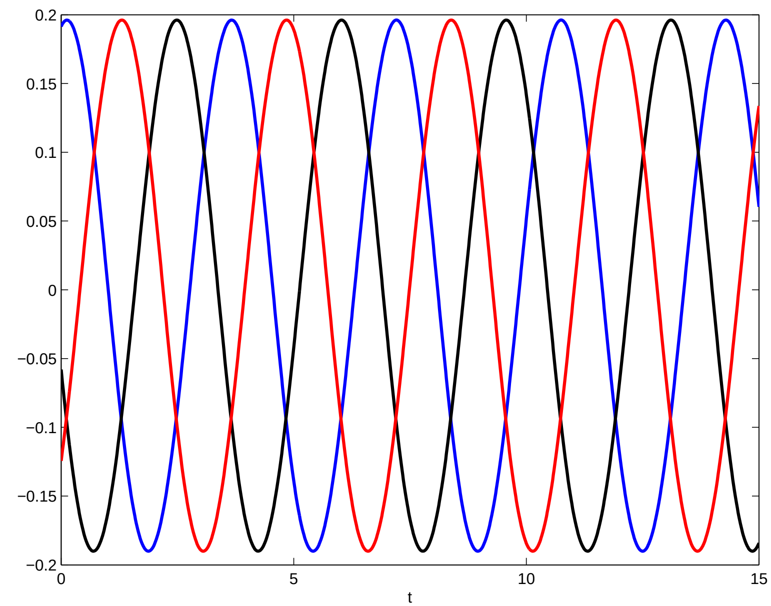
Three-Cell Unidirectional Ring: $\Gamma = \mathbf{Z}_3$



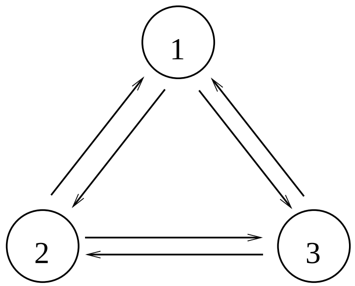
$$
\begin{aligned}
\dot{x}_1 &= f(x_1, x_3) \\
\dot{x}_2 &= f(x_2, x_1) \\
\dot{x}_3 &= f(x_3, x_2)
\end{aligned}
$$

- Discrete rotating waves





Three-Cell Bidirectional Ring: $\Gamma = \mathbf{S}_3$



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_3, x_1) \\ \dot{x}_3 &= f(x_3, x_1, x_2) \end{aligned}$$

