WHICH 2-HYPERSONAL 2-VARIABLE WEIGHTED SHIFTS ARE SUBNORMAL?

RAÚL E. CURTO, SANG HOON LEE, AND JASANG YOON

Abstract. It is well known that a 2-hyponormal unilateral weighted shift with two equal weights must be flat, and therefore subnormal. By contrast, a 2-hyponormal 2-variable weighted shift which is both horizontally flat and vertically flat need not be subnormal. In this paper we identify a large class $S$ of flat 2-variable weighted shifts for which 2-hyponormality is equivalent to subnormality. One measure of the size of $S$ is given by the fact that within $S$ there are hyponormal shifts which are not subnormal.

1. Statement of the Main Results

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. It is well known that the commutativity of the pair is necessary but not sufficient ([1], [18], [19], [20]), and it has recently been shown that the joint hyponormality of the pair is necessary but not sufficient [13]. Our previous work ([9], [10], [13], [14], [15], [24], [25]) has revealed that the nontrivial aspects of LPCS are best detected within the class $H_0$ of commuting pairs of subnormal operators; we thus focus our attention on this class. The class of subnormal pairs on Hilbert space will be denoted by $H_\infty$, and for an integer $k \geq 1$ the class of $k$-hyponormal pairs in $H_0$ will be denoted by $H_k$. Clearly, $H_\infty \subseteq \cdots \subseteq H_k \subseteq \cdots \subseteq H_1 \subseteq H_0$; the main results in [13] and [9] show that these inclusions are all proper. It is then natural to look for subclasses of $H_0$ on which subnormality and $k$-hyponormality agree, that is, classes on which subnormality can be detected with a matricial test.

In this paper we identify a large class $S \subseteq H_0$ on which 2-hyponormality and subnormality agree, that is, $S \cap H_2 = S \cap H_\infty$. Concretely, $S$ consists of all 2-variable weighted shifts $T \equiv (T_1, T_2) \in H_0$ such that $\alpha_{(k_1,0)} = \alpha_{(k_1+1,0)}$ and $\beta_{(0,k_2)} = \beta_{(0,k_2+1)}$ for some $k_1 \geq 1$ and $k_2 \geq 1$, where $\alpha$ and $\beta$ denote the weight sequences of $T_1$ and $T_2$, respectively. One measure of the size of $S$ is given by the fact that hyponormality and subnormality do not agree on $S$, that is $S \cap H_\infty \neq S \cap H_1$. Thus, $S$ consists of nontrivial shifts for which 2-hyponormality and subnormality are equivalent, but for which hyponormality and 2-hyponormality are different; that is, $S$ is small enough to ensure that 2-hyponormality implies subnormality, but large enough to separate hyponormality from subnormality.

Each hyponormal shift in $S$ is flat, that is, $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ and $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$. As a result, each hyponormal shift in $S$ belongs to the class $TC$ of shifts whose core is of tensor form (cf. Definition 2.8); $TC$ is a class that we have studied in detail in [11]. Previously, in [10] we had proved that there exist 2-variable weighted shifts in $TC$ which are hyponormal but not subnormal. By contrast, the 2-hyponormal shifts in $S$ are automatically subnormal.

We prove our main results by combining the 15-point Test for 2-hyponormality [9] with the Subnormal Backward Extension Criterion [13] and Smul’jan’s Test for positivity of operator matrices.

2000 Mathematics Subject Classification. Primary 47B20, 47B37, 47A13; Secondary 44A60, 47-04, 47A20, 28A50.

Key words and phrases. Jointly hyponormal pairs, 2-hyponormal pairs, subnormal pairs, 2-variable weighted shifts, flatness.

Research partially supported by NSF Grants DMS-0099357 and DMS-0400741.

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It is easy to see that \( W \) is never normal, and that it is hyponormal if and only if \( \alpha \geq 1 \) and \( k_2 \geq 1 \). (Here \( \alpha \) and \( \beta \) denote the weight sequences of \( T_1 \) and \( T_2 \), respectively; cf. Section 2).

**Theorem 1.2.** \( S \cap \mathcal{H}_2 = S \cap \mathcal{H}_\infty \).

The generic form of the weight diagram of a hyponormal 2-variable weighted shift in \( S \) is given in Figure 2(ii). Using that notation, we can sharpen Theorem 1.2 as follows.

