WHICH 2-HYPONORMAL 2-VARIABLE WEIGHTED SHIFTS ARE SUBNORMAL?

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Abstract. It is well known that a 2-hyponormal unilateral weighted shift with two equal weights must be flat, and therefore subnormal. By contrast, a 2-hyponormal 2-variable weighted shift which is both horizontally flat and vertically flat need not be subnormal. In this paper we identify a large class $\mathcal{S}$ of flat 2-variable weighted shifts for which 2-hyponormality is equivalent to subnormality. One measure of the size of $\mathcal{S}$ is given by the fact that within $\mathcal{S}$ there are hyponormal shifts which are not subnormal.

1. Statement of the Main Results

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. It is well known that the commutativity of the pair is necessary but not sufficient ([1], [18], [19], [20]), and it has recently been shown that the joint hyponormality of the pair is necessary but not sufficient [13]. Our previous work ([9], [10], [13], [14], [15], [24], [25]) has revealed that the nontrivial aspects of LPCS are best detected within the class $\mathfrak{H}_0$ of commuting pairs of subnormal operators; we thus focus our attention on this class. The class of subnormal pairs on Hilbert space will be denoted by $\mathfrak{H}_\infty$, and for an integer $k \geq 1$ the class of k-hyponormal pairs in $\mathfrak{H}_0$ will be denoted by $\mathfrak{H}_k$. Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$; the main results in [13] and [9] show that these inclusions are all proper. It is then natural to look for subclasses of $\mathfrak{H}_0$ on which subnormality and k-hyponormality agree, that is, classes on which subnormality can be detected with a matricial test.

In this paper we identify a large class $\mathcal{S} \subseteq \mathfrak{H}_0$ on which 2-hyponormality and subnormality agree, that is, $\mathcal{S} \bigcap \mathfrak{H}_2 = \mathcal{S} \bigcap \mathfrak{H}_\infty$. Concretely, $\mathcal{S}$ consists of all 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ such that $\alpha_{(k_1,0)} = \alpha_{(k_1+1,0)}$ and $\beta_{(0,k_2)} = \beta_{(0,k_2+1)}$ for some $k_1 \geq 1$ and $k_2 \geq 1$, where α and β denote the weight sequences of T_1 and T_2, respectively. One measure of the size of $\mathcal{S}$ is given by the fact that hyponormality and subnormality do not agree on $\mathcal{S}$, that is $\mathcal{S} \bigcap \mathfrak{H}_\infty \subsetneqq \mathcal{S} \bigcap \mathfrak{H}_1$. Thus, $\mathcal{S}$ consists of nontrivial shifts for which 2-hyponormality and subnormality are equivalent, but for which hyponormality and 2-hyponormality are different; that is, $\mathcal{S}$ is small enough to ensure that 2-hyponormality implies subnormality, but large enough to separate hyponormality from subnormality.

Each hyponormal shift in $\mathcal{S}$ is *flat*, that is, $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ and $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$. As a result, each hyponormal shift in $\mathcal{S}$ belongs to the class $\mathcal{TC}$ of shifts whose core is of tensor form (cf. Definition 2.8); $\mathcal{TC}$ is a class that we have studied in detail in [11]. Previously, in [10] we had proved that there exist 2-variable weighted shifts in $\mathcal{TC}$ which are hyponormal but not subnormal. By contrast, the 2-hyponormal shifts in $\mathcal{S}$ are automatically subnormal.

We prove our main results by combining the 15-point Test for 2-hyponormality [9] with the Subnormal Backward Extension Criterion [13] and Smul'jan's Test for positivity of operator matrices

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[21]. As a first step, we prove a propagation result for 2-hyponormal 2-variable weighted shifts (Theorem 2.10). We then seek conditions to guarantee the subnormality of $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_2$.

Our main results follow; first, we need a definition.

Definition 1.1. *$\mathcal{S} := \{\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0 : \alpha_{(k_1,0)} = \alpha_{(k_1+1,0)}$ and $\beta_{(0,k_2)} = \beta_{(0,k_2+1)}$ for some $k_1 \geq 1$ and $k_2 \geq 1\}$. (Here α and β denote the weight sequences of T_1 and T_2, respectively; cf. Section 2).*

Theorem 1.2. *$\mathcal{S} \bigcap \mathfrak{H}_2 = \mathcal{S} \bigcap \mathfrak{H}_\infty$.*

The generic form of the weight diagram of a hyponormal 2-variable weighted shift in $\mathcal{S}$ is given in Figure 2(ii). Using that notation, we can sharpen Theorem 1.2 as follows.

