THE QUADRATIC MOMENT PROBLEM FOR THE UNIT CIRCLE AND UNIT DISK*

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Dedicated to the memory of Velaho D. Bowman-Fialkow

For the quadratic complex moment problem $\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \leq i + j \leq 2)$, we obtain necessary and sufficient conditions for the existence of representing measures μ supported in the unit circle $\mathbf{T}$ or in the closed unit disk $\bar{\mathbf{D}}$. We explicitly construct all finitely atomic representing measures supported in $\mathbf{T}$ or $\bar{\mathbf{D}}$ which have the fewest atoms possible. For the quadratic $\bar{\mathbf{D}}$-moment problem in which the moment matrix $M(1)$ is positive and invertible, there exists an ellipse $\mathcal{E} \subseteq \mathbf{D}$ such that the minimal (3-atomic) representing measures are supported in the complement of the interior region of $\mathcal{E}$. Finally, we apply these results to obtain information on the location of the zeros of certain cubic polynomials.

1 Introduction

Given complex numbers $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$, and a closed subset K of the complex plane $\mathbf{C}$, the *quadratic complex K-moment problem* entails finding necessary and sufficient conditions for the existence and uniqueness of a positive Borel measure μ such that $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \;\; (0 \leq i+j \leq 2)$, and $\operatorname{supp} \mu \subseteq K$. In the present paper, we provide a comprehensive analysis of this problem for the cases when $K = \mathbf{T}$ (the unit circle) or $K = \bar{\mathbf{D}}$ (the closed unit disk). In each case, we provide existence criteria for representing measures μ expressed concretely in terms of the moment matrix $M(1) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}$ (Theorems 1.8 and

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3.1): such measures exist if and only if $M(1) \geq 0$ and $\gamma_{11} = \gamma_{00}$ (in the unit circle case) or $\gamma_{11} \leq \gamma_{00}$ (in the unit disk case). Our results go beyond existence criteria, in that we obtain a complete parameterization of minimal representing measures in each case (Remark 3.2, Corollary 4.12 and Theorem 5.7). For a prescribed point $z \in K$, we provide an explicit computational test for determining whether z is in the support of a representing measure, supported in K, with the fewest atoms (Theorem 4.2, Corollary 4.3, Theorem 4.11 and Corollary 4.12). For such a point z, we provide an algorithm for explicitly constructing all such *minimal K-representing measures*. Further, we present a geometric description of the location of the atoms of minimal K-representing measures (Corollary 4.12, Remark 4.1(ii) and Theorem 5.7). This parameterization of minimal representing measures is analogous to Hamburger's parameterization of representing measures for the full moment problem on the real line ([AhKr], [Akh], [KrNu], [ShTa]). We believe that our results for the disk comprise the first parameterization of minimal representing measures in a truncated moment problem for a set of positive planar measure.

In order to state our results in detail, we recall the moment matrix approach to truncated multivariable moment problems. Given a doubly indexed finite sequence of complex numbers $\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the *truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbf{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \qquad (0 \leq i + j \leq 2n);$$

μ is called a *representing measure* for γ. In [Cur], [CuFi2], [CuFi3], [CuFi4], [CuFi5], [Fi1] and [Fi2], we studied truncated moment problems (in one or several variables) using an approach based on positivity and extension properties of the *moment matrix* $M(n)$ associated to the data. If μ is a representing measure, then $\text{card supp}\, \mu \geq \text{rank}\, M(n)$ [CuFi2, Corollary 3.7]. For $n \geq 3$, not every minimal representing measure has $\text{rank}\, M(n)$ atoms ([CuFi4, Theorem 5.2], [Fi3]), but in a variety of cases we have established that minimal representing measures are indeed $\text{rank}\, M(n)$-atomic ([Cur], [CuFi2], [CuFi3], [CuFi4]). In particular, when $\gamma^{(2n)}$ is of *flat data type* (i.e., $M(n) \geq 0$ and $\text{rank}\, M(n) = \text{rank}\, M(n-1)$), a unique representing measure exists, which is $\text{rank}\, M(n)$-atomic [CuFi2].