**Theorem 1.3.** Let \( T \equiv (T_1, T_2) \in S \), with weight diagram given by Figure 2(ii), and assume that \( T \) is 2-hyponormal. Then \( T \) is subnormal, with Berger measure given as

\[
\mu = \frac{1}{\beta^2} \left\{ [b^2(1-x^2) - y^2(1-a^2)]\delta_{0,0} + y^2(1-a^2)\delta_{0,\beta^2} + b^2 x^2 - a^2 y^2]\delta_{(1,0)} + a^2 y^2\delta_{(1,\beta^2)} \right\}
\]

**Theorem 1.4.** \( S \cap \mathcal{H}_\infty \subseteq S \cap \mathcal{H}_1 \).

**Remark 1.5.** Hyponormality alone does not imply flatness. While it is true that in the presence of hyponormality the Six-point Test creates \( L \)-shaped propagation (i.e., \( \alpha_k + \varepsilon_1 = \alpha_k \Rightarrow \alpha_k = \alpha_k + \varepsilon_2 \) and \( \beta_k = \beta_k + \varepsilon_1 \); cf [15, Proof of Theorem 3.3]), without horizontal propagation (as guaranteed by the quadratic hyponormality of \( T_1 \)) this \( L \)-propagation does not result in vertical propagation, needed to eventually lead to flatness. The same phenomenon arises in one variable, where hyponormality is a very soft condition (\( \alpha_k \leq \alpha_{k+1} \) for all \( k \geq 0 \)), while 2-hyponormality is quite rigid. The work in [6] (extending the ideas in [22]) revealed that, for unilateral weighted shifts with two equal weights, 2-hyponormality and subnormality are identical notions. In two variables, however, the analogous result does not hold, as the present work shows.

**Acknowledgment.** Some of the results in this paper were motivated by calculations done with the software tool *Mathematica* [23].

## 2. Notation and Preliminaries

For \( \alpha \equiv \{\alpha_n\}_{n=0}^\infty \) a bounded sequence of positive real numbers (called *weights*), let \( W_\alpha \equiv shift(\alpha_0, \alpha_1, \cdots) : \ell^2(Z_+) \to \ell^2(Z_+) \) be the associated *unilateral weighted shift*, defined by \( W_\alpha e_n := \alpha_n e_{n+1} \) (all \( n \geq 0 \)), where \( \{e_n\}_{n=0}^\infty \) is the canonical orthonormal basis in \( \ell^2(Z_+) \). Two special cases of significant interest are \( U_+ := shift(1,1,\cdots) \) (the (unweighted) unilateral shift) and \( S_a := shift(a,1,1,\cdots) \) (\( 0 < a < 1 \)). The *moments* of \( \alpha \) are given as

\[
\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0 \end{cases}
\]

It is easy to see that \( W_\alpha \) is never normal, and that it is hyponormal if and only if \( \alpha_0 \leq \alpha_1 \leq \cdots \). Similarly, consider double-indexed positive bounded sequences \( \alpha_k, \beta_k \in \ell^\infty(Z^2_+) \), \( k \equiv (k_1, k_2) \in Z^2_+ := Z_+ \times Z_+ \) and let \( \ell^2(Z^2_+) \) be the Hilbert space of square-summable complex sequences indexed by \( Z^2_+ \). We define the *2-variable weighted shift* \( T \equiv (T_1, T_2) \) by

\[
T_1 e_k := \alpha_k e_{k+1} \\
T_2 e_k := \beta_k e_{k+\varepsilon_2}
\]
where \( \varepsilon_1 := (1,0) \) and \( \varepsilon_2 := (0,1) \). Clearly,
\[
T_1 T_2 = T_2 T_1 \iff \beta_{k+\varepsilon_1} \alpha_k = \alpha_{k+\varepsilon_2} \beta_k \quad (all \ k \in \mathbb{Z}_+^2).
\] (2.1)

Trivially, a pair of unilateral weighted shifts \( W_\alpha \) and \( W_\beta \) gives rise to a 2-variable weighted shift \( T \equiv (T_1,T_2) \). Let \( \alpha_{(k_1,k_2)} := \alpha_{k_1} \) and \( \beta_{(k_1,k_2)} := \beta_{k_2} \) (all \( k_1,k_2 \in \mathbb{Z}_+ \)). In this case, \( T \) is subnormal (resp. hyponormal) if and only if so are \( T_1 \) and \( T_2 \); in fact, under the canonical identification of \( \ell^2(\mathbb{Z}_+^2) \) with \( \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \), we have \( T_1 \equiv I \otimes W_\alpha \) and \( T_2 \equiv W_\beta \otimes I \), so \( T \) is also doubly commuting.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [4, III.8.16]), and independently established by Gellar and Wallen [16]: \( W_\alpha \) is subnormal if and only if there exists a probability measure \( \xi \) supported in \([0,\|W_\alpha\|^2]\), with \( \|W_\alpha\|^2 \in \text{supp} \xi \), and such that \( \gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int s^k \, d\xi(s) \quad (k \geq 1) \).