Theorem 1.3. *Let $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{S}$, with weight diagram given by Figure 2(ii), and assume that $\mathbf{T}$ is 2-hyponormal. Then $\mathbf{T}$ is subnormal, with Berger measure given as*

$$\begin{aligned} \mu \quad = \quad & \frac{1}{b^2}\{[b^2(1-x^2) - y^2(1-a^2)]\delta_{(0,0)} \\ & + y^2(1-a^2)\delta_{(0,b^2)} + (b^2x^2 - a^2y^2)\delta_{(1,0)} + a^2y^2\delta_{(1,b^2)}\}. \end{aligned}$$

Theorem 1.4. *$\mathcal{S} \bigcap \mathfrak{H}_\infty \subsetneq \mathcal{S} \bigcap \mathfrak{H}_1$.*

Remark 1.5. Hyponormality alone does not imply flatness. While it is true that in the presence of hyponormality the Six-point Test creates L-shaped propagation (i.e., $\alpha_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}} \Longrightarrow \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}$ and $\beta_{\mathbf{k}} = \beta_{\mathbf{k}+\varepsilon_1}$; cf [15, Proof of Theorem 3.3]), without horizontal propagation (as guaranteed by the quadratic hyponormality of T_1) this L-propagation does not result in vertical propagation, needed to eventually lead to flatness. The same phenomenon arises in one variable, where hyponormality is a very soft condition ($\alpha_k \leq \alpha_{k+1}$ for all $k \geq 0$), while 2-hyponormality is quite rigid. The work in [6] (extending the ideas in [22]) revealed that, for unilateral weighted shifts with two equal weights, 2-hyponormality and subnormality are identical notions. In two variables, however, the analogous result does not hold, as the present work shows.

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2. Notation and Preliminaries

For $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha \equiv shift(\alpha_0, \alpha_1, \cdots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. Two special cases of significant interest are $U_+ := shift(1, 1, \cdots)$ (the (unweighted) unilateral shift) and $S_a := shift(a, 1, 1, \cdots)$ $(0 < a < 1)$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cl} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. We define the 2-*variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

$$T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \ \text{ (all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{2.1}$$

Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+$). In this case, $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ with $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, we have $T_1 \cong I \bigotimes W_\alpha$ and $T_2 \cong W_\beta \bigotimes I$, so $\mathbf{T}$ is also doubly commuting.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [4, III.8.16]), and independently established by Gellar and Wallen [16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$, with $\|W_\alpha\|^2 \in supp\xi$, and such that $\gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int s^k \, d\xi(s)$ $(k \geq 1)$.

We also recall the notion of moment of order $\mathbf{k}$ for a pair (α, β) satisfying (2.1). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha,\beta) := \left\{ \begin{array}{cc} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{array} \right\}. \tag{2.2}$$

We remark that, due to the commutativity condition (2.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1, k_2). Moreover, $\mathbf{T}$ is subnormal if and only if there is a regular Borel probability measure μ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ $(a_i := \|T_i\|^2)$ such that

$$\gamma_{\mathbf{k}} = \iint_R s^{k_1} t^{k_2} \, d\mu(s,t) \ \text{(all } \mathbf{k} \in \mathbb{Z}_+^2)[17]. \tag{2.3}$$

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S,T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix $[\mathbf{T}^*, \mathbf{T}] := ([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [2], [12]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Longrightarrow$ subnormal $\Longrightarrow$ hyponormal. Moreover, the restriction of a hyponormal n-tuple to an invariant subspace is again hyponormal. The Bram-Halmos criterion states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if the k -tuple $(T, T^2, \cdots, T^k)$ is hyponormal for all $k \geq 1$; we say that T is k-hyponormal when the latter condition holds. On the other hand, for a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ of operators on Hilbert space we have

Definition 2.1. *(cf.* [9]*) A commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is called k-hyponormal if $\mathbf{T}(k) := (T_1, T_2, T_1^2, T_2T_1, T_2^2, \cdots, T_1^k, T_2T_1^{k-1}, \cdots, T_2^k)$ is hyponormal, or equivalently*

$$([(T_2^qT_1^p)^*, T_2^nT_1^m])_{\substack{1 \leq m+n \leq k \\ 1 \leq p+q \leq k}} \geq 0.$$

Clearly, subnormal $\Longrightarrow$ $(k+1)$-hyponormal $\Longrightarrow$ k-hyponormal for every $k \geq 1$, and of course 1-hyponormality agrees with the usual definition of joint hyponormality (as above). In [9] we obtained the following multivariable version of the Bram-Halmos criterion for subnormality, which provided an abstract answer to the LPCS, by showing that no matter how k-hyponormal the pair $\mathbf{T}$ might be, it may still fail to be subnormal.

Theorem 2.2. *(*[9, Theorem 2.3]*) Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of subnormal operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent.*

(i) $\mathbf{T}$ *is subnormal.*
(ii) $\mathbf{T}$ *is* k*-hyponormal for all* $k \in \mathbb{Z}_+$.

In the single variable case, there are useful criteria for k-hyponormality ([6], [8]); for 2-variable weighted shifts, a simple criterion for joint hyponormality was given in([5]). The following characterization of k-hyponormality for 2-variable weighted shifts was given in [9, Theorem 2.4].

Theorem 2.3. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a 2-variable weighted shift with weight sequences* $\alpha \equiv \{\alpha_{\mathbf{k}}\}$ *and* $\beta \equiv \{\beta_{\mathbf{k}}\}$. *The following statements are equivalent.*
(a) $\mathbf{T}$ *is* k*-hyponormal.*
(b) $M_{\mathbf{u}}(k) := (\gamma_{\mathbf{u}+(m,n)+(p,q)})_{\substack{0\le m+n\le k\\ 0\le p+q\le k}} \ge 0$ *for all* $\mathbf{u} \in \mathbb{Z}_+^2$. *(For a subnormal pair* $\mathbf{T}$*, the matrix* $M_{\mathbf{u}}(k)$ *is the truncation of the moment matrix associated to the Berger measure of* $\mathbf{T}$*.)*

The following special cases of Theorem 2.3 will be essential for our work.