$$f(x_2, x_1, x_3) = f(x_2, x_3, x_1)$$

- Discrete rotating waves

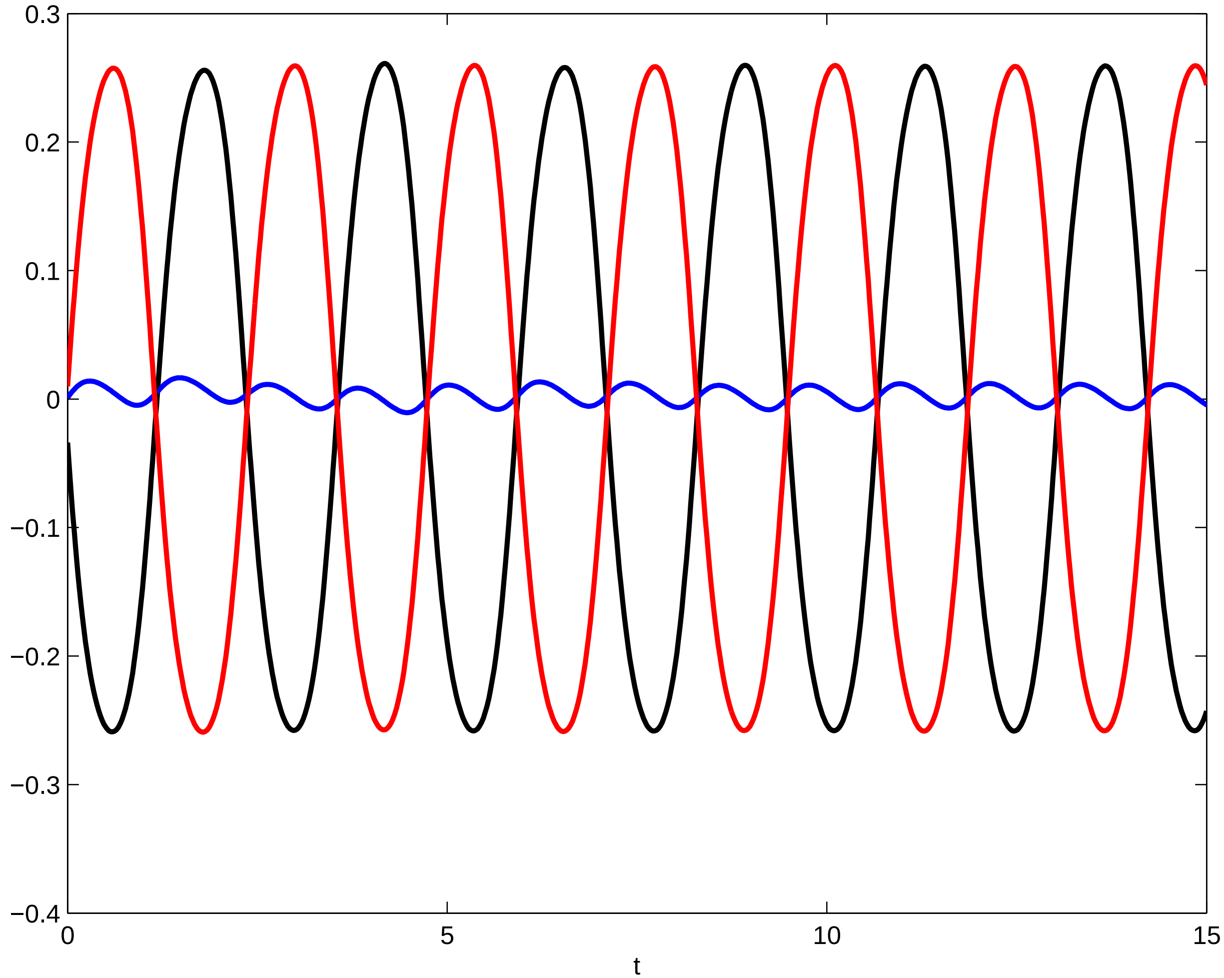
  In-phase periodic solutions: $x_3(t) = x_1(t)$

  Out-of-phase periodic solutions:

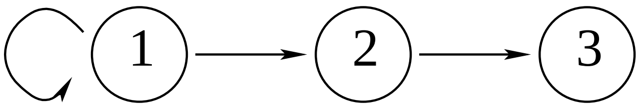
  $x_3(t) = x_1\left(t + \frac{T}{2}\right)$ and $x_2(t) = x_2\left(t + \frac{T}{2}\right)$

G. and Stewart (1986)

Bidirectional Three-Cell Ring (2)



Three-Cell Feed-Forward Network

- 

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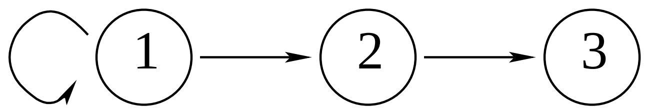
Three-Cell Feed-Forward Network

- 

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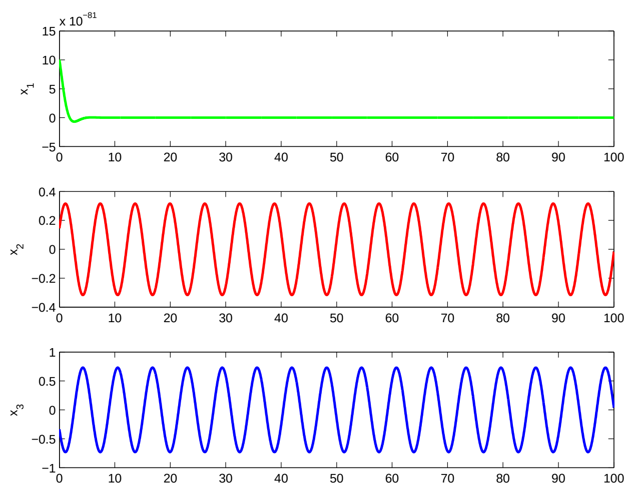
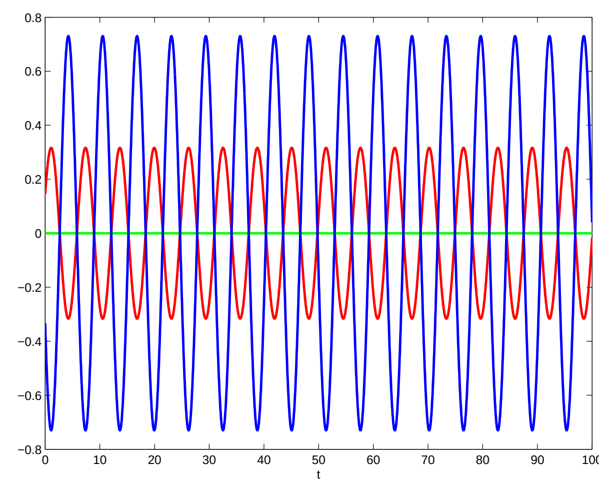
- Network supports solution by Hopf bifurcation where
  $x_1(t)$ equilibrium $\qquad x_2(t), x_3(t)$ time periodic