We also recall the notion of moment of order \( k \) for a pair \((\alpha, \beta)\) satisfying (2.1). Given \( k \in \mathbb{Z}_+^2 \), the moment of \((\alpha, \beta)\) of order \( k \) is
\[
\gamma_k \equiv \gamma_{k}(\alpha, \beta) := \begin{cases} 
1 & \text{if } k = 0 \\
\alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\
\beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\
\alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1
\end{cases}
\] (2.2)

We remark that, due to the commutativity condition (2.1), \( \gamma_k \) can be computed using any nondecreasing path from \((0,0)\) to \((k_1,k_2)\). Moreover, \( T \) is subnormal if and only if there is a regular Borel probability measure \( \mu \) defined on the 2-dimensional rectangle \( R = [0,a_1] \times [0,a_2] \) \((a_i := \|T_i\|^2)\) such that
\[
\gamma_k = \int_R s^{k_1} t^{k_2} \, d\mu(s,t) \quad (all \ k \in \mathbb{Z}_+^2)[17].
\] (2.3)

Let \( \mathcal{H} \) be a complex Hilbert space and let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of bounded linear operators on \( \mathcal{H} \). For \( S, T \in \mathcal{B}(\mathcal{H}) \) let \( [S,T] := ST - TS \). We say that an \( n \)-tuple \( T = (T_1, \cdots, T_n) \) of operators on \( \mathcal{H} \) is (jointly) hyponormal if the operator matrix \( [T^*, T] := ([T^*_i, T_i])_{i,j=1}^n \) is positive on the direct sum of \( n \) copies of \( \mathcal{H} \) (cf. [2], [12]). The \( n \)-tuple \( T \) is said to be normal if \( T \) is commuting and each \( T_i \) is normal, and \( T \) is subnormal if \( T \) is the restriction of a normal \( n \)-tuple to a common invariant subspace. Clearly, normal \( \implies \) subnormal \( \implies \) hyponormal. Moreover, the restriction of a hyponormal \( n \)-tuple to an invariant subspace is again hyponormal. The Bram-Halmos criterion states that an operator \( T \in \mathcal{B}(\mathcal{H}) \) is subnormal if and only if the \( k \)-tuple \((T, T^2, \cdots, T^k)\) is hyponormal for all \( k \geq 1 \); we say that \( T \) is \( k \)-hyponormal when the latter condition holds. On the other hand, for a commuting pair \( T \equiv (T_1, T_2) \) of operators on Hilbert space we have

**Definition 2.1.** (cf. [9]) A commuting pair \( T \equiv (T_1, T_2) \) is called \( k \)-hyponormal if \( T(k) := (T_1, T_2, T_1^2, T_2 T_1, T_2^2, \cdots, T_1^k, T_2 T_1^{k-1}, \cdots, T_2^k) \) is hyponormal, or equivalently
\[
([T_1^2 T_2^* T_2^* T_1^* T_1^* T_1, T_2^{m+n-1}, T_2^m T_1^n])_{1 \leq p+q \leq k} \geq 0.
\]

Clearly, subnormal \( \implies \) \((k+1)\)-hyponormal \( \implies \) \( k \)-hyponormal for every \( k \geq 1 \), and of course \( 1 \)-hyponormality agrees with the usual definition of joint hyponormality (as above). In [9] we obtained the following multivariable version of the Bram-Halmos criterion for subnormality, which provided an abstract answer to the LPCS, by showing that no matter how \( k \)-hyponormal the pair \( T \) might be, it may still fail to be subnormal.

**Theorem 2.2.** ([9, Theorem 2.3]) Let \( T \equiv (T_1, T_2) \) be a commuting pair of subnormal operators on a Hilbert space \( \mathcal{H} \). The following statements are valid.
(i) $T$ is subnormal.  
(ii) $T$ is $k$-hyponormal for all $k \in \mathbb{Z}_+$.  

In the single variable case, there are useful criteria for $k$-hyponormality ([6], [8]); for 2-variable weighted shifts, a simple criterion for joint hyponormality was given in (5). The following characterization of $k$-hyponormality for 2-variable weighted shifts was given in [9, Theorem 2.4].

**Theorem 2.3.** Let $T \equiv (T_1, T_2)$ be a 2-variable weighted shift with weight sequences $\alpha \equiv \{\alpha_k\}$ and $\beta \equiv \{\beta_k\}$. The following statements are equivalent.

(a) $T$ is $k$-hyponormal.

(b) $M_u(k) := (\gamma_{u + (m,n) + (p,q)})_{0 \leq m+n \leq k} \geq 0$ for all $u \in \mathbb{Z}_+^2$. (For a subnormal pair $T$, the matrix $M_u(k)$ is the truncation of the moment matrix associated to the Berger measure of $T$.)

The following special cases of Theorem 2.3 will be essential for our work.