Lemma 2.4. *([5])(Six-point Test; cf. Figure 1(i)) Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a 2-variable weighted shift, with weight sequences* α *and* β*. Then*

$$\begin{aligned}
[\mathbf{T}^*, \mathbf{T}] \quad \ge \quad 0 &\iff M_{\mathbf{k}}(1) \ge 0 \ (\text{all } \mathbf{k} \in \mathbb{Z}_+^2) \\
&\iff \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \ge 0 \ (\text{all } \mathbf{k} \in \mathbb{Z}_+^2).
\end{aligned}$$

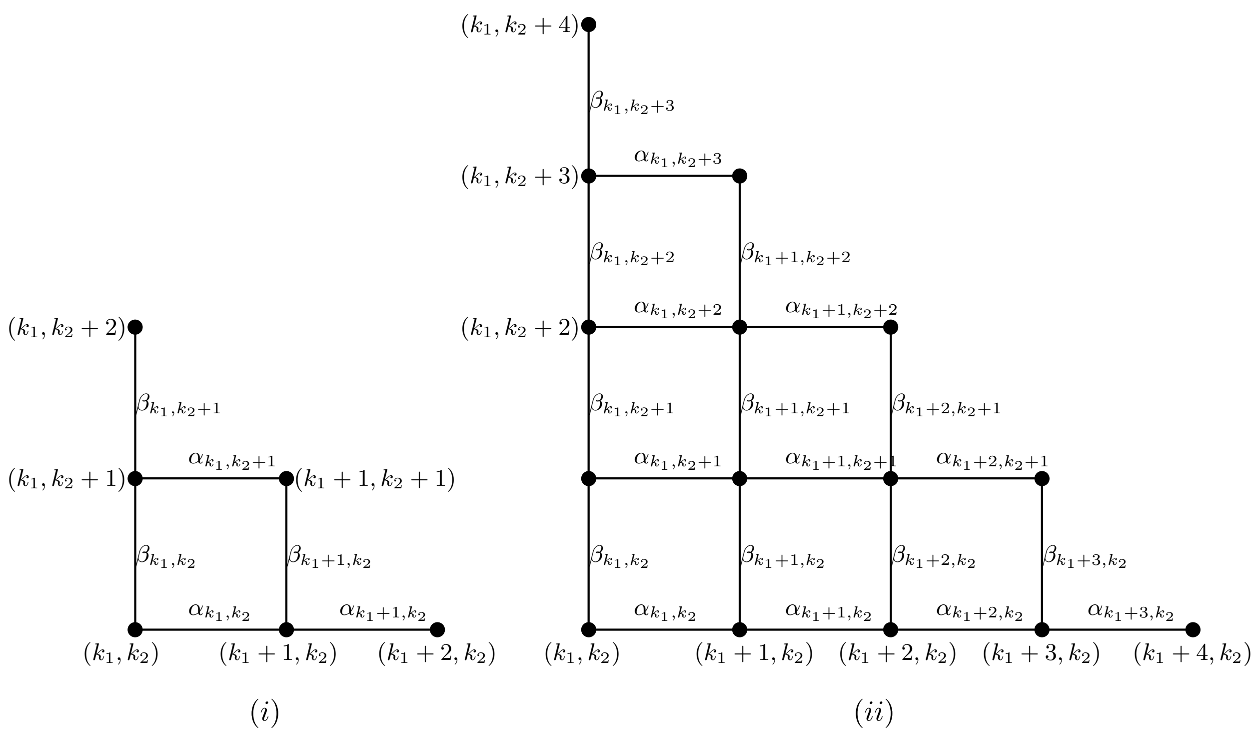

FIGURE 1. Weight diagrams used in the Six-point Test and 15-point Test, respectively.

Lemma 2.5. *([9]) (15-point Test; cf. Figure 1(ii)) If $\mathbf{T}\equiv(T_1,T_2)$ is 2-variable weighted shift with weight sequence $\alpha\equiv\{\alpha_{\mathbf{k}}\}$ and $\beta\equiv\{\beta_{\mathbf{k}}\}$, then $\mathbf{T}$ is 2-hyponormal if and only if*

$$M_{(k_1,k_2)}(2)\equiv\begin{pmatrix}\gamma_{k_1,k_2} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2}\\ \gamma_{k_1+1,k_2} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1+3,k_2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2}\\ \gamma_{k_1,k_2+1} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1,k_2+3}\\ \gamma_{k_1+2,k_2} & \gamma_{k_1+3,k_2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+4,k_2} & \gamma_{k_1+3,k_2+1} & \gamma_{k_1+2,k_2+2}\\ \gamma_{k_1+1,k_2+1} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1+3,k_2+1} & \gamma_{k_1+2,k_2+2} & \gamma_{k_1+1,k_2+3}\\ \gamma_{k_1,k_2+2} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1,k_2+3} & \gamma_{k_1+2,k_2+2} & \gamma_{k_1+1,k_2+3} & \gamma_{k_1,k_2+4}\end{pmatrix}\geq 0,$$

for all $\mathbf{k}\in\mathbb{Z}_+^2$. We now recall the following notation and terminology from [13]:
(i) given a probability measure μ on $X\times Y\equiv\mathbb{R}_+\times\mathbb{R}_+$, with $\frac{1}{t}\in L^1(\mu)$, the extremal measure μ_{ext} (which is also a probability measure) on $X\times Y$ is given by $d\mu_{ext}(s,t):=(1-\delta_0(t))\frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}}d\mu(s,t)$; and
(ii) given a measure μ on $X\times Y$, the marginal measure μ^X is given by $\mu^X:=\mu\circ\pi_X^{-1}$, where $\pi_X: X\times Y\to X$ is the canonical projection onto X. Thus, $\mu^X(E)=\mu(E\times Y)$, for every $E\subseteq X$.