- $x_2(t) \approx \lambda^{1/2} \qquad x_3(t) \approx \lambda^{1/6}$





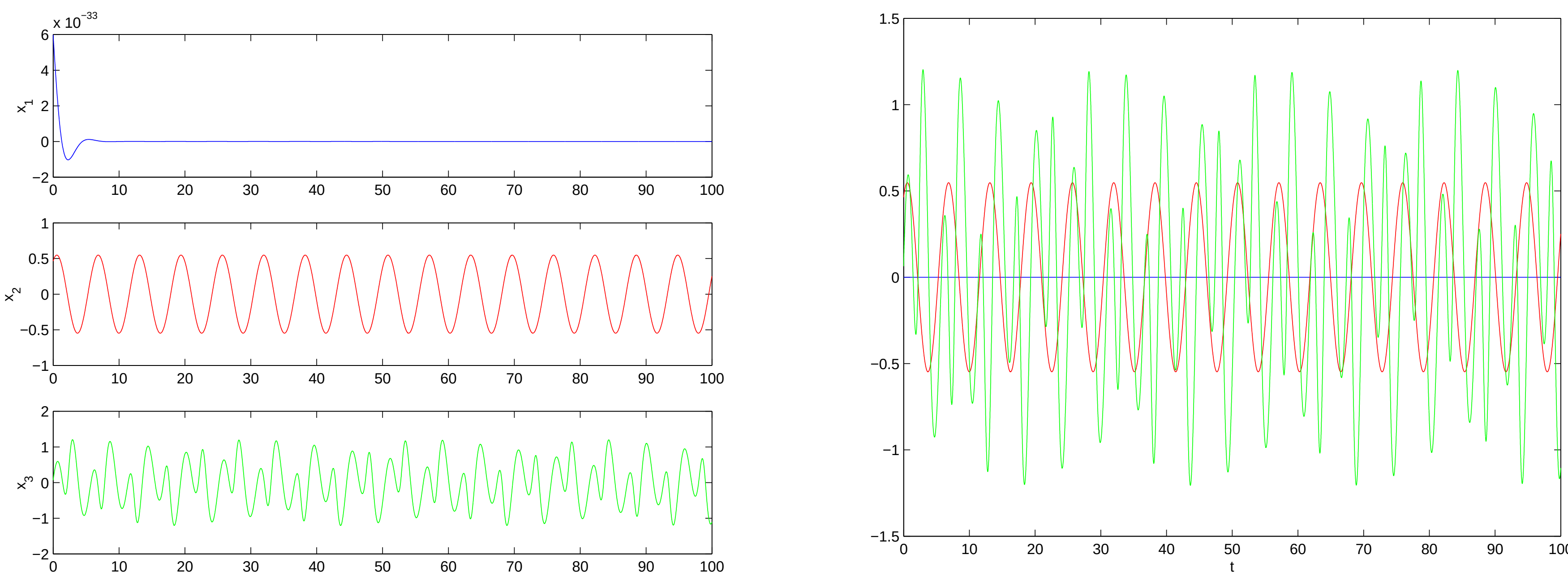
G., Nicol, and Stewart (2004); Elmhirst and G. (2005)