**Lemma 2.4.** ([5]) (Six-point Test; cf. Figure 1(i)) Let $T \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences $\alpha$ and $\beta$. Then

$$[T^*, T] \geq 0 \iff M(k) \geq 0 \text{ (all } k \in \mathbb{Z}_+^2)$$

$$\iff \begin{pmatrix} \alpha_k^2 - \alpha_k^2 & \alpha_k \beta_k - \alpha_k \beta_k & \beta_k^2 - \beta_k^2 \\ \alpha_k \beta_k - \alpha_k \beta_k & \alpha_k^2 - \alpha_k^2 & \beta_k^2 - \beta_k^2 \\ \beta_k^2 - \beta_k^2 & \beta_k^2 - \beta_k^2 & \alpha_k^2 - \alpha_k^2 \end{pmatrix} \geq 0 \text{ (all } k \in \mathbb{Z}_+^2).$$

**Figure 1.** Weight diagrams used in the Six-point Test and 15-point Test, respectively.
Lemma 2.5. ([9]) (15-point Test; cf. Figure 1(ii)) If $T \equiv (T_1, T_2)$ is 2-variable weighted shift with weight sequence $\alpha \equiv \{\alpha_k\}$ and $\beta \equiv \{\beta_k\}$, then $T$ is 2-hyponormal if and only if

$$M(k_1, k_2)(2) \equiv \begin{pmatrix} \gamma_{k_1, k_2} & \gamma_{k_1+1, k_2} & \gamma_{k_1+2, k_2} & \gamma_{k_1+1, k_2+1} & \gamma_{k_1+2, k_2+1} \\ \gamma_{k_1+1, k_2+1} & \gamma_{k_1+2, k_2+1} & \gamma_{k_1+3, k_2+1} & \gamma_{k_1+2, k_2+2} & \gamma_{k_1+3, k_2+2} \\ \gamma_{k_1+2, k_2+1} & \gamma_{k_1+3, k_2+2} & \gamma_{k_1+4, k_2+2} & \gamma_{k_1+3, k_2+3} & \gamma_{k_1+4, k_2+3} \\ \gamma_{k_1+1, k_2+2} & \gamma_{k_1+2, k_2+3} & \gamma_{k_1+3, k_2+3} & \gamma_{k_1+2, k_2+4} & \gamma_{k_1+3, k_2+4} \end{pmatrix} \geq 0,$$

for all $k \in \mathbb{Z}_+^2$. We now recall the following notation and terminology from [13]:

(i) given a probability measure $\mu$ on $X \times Y \equiv \mathbb{R} \times \mathbb{R_+}$, with $\frac{1}{t} \in L^1(\mu)$, the extremal measure $\mu_{\text{ext}}$ (which is also a probability measure) on $X \times Y$ is given by $d\mu_{\text{ext}}(s, t) := (1 - \delta_0(t)) \frac{1}{t} \| \frac{1}{t} \|_{L^1(\mu)} d\mu(s, t)$; and

(ii) given a measure $\mu$ on $X \times Y$, the marginal measure $\mu^X$ is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto $X$. Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$.

We now list some results which are needed in the proof of Theorem 3.1.

Lemma 2.6. (cf. [21], [7, Proposition 2.2]) Let $M \equiv \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}$ be a $2 \times 2$ operator matrix, where $A$ and $C$ are square matrices and $B$ is a rectangular matrix. Then

$$M \geq 0 \iff \text{there exists } W \text{ such that } \begin{cases} C \geq 0 \\ B = CW \\ A \geq W^*CW. \end{cases}$$

For Lemma 2.7, Definition 2.8 and Theorem 3.1 the following two subspaces of $\ell^2(\mathbb{Z}_+^2)$ will be needed: $M := \bigvee \{e_k : k_2 \geq 1\}$ and $N := \bigvee \{e_k : k_1 \geq 1\}$.

Lemma 2.7. ([13]) (Subnormal backward extension of a 2-variable weighted shift) Consider the 2-variable weighted shift whose weight diagram is given in Figure 2(i), and let $T_{\mathcal{M}}$ denote the restriction of $T$ to $\mathcal{M}$. Assume that $T_{\mathcal{M}}$ is subnormal with associated measure $\mu_{\mathcal{M}}$, and that $W_0 := \text{shift}(\alpha_{00}, \alpha_{10}, \cdots)$ is subnormal with associated measure $\nu$. Then $T$ is subnormal if and only if (i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$; (ii) $\beta_{00}^2 \leq \left( \| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}})} \right)^{-1}$; (iii) $\beta_{00}^2 \| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}})} \|_{L^1(\mu_{\mathcal{M}})} < \nu$. Moreover, if $\beta_{00}^2 \| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}})} = 1$, then $(\mu_{\mathcal{M}})_{\text{ext}} = \nu$. In the case when $T$ is subnormal, Berger measure $\mu$ of $T$ is given by

$$d\mu(s, t) = \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})(s, t) + (dv(s) - \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})(s, t))d\delta_0(t).$$