We now list some results which are needed in the proof of Theorem 3.1.

Lemma 2.6. *(cf. [21], [7, Proposition 2.2]) Let $M\equiv\begin{pmatrix}A & B^*\\ B & C\end{pmatrix}$ be a 2×2 operator matrix, where A and C are square matrices and B is a rectangular matrix. Then*

$$M\geq 0 \iff \textit{there exists } W \textit{ such that } \begin{cases} C\geq 0\\ B=CW\\ A\geq W^*CW.\end{cases}$$

For Lemma 2.7, Definition 2.8 and Theorem 3.1 the following two subspaces of $\ell^2(\mathbb{Z}_+^2)$ will be needed: $\mathcal{M}:=\bigvee\{e_{\mathbf{k}}:k_2\geq 1\}$ and $\mathcal{N}:=\bigvee\{e_{\mathbf{k}}:k_1\geq 1\}$.

Lemma 2.7. *([13]) (Subnormal backward extension of a 2-variable weighted shift) Consider the 2-variable weighted shift whose weight diagram is given in Figure 2(i), and let $\mathbf{T}|_{\mathcal{M}}$ denote the restriction of $\mathbf{T}$ to $\mathcal{M}$. Assume that $\mathbf{T}|_{\mathcal{M}}$ is subnormal with associated measure $\mu_{\mathcal{M}}$, and that $W_0:=shift(\alpha_{00},\alpha_{10},\cdots)$ is subnormal with associated measure ν. Then $\mathbf{T}$ is subnormal if and only if (i) $\frac{1}{t}\in L^1(\mu_{\mathcal{M}})$; (ii) $\beta_{00}^2\leq(\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$; (iii) $\beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}(\mu_{\mathcal{M}})_{ext}^X\leq\nu$. Moreover, if $\beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}=1$, then $(\mu_{\mathcal{M}})_{ext}^X=\nu$. In the case when $\mathbf{T}$ is subnormal, Berger measure μ of $\mathbf{T}$ is given by*

$$d\mu(s,t)=\beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}d(\mu_{\mathcal{M}})_{ext}(s,t)+(d\nu(s)-\beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}d(\mu_{\mathcal{M}})_{ext}^X(s))d\delta_0(t).$$

J. Stampfli showed in [22] that for subnormal weighted shifts W_α a *propagation* phenomenon occurs that forces the flatness of W_α whenever two equal weights are present. Later, R.E. Curto proved in [6] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: If W_α is quadratically hyponormal and $\alpha_n=\alpha_{n+1}=\alpha_{n+2}$ for some $n\geq 0$, then $\alpha_1=\alpha_2=\alpha_3=\cdots$. Y. Choi [3] improved this result, that is, if W_α be quadratically hyponormal and $\alpha_n=\alpha_{n+1}$ for some $n\geq 1$, then W_α is *flat*, that is, $W_\alpha\equiv shift(\alpha_0,\alpha_1,\alpha_1,\cdots)$. In Theorem 2.10 we show that similar propagation phenomena occur for 2-hyponormal 2-variable weighted shifts $\mathbf{T}$. In two variables, the flatness of $\mathbf{T}$ is captured by the so-called *core* of $\mathbf{T}$, $c(\mathbf{T})$ (cf. Definition 2.8 below), as follows: $\mathbf{T}$ is flat when $c(\mathbf{T})$ is a 2-variable weighted shift of tensor form.

Definition 2.8. *(i) The core of a 2-variable weighted shift* $\mathbf{T}$ *is the restriction of* $\mathbf{T}$ *to* $\mathcal{M}\cap\mathcal{N}$*, in symbols,* $c(\mathbf{T}) := \mathbf{T}|_{\mathcal{M}\cap\mathcal{N}}$.
(ii) A 2-variable weighted shift $\mathbf{T}$ *is said to be of tensor form if* $\mathbf{T} \cong (I\otimes W_\alpha, W_\beta\otimes I)$*. When* $\mathbf{T}$ *is subnormal, this is equivalent to requiring that the Berger measure be a Cartesian product* $\xi\times\eta$.
(iii) The class of all 2-variable weighted shift $\mathbf{T}\in\mathfrak{H}_0$ *whose core is of tensor form will be denoted by* $\mathcal{TC}$*, that is,* $\mathcal{TC} := \{\mathbf{T}\in\mathfrak{H}_0 : c(\mathbf{T}) \text{ is of tensor form}\}$.