Quasiperiodic Solutions in FF Network

Network supports solution where

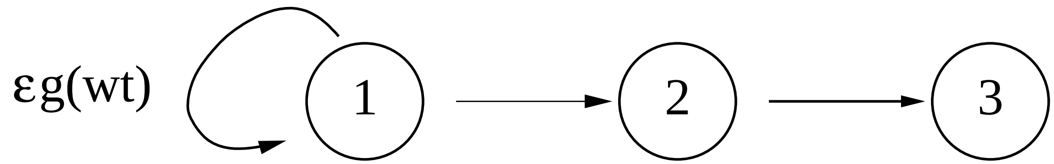
$x_1(t)$ equilibrium, $x_2(t)$ time periodic, $x_3(t)$ quasiperiodic

$$f(y_1, y_2) = (i + 0.3 - |y_1|^2)y_1 - y_2 - 1.83|y_2|^2 y_2 + (2.33 + 4.71i)|y_2|^2 y_1$$



G., Nicol, and Stewart (2004); Broer and Vegter (2007)

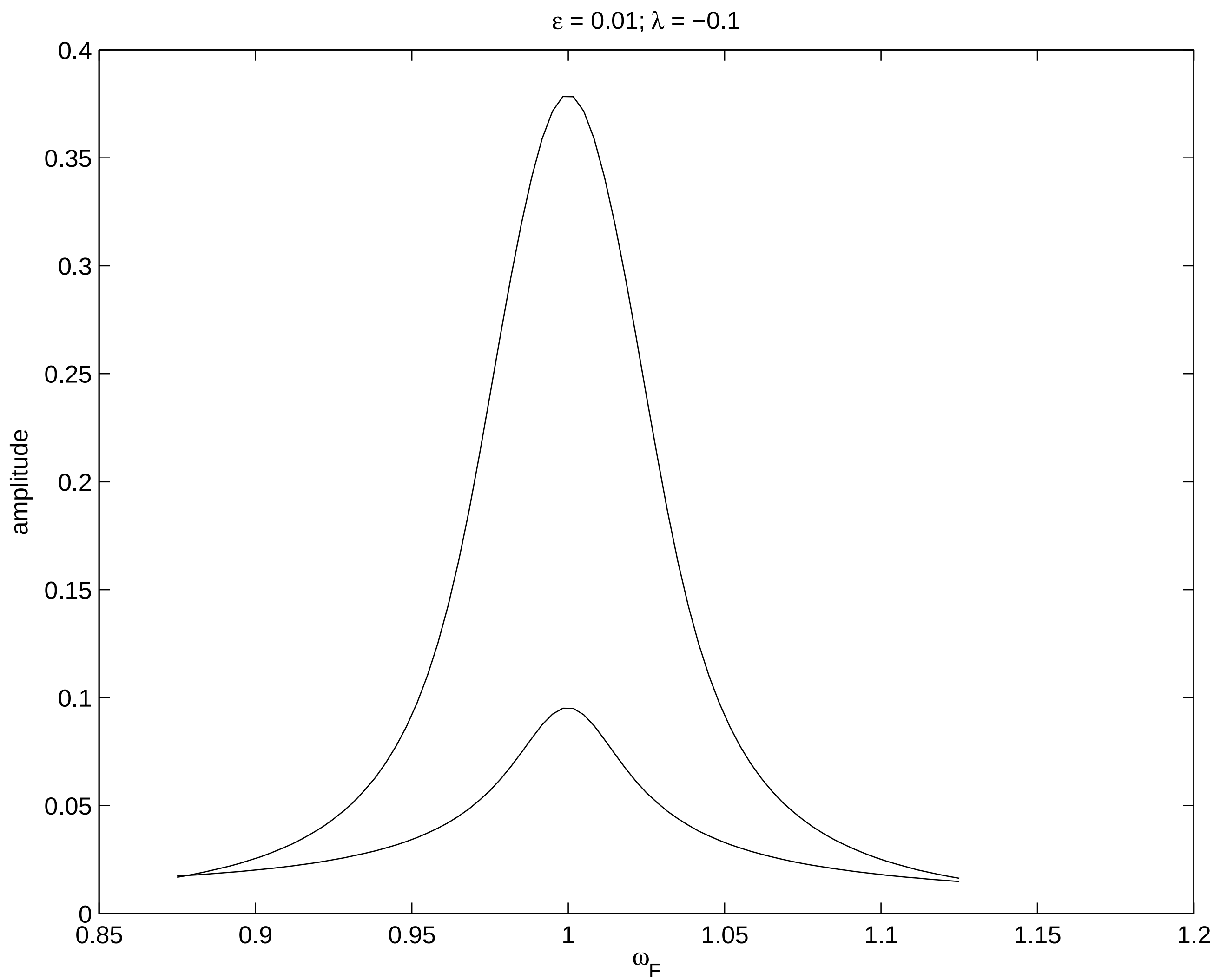
Forced Feed Forward Network