J. Stampfli showed in [22] that for subnormal weighted shifts $W_\alpha$, a propagation phenomenon occurs that forces the flatness of $W_\alpha$ whenever two equal weights are present. Later, R.E. Curto proved in [6] that a hyponormal weighted shift with three equal weights cannot be quadratically hyponormal without being flat: If $W_\alpha$ is quadratically hyponormal and $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$. Y. Choi [3] improved this result, that is, if $W_\alpha$ be quadratically hyponormal and $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then $W_\alpha$ is flat, that is, $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \alpha_1, \cdots)$. In Theorem 2.10 we show that similar propagation phenomena occur for 2-hyponormal 2-variable weighted shifts $T$. In two variables, the flatness of $T$ is captured by the so-called core of $T$, $c(T)$ (cf. Definition 2.8 below), as follows: $T$ is flat when $c(T)$ is a 2-variable weighted shift of tensor form.
Definition 2.8. (i) The core of a 2-variable weighted shift $T$ is the restriction of $T$ to $M \cap N$, in symbols, $c(T) := T|_{M \cap N}$.

(ii) A 2-variable weighted shift $T$ is said to be of tensor form if $T \cong (I \otimes W_\alpha, W_\beta \otimes I)$. When $T$ is subnormal, this is equivalent to requiring that the Berger measure be a Cartesian product $\xi \otimes \eta$.

(iii) The class of all 2-variable weighted shift $T \in \mathcal{H}_0$ whose core is of tensor form will be denoted by $\mathcal{C}$, that is, $\mathcal{T} := \{T \in \mathcal{H}_0 : c(T) \text{ is of tensor form}\}$.

We now recall that a 2-variable weighted shift $T$ is said to be horizontally flat when $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geq 1$; and we call $T$ vertically flat when $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$. We say $T$ is flat if $T$ is horizontally and vertically flat, and that $T$ is symmetrically flat if $T$ is flat and $\alpha_{(1,1)} = \beta_{(1,1)}$.

The next result shows the extent to which propagation holds in the presence of (joint) hyponormality.

Proposition 2.9. ([15]) Let $T$ be a commuting hyponormal 2-variable weighted shift whose weight diagram is given in Figure 2(i). Then

(i) If $\alpha_{k+\varepsilon_1} = \alpha_k$ for some $k$, then $\alpha_{k+\varepsilon_2} = \alpha_k$ and $\beta_{k+\varepsilon_1} = \beta_k$;

(ii) If $T_1$ is quadratically hyponormal, if $T_2$ is subnormal, and if $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 0$, then $T$ is horizontally flat.

On the other hand, under the assumption of 2-hyponormality (and without assuming the subnormality of either $T_1$ or $T_2$), we can prove that $T$ is flat whenever $\alpha_{k+\varepsilon_1} = \alpha_k$ and $\beta_{m+\varepsilon_2} = \beta_m$ for some $k$ and $m$. We do this using the 15-point Test (Lemma 2.5).

Theorem 2.10. Let $T$ be a 2-hyponormal 2-variable weighted shift. If $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $T$ is horizontally flat. If instead, $\beta_{(k_1,k_2)+\varepsilon_2} = \beta_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $T$ is vertically flat.

Proof. Without loss of generality, assume $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} = 1$. Then by the above mentioned result of Y. Choi and Proposition 2.9(i), we can see that $T_1$ is of tensor form when restricted to the subspace $\mathcal{V}\{e_m : m_1 \geq 1 \text{ and } m_2 \geq 2\}$. If $k_2 = 1$ we are done, so assume $k_2 \geq 2$ and consider the matrix

$$M_{(k_1,k_2-2)(2)} = \begin{pmatrix}
\gamma_{k_1,k_2-2} & \gamma_{k_1+1,k_2-2} & \gamma_{k_1,k_2-1} & \gamma_{k_1+2,k_2-2} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1,k_2} \\
\gamma_{k_1+1,k_2-2} & \gamma_{k_1+2,k_2-2} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1+3,k_2-2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} \\
\gamma_{k_1,k_2-1} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1,k_2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} & \gamma_{k_1+1,k_2+1} \\
\gamma_{k_1+2,k_2-2} & \gamma_{k_1+3,k_2-2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+4,k_2-2} & \gamma_{k_1+3,k_2-1} & \gamma_{k_1+2,k_2+1} \\
\gamma_{k_1+1,k_2-1} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} & \gamma_{k_1+3,k_2-1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} \\
\gamma_{k_1,k_2} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1+2,k_2+2}
\end{pmatrix},$$

which we know is positive semidefinite since $T$ is 2-hyponormal (Lemma 2.5). It suffices to prove that $\alpha_{k-\varepsilon_2} = 1$. Let us focus on the principal submatrix $M \geq 1$ determined by rows and columns 1, 3 and 5. From (2.2) we have

$$M = \begin{pmatrix}
1 & \beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2}
\beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2}
\beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2}
\beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2} & \beta^2_{k-2\varepsilon_2}
\end{pmatrix} \geq 1.
$$