We now recall that a 2-variable weighted shift $\mathbf{T}$ is said to be *horizontally flat* when $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1,k_2\geq 1$; and we call $\mathbf{T}$ *vertically flat* when $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1,k_2\geq 1$. We say $\mathbf{T}$ is *flat* if $\mathbf{T}$ is horizontally and vertically flat, and that $\mathbf{T}$ is symmetrically flat if $\mathbf{T}$ is flat and $\alpha_{(1,1)} = \beta_{(1,1)}$. The next result shows the extent to which propagation holds in the presence of (joint) hyponormality.

Proposition 2.9. *([15]) Let* $\mathbf{T}$ *be a commuting hyponormal 2-variable weighted shift whose weight diagram is given in Figure 2(i). Then*
(i) If $\alpha_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}}$ *for some* $\mathbf{k}$*, then* $\alpha_{\mathbf{k}+\varepsilon_2} = \alpha_{\mathbf{k}}$ *and* $\beta_{\mathbf{k}+\varepsilon_1} = \beta_{\mathbf{k}}$*;*
(ii) If T_1 *is quadratically hyponormal, if* T_2 *is subnormal, and if* $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ *for some* $k_1,k_2\geq 0$*, then* $\mathbf{T}$ *is horizontally flat.*

On the other hand, under the assumption of 2-hyponormality (and without assuming the subnormality of either T_1 or T_2), we can prove that $\mathbf{T}$ is flat whenever $\alpha_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}}$ and $\beta_{\mathbf{m}+\varepsilon_2} = \beta_{\mathbf{m}}$ for some $\mathbf{k}$ and $\mathbf{m}$. We do this using the 15-point Test (Lemma 2.5).

Theorem 2.10. *Let* $\mathbf{T}$ *be a 2-hyponormal 2-variable weighted shift. If* $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ *for some* $k_1,k_2\geq 1$*, then* $\mathbf{T}$ *is horizontally flat. If instead,* $\beta_{(k_1,k_2)+\varepsilon_2} = \beta_{(k_1,k_2)}$ *for some* $k_1,k_2\geq 1$*, then* $\mathbf{T}$ *is vertically flat.*

Proof. Without loss of generality, assume $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} = 1$. Then by the above mentioned result of Y. Choi and Proposition 2.9(i), we can see that T_1 is of tensor form when restricted to the subspace $\bigvee\{e_{\mathbf{m}} : m_1\geq 1 \text{ and } m_2\geq k_2\}$. If $k_2 = 1$ we are done, so assume $k_2\geq 2$ and consider the matrix

$$M_{(k_1,k_2-2)}(2) = \begin{pmatrix} \gamma_{k_1,k_2-2} & \gamma_{k_1+1,k_2-2} & \gamma_{k_1,k_2-1} & \gamma_{k_1+2,k_2-2} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1,k_2} \\ \gamma_{k_1+1,k_2-2} & \gamma_{k_1+2,k_2-2} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1+3,k_2-2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} \\ \gamma_{k_1,k_2-1} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1,k_2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} \\ \gamma_{k_1+2,k_2-2} & \gamma_{k_1+3,k_2-2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+4,k_2-2} & \gamma_{k_1+3,k_2-1} & \gamma_{k_1+2,k_2} \\ \gamma_{k_1+1,k_2-1} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} & \gamma_{k_1+3,k_2-1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} \\ \gamma_{k_1,k_2} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2} \end{pmatrix},$$

which we know is positive semidefinite since $\mathbf{T}$ is 2-hyponormal (Lemma 2.5). It suffices to prove that $\alpha_{\mathbf{k}-\varepsilon_2} = 1$. Let us focus on the principal submatrix $M\geq 0$ determined by rows and columns 1, 3 and 5. From (2.2) we have

$$M = \begin{pmatrix} 1 & \beta^2_{\mathbf{k}-2\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\alpha^2_{\mathbf{k}-\varepsilon_2} \\ \beta^2_{\mathbf{k}-2\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\beta^2_{\mathbf{k}-\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\beta^2_{\mathbf{k}-\varepsilon_2} \\ \beta^2_{\mathbf{k}-2\varepsilon_2}\alpha^2_{\mathbf{k}-\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\beta^2_{\mathbf{k}-\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\beta^2_{\mathbf{k}-\varepsilon_2} \end{pmatrix} \geq 0. \qquad (2.4)$$

For notational convenience, let $a := \beta^2_{\mathbf{k}-2\varepsilon_2}$, $b := \beta^2_{\mathbf{k}-\varepsilon_2}$ and $c := \alpha^2_{\mathbf{k}-\varepsilon_2}$. Thus,

$$M \equiv \begin{pmatrix} 1 & a & ac \\ a & ab & ab \\ ac & ab & ab \end{pmatrix} \geq 0 \iff N := \begin{pmatrix} ab-a^2 & ab-a^2c \\ ab-a^2c & ab-a^2c^2 \end{pmatrix} \geq 0.$$

If $a = b$ then we must necessarily have $c = 1$, that is, $\alpha_{\mathbf{k}-\varepsilon_2} = 1$, as desired. Assume, therefore, that $a < b$. It follows that

$$\begin{aligned} N &\geq 0 \iff \det N \geq 0 \iff (ab - a^2)(ab - a^2c^2) - (ab - a^2c)^2 \geq 0 \\ &\iff -a^3b\,(c-1)^2 \geq 0 \iff c = 1, \end{aligned}$$

that is, $\alpha_{\mathbf{k}-\varepsilon_2} = 1$, as desired. □

3. Proofs of the Main Results

We are now ready to prove our main results, which we restate for the reader's convenience.