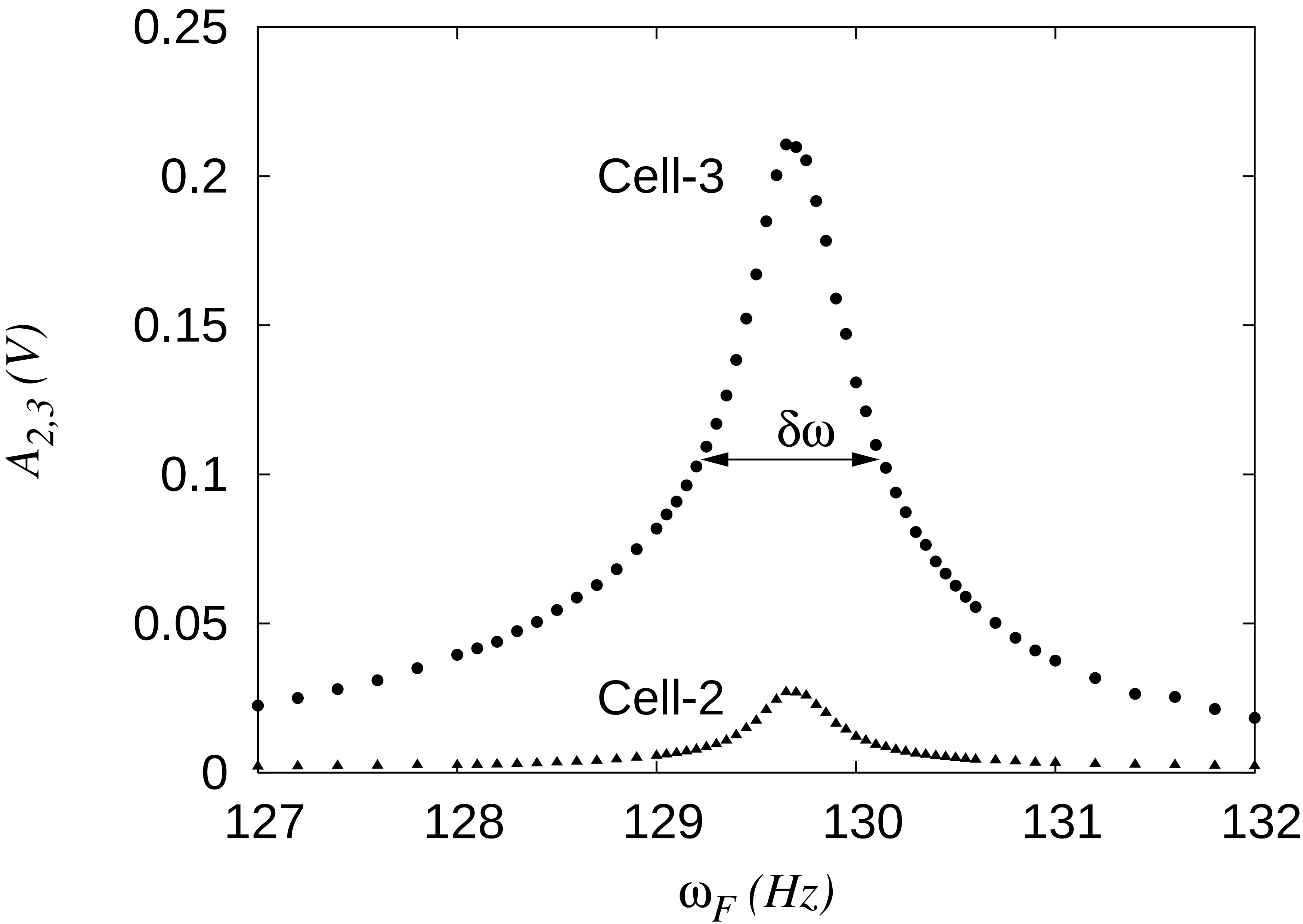
- forcing at frequency ω_f and amplitude ε
- network tuned near Hopf bifurcation with frequency ω_h
- $\lambda < 0$ so that equilibrium is stable
- Three parameters: λ, ϵ, $\omega_f - \omega_h$

Numerics with Aronson

$$g(t) = \varepsilon(e^{i\omega_F t} + 2e^{2i\omega_F t} - 0.5e^{3i\omega_F t}) \quad \lambda = -0.1 \quad \varepsilon = 0.01$$