(2.4)

For notational convenience, let $a := \beta^2_{k-2\varepsilon_2}$, $b := \beta^2_{k-\varepsilon_2}$ and $c := \alpha^2_{k-\varepsilon_2}$. Thus,

$$M \equiv \begin{pmatrix}
1 & a & ac & ab & ab \\
a & ab & ab & ab & ab \\
ac & ab & ab & ab & ab \\
ab & ab & ab & ab & ab \\
ab & ab & ab & ab & ab
\end{pmatrix} \geq 0 \iff N := \begin{pmatrix}
ab - a^2 & ab & ab & ab \\
ab & ab & ab & ab \\
ab & ab & ab & ab \\
ab & ab & ab & ab \\
ab & ab & ab & ab
\end{pmatrix} \geq 0.$$
If \( a = b \) then we must necessarily have \( c = 1 \), that is, \( \alpha_{k-\varepsilon_2} = 1 \), as desired. Assume, therefore, that \( a < b \). It follows that

\[
N \geq 0 \iff \det N \geq 0 \iff (ab - a^2)(ab - a^2e^2) - (ab - a^2e^2)^2 \geq 0 \\
\iff -a^3b(c - 1)^2 \geq 0 \iff c = 1,
\]

that is, \( \alpha_{k-\varepsilon_2} = 1 \), as desired. \( \square \)

3. Proofs of the Main Results

We are now ready to prove our main results, which we restate for the reader’s convenience.

**Theorem 3.1.** \( S \cap \delta_2 = S \cap \delta_{\infty} \).

**Proof.** By Proposition 2.9 and Theorem 2.10, we can assume, without loss of generality, that \( \alpha_{(k_1, k_2)} = \alpha_{(1,0)} = 1 \) (all \( k_1 \geq 1, k_2 \geq 0 \)) and \( \beta_{(k_1, k_2)} = \beta_{(0,1)} \leq 1 \) (all \( k_1 \geq 0, k_2 \geq 1 \)). For notational convenience, we let \( a := \alpha_{(0,1)} < 1 \), \( b := \beta_{(0,1)} \leq 1 \), \( x := \alpha_{(0,0)} < 1 \), and \( y := \beta_{(0,0)} < 1 \). We summarize this information in Figure 2(ii).

![Figure 2](image)

**Figure 2.** Weight diagrams of the 2-variable weighted shifts in Lemma 2.7 and Theorem 3.1, respectively.

Let \( C_k := \{(a, b, x, y) : T \in \delta_k \} \) (\( k = 1, 2, \infty \)). Clearly, \( C_\infty \subseteq C_2 \subseteq C_1 \). We first describe concretely the set \( C_2 \) in terms of necessary and sufficient conditions on the four parameters \( a, b, x \) and \( y \) that guarantee the 2-hyponormality of the pair \( T \). We then establish that \( C_2 \subseteq C_\infty \). Finally, in Theorem 3.3 we will show that \( C_2 \not\subseteq C_1 \).

It is straightforward to verify that \( T|_M \cong (I \otimes shift(a, 1, 1, \cdots), bU_+ \otimes I) \) and \( T|_N \cong (I \otimes U_+, shift(\frac{a}{y}, b, b, \cdots) \otimes I) \); thus, \( T|_M \) and \( T|_N \) are subnormal. As a consequence, to describe \( C_2 \)
we only need to apply the 15-point Test (Lemma 2.5) at \(k = (0,0)\), that is, we need to guarantee that \(M_{(0,0)}(2) \geq 0\). We thus have:

\[ \mathbf{T} \in \mathcal{S}_2 \iff M_{(0,0)}(2) \geq 0. \]