Theorem 3.1. $\mathcal{S} \bigcap \mathfrak{H}_2 = \mathcal{S} \bigcap \mathfrak{H}_\infty$.

Proof. By Proposition 2.9 and Theorem 2.10, we can assume, without loss of generality, that $\alpha_{(k_1,k_2)} = \alpha_{(1,0)} = 1$ (all $k_1 \geq 1, k_2 \geq 0$) and $\beta_{(k_1,k_2)} = \beta_{(0,1)} \leq 1$ (all $k_1 \geq 0, k_2 \geq 1$). For notational convenience, we let $a := \alpha_{(0,1)} < 1$, $b := \beta_{(0,1)} \leq 1$, $x := \alpha_{(0,0)} < 1$, and $y := \beta_{(0,0)} < 1$. We summarize this information in Figure 2(ii).

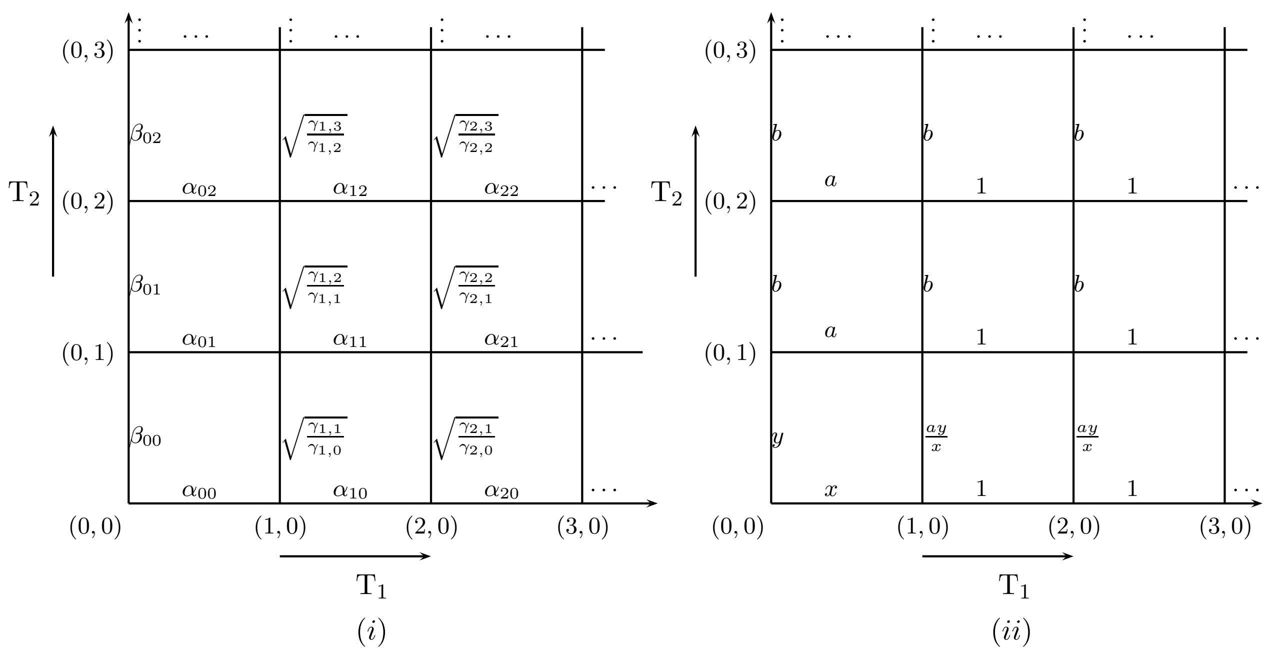

Figure 2. Weight diagrams of the 2-variable weighted shifts in Lemma 2.7 and Theorem 3.1, respectively.

Let $C_k := \{(a, b, x, y) : \mathbf{T} \in \mathfrak{H}_k\}$ $(k = 1, 2, \infty)$. Clearly, $C_\infty \subseteq C_2 \subseteq C_1$. We first describe concretely the set C_2 in terms of necessary and sufficient conditions on the four parameters a, b, x and y that guarantee the 2-hyponormality of the pair $\mathbf{T}$. We then establish that $C_2 \subseteq C_\infty$. Finally, in Theorem 3.3 we will show that $C_2 \subsetneq C_1$.

It is straightforward to verify that $\mathbf{T}|_{\mathcal{M}} \cong (I \otimes shift(a, 1, 1, \cdots), bU_+ \otimes I)$ and $\mathbf{T}|_{\mathcal{N}} \cong (I \otimes U_+, shift(\frac{ay}{x}, b, b, \cdots) \otimes I)$; thus, $\mathbf{T}|_{\mathcal{M}}$ and $\mathbf{T}|_{\mathcal{N}}$ are subnormal. As a consequence, to describe C_2

we only need to apply the 15-point Test (Lemma 2.5) at $\mathbf{k} = (0,0)$, that is, we need to guarantee that $M_{(0,0)}(2) \geq 0$. We thus have:

$$\mathbf{T} \in \mathfrak{H}_2 \Longleftrightarrow M_{(0,0)}(2) \geq 0.$$

Now, since the moments $\gamma_{\mathbf{k}}$ ($\mathbf{k} \in \mathbb{Z}_+^2$) associated with $\mathbf{T}$ are