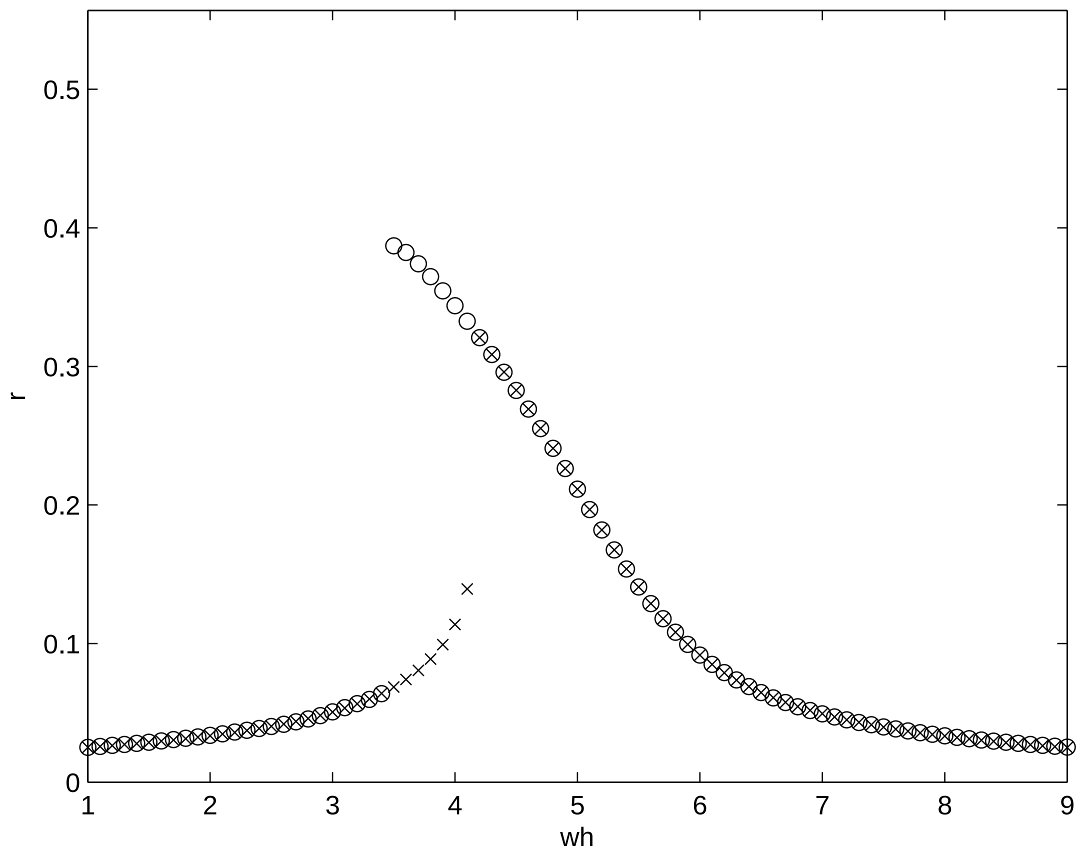


McCullen-Mullin Experiment



More Precisely

- $\omega_f = 5$, $\lambda = -0.109$, $\varepsilon = 0.1$, $\gamma = \mathbf{10}$
- $\dot{z} = (\lambda + \omega_H i - (1 + i\gamma)|z|^2)z + \varepsilon e^{2\pi i \omega_f t}$



Best Guess

- Fix $\lambda < 0$ and $\varepsilon > 0$ near 0
- For all $\gamma > \gamma_c$ there is a region of multiple small amplitude periodic solutions near ω_0 as ω_F is varied
- $\omega_0 \to \omega_H$ and $\gamma_c \to \sqrt{3}$ as $\lambda, \varepsilon \to 0$

Postlethwaite and G. (2008)

Many Thanks To Ian Stewart

Feedforward

Toby Elmhirst *Cook's U*
Matt Nicol *Houston*
Nick McCullen *Bath*
Tom Mullin *Manchester*
Don Aronson *Minnesota*

Quadrupedal Gaits

Luciano Buono *Oshawa*
Jim Collins *Boston U*

Coupled Cells

Reiner Lauterbach *Hamburg*
Maria Leite *Oklahoma*
Marcus Pivato *Trent*
Andrew Török *Houston*

Pictures of Patterns

Mike Gorman *Houston*
Steve Morris *Toronto*