Now, since the moments \(\gamma_k\) \((k \in \mathbb{Z}_+^2)\) associated with \(\mathbf{T}\) are

\[
\gamma_k = \begin{cases} 
1 & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\
x^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\
y^2 b^{2(k_2-1)} & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\
a^2 y^2 b^{2(k_2-1)} & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1,
\end{cases}
\]

it follows that

\[
M_{(0,0)}(2) \equiv \begin{pmatrix} 
x^2 & y^2 & x^2 & a^2 y^2 & b^2 y^2 \\
y^2 & a^2 y^2 & x^2 & a^2 y^2 & a^2 b^2 y^2 \\
x^2 & a^2 y^2 & b^2 y^2 & a^2 y^2 & b^4 y^2 \\
a^2 y^2 & b^2 y^2 & a^2 b^2 y^2 & a^2 b^2 y^2 & b^4 y^2 \\
b^2 y^2 & a^2 b^2 y^2 & b^4 y^2 & a^2 b^2 y^2 & a^2 b^2 y^2 \\
b^2 y^2 & a^2 b^2 y^2 & b^4 y^2 & a^2 b^2 y^2 & a^2 b^2 y^2
\end{pmatrix} \geq 0
\]

\[
\iff M := \begin{pmatrix} 
x^2 & y^2 & a^2 y^2 \\
y^2 & a^2 y^2 & b^2 y^2 \\
a^2 y^2 & b^2 y^2 & a^2 b^2 y^2 \\
a^2 y^2 & b^2 y^2 & a^2 b^2 y^2 \\
b^2 y^2 & a^2 b^2 y^2 & b^4 y^2 \\
b^2 y^2 & a^2 b^2 y^2 & a^2 b^2 y^2
\end{pmatrix} \geq 0
\]

(since the sixth row is a multiple of the third row, and the second and fourth rows are identical).

If we now interchange the second and third rows and columns, we see that the positivity of \(M\) is determined by the positivity of

\[
\left( \begin{array}{cc}
1 & y^2 \\
y^2 & b^2 y^2 \\
x^2 & a^2 y^2 \\
a^2 y^2 & b^2 y^2 \\
b^2 y^2 & a^2 b^2 y^2 \\
a^2 b^2 y^2 & a^2 b^2 y^2 \\
\end{array} \right)
\left( \begin{array}{cc}
x^2 & a^2 y^2 \\
a^2 y^2 & b^2 y^2 \\
a^2 b^2 y^2 & a^2 b^2 y^2 \\
\end{array} \right).
\]

Thus, from Lemma 2.6 (with \(C = B\) and \(W = I\)), we have

\[ M \geq 0 \iff A \geq B \geq 0. \]

Now observe that \(B \geq 0 \iff ay \leq bx\) and

\[ A - B = \left( \begin{array}{cc}
1 - x^2 & y^2(1 - a^2) \\
y^2(1 - a^2) & b^2 y^2(1 - a^2) \\
\end{array} \right). \]

Since the \((1,1)\)-entry of \(A - B\) is always positive, the positivity of \(A - B\) is completely determined by its determinant; that is,

\[ A - B \geq 0 \iff \det(A - B) \geq 0 \iff y^2(1 - a^2) \leq b^2(1 - x^2). \]

It follows that

\[ \mathbf{T} \in \mathcal{S}_2 \iff ay \leq bx \text{ and } y^2(1 - a^2) \leq b^2(1 - x^2). \tag{3.1} \]

We thus see that

\[
\begin{align*}
C_2 &= \{(a, b, x, y) : 0 < x < 1, \ 0 < y < 1, \ ay \leq bx, \text{ and } y^2(1 - a^2) \leq b^2(1 - x^2)\}. \tag{3.2}
\end{align*}
\]
We will now prove that \( C_2 \subseteq C_\infty \). Let \((a, b, x, y) \in C_2\). Let \( z := \frac{ay}{b} \).

**Case 1.** If \( z = b \), then \( T|_{\mathcal{N}} \equiv (I \otimes \mathcal{U}_b, b_{\mathcal{U}_b} \otimes I) \), and a straightforward application of Lemma 2.7 in the \( s \) direction shows that \( T \) is subnormal if and only if \( bx \leq y \) if and only if \( a \leq 1 \) (since \( bx = ay \)), which is true. Thus, \((a, b, x, y) \in C_\infty\).

**Case 2.** Assume now that \( z < 1 \). Since \( T|_{\mathcal{N}} \equiv (I \otimes \mathcal{U}_b, bS_{z/b} \otimes I) \) is subnormal with Berger measure \( \mu_{\mathcal{N}} \equiv \delta_1 \times (1 - \frac{z^2}{b^2})\delta_0 + \frac{z^2}{b^2}\delta_{b^2} \), we can think of \( T \) as a backward extension of \( T|_{\mathcal{N}} \) (in the \( s \) direction) and apply Lemma 2.7. Note that \( \frac{1}{b} \| \frac{1}{b}^{1/2} \|_{L^1(\mu_{\mathcal{N}})} = 1 \), so \( d(\mu_{\mathcal{N}})_{\text{ext}}(s, t) \equiv (1 - \delta_0(s))\frac{1}{b}d\mu_{\mathcal{N}}(s, t) = d\delta_1(s)[(1 - \frac{z^2}{b^2})\delta_0(t) + \frac{z^2}{b^2}d\delta_{b^2}(t)] \) and \( \alpha_{b, \theta}^2 \frac{1}{b} \| \frac{1}{b}^{1/2} \|_{L^1(\mu_{\mathcal{N}})^Y} = \frac{b^2}{1 - \delta_0} x^2 [(1 - \frac{z^2}{b^2})\delta_0 + \frac{z^2}{b^2}\delta_{b^2}] \). Thus, by Lemma 2.7,

\[
T \text{ is subnormal } \iff \alpha_{b, \theta}^2 \frac{1}{b} \| \frac{1}{b}^{1/2} \|_{L^1(\mu_{\mathcal{N}})^Y} \leq \eta_0
\]