$$\gamma_{\mathbf{k}} = \begin{cases} 1 & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ x^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ y^2 b^{2(k_2-1)} & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ a^2 y^2 b^{2(k_2-1)} & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1, \end{cases}$$

it follows that

$$M_{(0,0)}(2) \equiv \begin{pmatrix} 1 & x^2 & y^2 & x^2 & a^2y^2 & b^2y^2 \\ x^2 & x^2 & a^2y^2 & x^2 & a^2y^2 & a^2b^2y^2 \\ y^2 & a^2y^2 & b^2y^2 & a^2y^2 & a^2b^2y^2 & b^4y^2 \\ x^2 & x^2 & a^2y^2 & x^2 & a^2y^2 & a^2b^2y^2 \\ a^2y^2 & a^2y^2 & a^2b^2y^2 & a^2y^2 & a^2b^2y^2 & a^2b^4y^2 \\ b^2y^2 & a^2b^2y^2 & b^4y^2 & a^2b^2y^2 & a^2b^4y^2 & b^6y^2 \end{pmatrix} \geq 0$$

$$\Longleftrightarrow M := \begin{pmatrix} 1 & x^2 & y^2 & a^2y^2 \\ x^2 & x^2 & a^2y^2 & a^2y^2 \\ y^2 & a^2y^2 & b^2y^2 & a^2b^2y^2 \\ a^2y^2 & a^2y^2 & a^2b^2y^2 & a^2b^2y^2 \end{pmatrix} \geq 0$$

(since the sixth row is a multiple of the third row, and the second and fourth rows are identical). If we now interchange the second and third rows and columns, we see that the positivity of M is determined by the positivity of

$$\begin{pmatrix} \begin{pmatrix} 1 & y^2 \\ y^2 & b^2y^2 \end{pmatrix} & \begin{pmatrix} x^2 & a^2y^2 \\ a^2y^2 & a^2b^2y^2 \end{pmatrix} \\ \begin{pmatrix} x^2 & a^2y^2 \\ a^2y^2 & a^2b^2y^2 \end{pmatrix} & \begin{pmatrix} x^2 & a^2y^2 \\ a^2y^2 & a^2b^2y^2 \end{pmatrix} \end{pmatrix} := \begin{pmatrix} A & B \\ B & B \end{pmatrix} \geq 0.$$

Thus, from Lemma 2.6 (with $C = B$ and $W = I$), we have

$$M \geq 0 \Longleftrightarrow A \geq B \geq 0.$$

Now observe that $B \geq 0 \iff ay \leq bx$ and

$$A - B = \begin{pmatrix} 1 - x^2 & y^2(1-a^2) \\ y^2(1-a^2) & b^2y^2(1-a^2) \end{pmatrix}.$$

Since the $(1,1)$-entry of $A - B$ is always positive, the positivity of $A - B$ is completely determined by its determinant; that is,

$$\begin{aligned} A - B \quad &\geq \quad 0 \Longleftrightarrow \det(A-B) \geq 0 \\ &\Longleftrightarrow \quad y^2(1-a^2) \leq b^2(1-x^2). \end{aligned}$$

It follows that

$$\mathbf{T} \in \mathfrak{H}_2 \iff ay \leq bx \text{ and } y^2(1-a^2) \leq b^2(1-x^2). \tag{3.1}$$

We thus see that

$$\begin{aligned} C_2 \quad &= \quad \{(a,b,x,y) : 0 < x < 1,\ 0 < y < 1, \\ ay \quad &\leq \quad bx, \text{ and } y^2(1-a^2) \leq b^2(1-x^2)\}. \end{aligned} \tag{3.2}$$

We will now prove that $C_2 \subseteq C_\infty$. Let $(a,b,x,y) \in C_2$. Let $z := \frac{ay}{x}$.

Case 1. If $z = b$, then $\mathbf{T}|_{\mathcal{N}} \equiv (I \otimes U_+, bU_+ \otimes I)$, and a straightforward application of Lemma 2.7 in the s direction shows that $\mathbf{T}$ is subnormal if and only if $bx \le y$ if and only if $a \le 1$ (since $bx = ay$), which is true. Thus, $(a,b,x,y) \in C_\infty$.

Case 2. Assume now that $z < 1$. Since $\mathbf{T}|_{\mathcal{N}} \equiv (I \otimes U_+, bS_{z/b} \otimes I)$ is subnormal with Berger measure $\mu_{\mathcal{N}} \equiv \delta_1 \times [(1-\frac{z^2}{b^2})\delta_0 + \frac{z^2}{b^2}\delta_{b^2}]$, we can think of $\mathbf{T}$ as a backward extension of $\mathbf{T}|_{\mathcal{N}}$ (in the s direction) and apply Lemma 2.7. Note that $\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} = 1$, so $d(\mu_{\mathcal{N}})_{ext}(s,t) \equiv (1-\delta_0(s))\frac{1}{s}d\mu_{\mathcal{N}}(s,t) = d\delta_1(s)[(1-\frac{z^2}{b^2})d\delta_0(t) + \frac{z^2}{b^2}d\delta_{b^2}(t)]$ and $\alpha_{00}^2 \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} (\mu_{\mathcal{N}})_{ext}^Y = x^2[(1-\frac{z^2}{b^2})\delta_0 + \frac{z^2}{b^2}\delta_{b^2}]$. Thus, by Lemma 2.7,

$$
\begin{aligned}
&\mathbf{T} \text{ is subnormal } \Longleftrightarrow \alpha_{00}^2 \left\|\tfrac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} (\mu_{\mathcal{N}})_{ext}^Y \le \eta_0 \\
&(\text{where } \eta_0 \text{ denotes the Berger measure of } shift(y,b,b,\cdots)) \\
&\Longleftrightarrow x^2[(1-\tfrac{z^2}{b^2})\delta_0 + \tfrac{z^2}{b^2}\delta_{b^2}] \le (1-\tfrac{y^2}{b^2})\delta_0 + \tfrac{y^2}{b^2}\delta_{b^2} \\
&\Longleftrightarrow x^2(1-\tfrac{z^2}{b^2}) \le (1-\tfrac{y^2}{b^2}) \text{ and } \tfrac{x^2z^2}{b^2} \le \tfrac{y^2}{b^2} \\
&\Longleftrightarrow y^2(1-a^2) \le b^2(1-x^2),
\end{aligned} \tag{3.3}
$$

as in the last condition in (3.2). Therefore, $\mathbf{T}$ is subnormal, and the proof is complete. $\square$

Theorem 3.2. *Let $\mathbf{T} \equiv (T_1,T_2) \in \mathcal{S}$, with weight diagram given by Figure 2(ii), and assume that $\mathbf{T}$ is 2-hyponormal. Then $\mathbf{T}$ is subnormal, with Berger measure given as*

$$
\begin{aligned}
\mu \;=\; &\frac{1}{b^2}\{[b^2(1-x^2) - y^2(1-a^2)]\delta_{(0,0)} \\
&+y^2(1-a^2)\delta_{(0,b^2)} + (b^2x^2 - a^2y^2)\delta_{(1,0)} + a^2y^2\delta_{(1,b^2)}\}.
\end{aligned} \tag{3.4}
$$

Proof. We apply the main result in [11], in the special form needed here; cf. [11, Proposition 3.1]. For a 2-variable subnormal weighted shift with weight diagram given by Figure 2(ii), the Berger measure is

$$\mu = \varphi \times (\delta_0 - \delta_{b^2}) + y^2 \left\|\frac{1}{t}\right\|_{L^1(\psi)} \delta_0 \times (\tilde{\psi} - \delta_{b^2}) + \xi_x \times \delta_{b^2}, \tag{3.5}$$

where $\psi = (1-a^2)\delta_{b^2}$, $\varphi = \xi_x - y^2\frac{1-a^2}{b^2}\delta_0 - \frac{a^2y^2}{b^2}\delta_1$, $d\tilde{\psi}(t) := \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\psi)}}d\psi(t)$ and ξ_x is the Berger measure of S_x. A straightforward calculation shows that $\tilde{\psi} = \delta_{b^2}$, so that (3.5) becomes

$$
\begin{aligned}
\mu \;=\; &\{[\frac{b^2(1-x^2) - y^2(1-a^2)}{b^2}]\delta_0 + \frac{b^2x^2 - a^2y^2}{b^2}\delta_1\} \times (\delta_0 - \delta_{b^2}) \\
&+[(1-x^2)\delta_0 + x^2\delta_1] \times \delta_{b^2},
\end{aligned}
$$

which easily leads to the desired formula (3.4). $\square$

We conclude this section with a proof of Theorem 1.4, which we reformulate in terms of C_1 and C_2. Toward this end, we need a description of $C_1 := \{(a,b,x,y) : \mathbf{T} \in \mathfrak{H}_1\}$: by the Six-point Test (Lemma 2.4), $\mathbf{T}$ is hyponormal if and only if

$$M_{(0,0)}(1) \equiv \begin{pmatrix} 1 & x^2 & y^2 \\ x^2 & x^2 & a^2y^2 \\ y^2 & a^2y^2 & b^2y^2 \end{pmatrix} \ge 0.$$

Theorem 3.3. $C_2 \subsetneq C_1$.

Proof. Since $x<1$, $M_{(0,0)}(1)\geq 0$ if and only if det $M_{(0,0)}(1)\geq 0$, that is, if and only if

$$P_1:=y^2(b^2x^2-a^4y^2-b^2x^4+2a^2x^2y^2-x^2y^2)\geq 0.$$

Let $x>\frac{\sqrt{2}}{2}$ and let $a:=\sqrt{2x^2-1}$. It follows that $1-a^2=2(1-x^2)$, so that

$$P_2:=b^2(1-x^2)-y^2(1-a^2)=b^2(1-x^2)-2y^2(1-x^2)=(b^2-2y^2)(1-x^2),$$

and it suffices to choose y such that $\frac{\sqrt{2}}{2}b<y<b$ to make $P_2<0$, and thus break 2-hyponormality (cf. (3.1)). On the other hand,

$$\begin{aligned} P_1 &\equiv P_2x^2y^2+y^4a^2(x^2-a^2) \\ &= y^2(b^2x^2-b^2x^4-y^2+x^2y^2) \\ &= (b^2x^2-y^2)(1-x^2), \end{aligned}$$

which can be made nonnegative by taking x close to 1. This shows that $C_2\subsetneq C_1$. $\square$

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