(3.3)

as in the last condition in (3.2). Therefore, \( T \) is subnormal, and the proof is complete.

**Theorem 3.2.** Let \( T \equiv (T_1, T_2) \in \mathcal{S}_2 \), with weight diagram given by Figure 2(ii), and assume that \( T \) is 2-hyponormal. Then \( T \) is subnormal, with Berger measure given as

\[
\mu = \frac{1}{b^2} \{ [b^2(1 - x^2) - y^2(1 - a^2)]\delta_{(0, 0)} + y^2(1 - a^2)\delta_{(0, b^2)} + (y^2 x^2 - a^2 y^2)\delta_{(1, 0)} + a^2 y^2 \delta_{(1, b^2)} \}
\]

(3.4)

**Proof.** We apply the main result in [11], in the special form needed here; cf. [11, Proposition 3.1]. For a 2-variable subnormal weighted shift with weight diagram given by Figure 2(ii), the Berger measure is

\[
\mu = \varphi \times (\delta_0 - \delta_{b^2}) + y^2 \frac{1}{t} \| \frac{1}{t} \|_{L^1(\psi)} \delta_0 \times (\tilde{\psi} - \delta_{b^2}) + \xi_x \times \delta_{b^2},
\]

(3.5)

where \( \psi = (1 - a^2)\delta_{b^2}, \varphi = \xi_x - y^2 \frac{1-a^2}{b^2} \delta_0 - a^2 \frac{z^2}{b^2} \delta_1, \quad d\tilde{\psi}(t) := \frac{1}{t} \| \frac{1}{t} \|_{L^1(\psi)}d\psi(t) \) and \( \xi_x \) is the Berger measure of \( S_x \). A straightforward calculation shows that \( \tilde{\psi} = \delta_{b^2}, \) so that (3.5) becomes

\[
\mu = \{ [b^2(1 - x^2) - y^2(1 - a^2)]\delta_0 + b^2 x^2 - a^2 y^2 \delta_1 \} \times (\delta_0 - \delta_{b^2})
\]

\[+ [(1 - x^2)\delta_0 + x^2 \delta_1] \times \delta_{b^2},
\]

which easily leads to the desired formula (3.4).

We conclude this section with a proof of Theorem 1.4, which we reformulate in terms of \( C_1 \) and \( C_2 \). Toward this end, we need a description of \( C_1 := \{(a, b, x, y) : T \in \mathcal{S}_1\} \): by the Six-point Test (Lemma 2.4), \( T \) is hyponormal if and only if

\[
M_{(0, 0)}(1) \equiv \begin{pmatrix}
1 & x^2 & y^2 \\
x^2 & a^2 y^2 \\
y^2 & a^2 y^2 & b^2 y^2
\end{pmatrix} \geq 0.
\]

**Theorem 3.3.** \( C_2 \subseteq C_1 \).
Proof. Since $x < 1$, $M_{(0,0)}(1) \geq 0$ if and only if $\det M_{(0,0)}(1) \geq 0$, that is, if and only if

$$P_1 := y^2(b^2x^2 - a^4y^2 - b^2x^4 + 2a^2x^2y^2 - x^2y^2) \geq 0.$$ 

Let $x > \frac{\sqrt{2}}{2}$ and let $a := \sqrt{2x^2 - 1}$. It follows that $1 - a^2 = 2(1 - x^2)$, so that

$$P_2 := b^2(1 - x^2) - y^2(1 - a^2) = b^2(1 - x^2) - 2y^2(1 - x^2) = (b^2 - 2y^2)(1 - x^2),$$

and it suffices to choose $y$ such that $\frac{\sqrt{2}}{2} b < y < b$ to make $P_2 < 0$, and thus break 2-hyponormality (cf. (3.1)). On the other hand,

$$P_1 \equiv P_2x^2y^2 + y^4a^2(x^2 - a^2) = y^2(b^2x^2 - b^2x^4 - y^2 + x^2y^2) = (b^2x^2 - y^2)(1 - x^2),$$

which can be made nonnegative by taking $x$ close to 1. This shows that $C_2 \subsetneq C_1$. \hfill \Box

References


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