This result is compatible with our previous results for measures supported on the real line, nonnegative real line, or some prescribed finite interval (truncated Hamburger, Stieltjes and Hausdorff moment problems), and for measures supported on the unit circle (truncated Toeplitz moment problems) [CuFi1]. In these cases, the concept of *recursiveness* for positive Hankel matrices motivated the development in [CuFi2] of a "functional calculus" for the columns of $M(n)$, which plays a key role in establishing the following basic tool for constructing $\text{rank}\, M(n)$-atomic representing measures. In the sequel, we let $\mathcal{C}_{M(n)}$ denote the column space of $M(n)$, we denote the successive columns of $M(n)$ by $\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, ..., Z^n, ..., \bar{Z}^n$, and we let $\mathcal{P}_n$ denote the complex polynomials in z and $\bar{z}$ of degree at most n. For $p \in \mathcal{P}_n$, $p \equiv \sum a_{ij} \bar{z}^i z^j$, we let $p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j \in \mathcal{C}_{M(n)}$.

Theorem 1.1 (*cf.* [CuFi2, Theorem 5.4, Corollary 5.12, Theorem 5.13, and Corollary 5.15]) *Suppose $M(n)$ is positive and admits a flat (i.e., rank-preserving) extension $M(n+1)$, so* *that* $Z^{n+1} = p(Z, \bar{Z})$ *in* $\mathcal{C}_{M(n+1)}$ *for some* $p \in \mathcal{P}_n$. *Then there exist unique, successive flat moment matrix extensions* $M(n+2)$, $M(n+3)$, ..., *which are determined by the relations*

$$Z^{n+k} = (z^{k-1}p)(Z, \bar{Z}) \in \mathcal{C}_{M(n+k)} \ \ (k \geq 2).$$

Let $r := \operatorname{rank} M(n)$. *Then there exist unique scalars* $a_0, ..., a_{r-1}$ *such that in* $\mathcal{C}_{M(r)}$,

$$Z^r = a_0 1 + ... + a_{r-1} Z^{r-1}.$$

The generating polynomial $g_\gamma(z) \equiv z^r - (a_0 + ... + a_{r-1}z^{r-1})$ *has* r *distinct roots,* $z_0, ..., z_{r-1}$, *and* γ *has a* $\operatorname{rank} M(n)$*-atomic (minimal) representing measure of the form* $\nu \equiv \nu[M(n+1)] = \sum \rho_i \delta_{z_i}$, *where the densities* $\rho_i > 0$ *are determined by the Vandermonde equation*

$$V(z_0, ..., z_{r-1})(\rho_0, ..., \rho_{r-1})^T = (\gamma_{00}, ..., \gamma_{0,r-1})^T,$$

and where δ_w *denotes the unit-mass atomic measure supported at* w. *The measure* $\nu[M(n+1)]$ *is the unique representing measure for* $\gamma^{(2n+2)}$.

In [CuFi5] we studied the *truncated complex* K*-moment problem*, in which the support of a representing measure is required to be contained in a prescribed closed set $K \subseteq \mathbf{C}$. For a polynomial $q(z, \bar{z})$, let $K_q := \{z \in C : q(z, \bar{z}) \geq 0\}$ and let $k := \left[\frac{1+\deg q}{2}\right]$. Corresponding to a flat extension $M(n+1)$ of $M(n) \geq 0$ we have the extension $M(n+k)$ (Theorem 1.1); we let $M_q(n+k)$ denote the *localizing matrix* for $M(n+k)$ relative to q (see below for the definition of localizing matrix).

Theorem 1.2 [CuFi5, Theorem 1.1] *There exists a* $\operatorname{rank} M(n)$*-atomic representing measure for* $\gamma^{(2n)}$ *supported in* K_q *if and only if* $M(n) \geq 0$ *and there is some flat extension* $M(n+1)$ *for which* $M_q(n+k) \geq 0$. *In this case,* $\nu[M(n+1)]$ *is a* $\operatorname{rank} M(n)$*-atomic representing measure supported in* K_q, *with precisely* $\operatorname{rank} M(n) - \operatorname{rank} M_q(n+k)$ *atoms in* $\mathcal{Z}(q) := \{z \in \mathbf{C} : q(z, \bar{z}) = 0\}$.

For $q(z, \bar{z}) := 1 - \bar{z}z$, Theorem 1.2 provides an *abstract* solution to the truncated complex K-moment problem for $K = \bar{\mathbf{D}}$ or $K = \mathbf{T}$, but establishing the existence of the desired flat extension $M(n+1)$ is itself a difficult problem [CuFi5]. Similarly, Atzmon's solution to the full complex moment problem for $\bar{\mathbf{D}}$ [Atz], Putinar's elegant alternative approach [Put], and Schmüdgen's subsequent solution to the full K-moment problem for compact semi-algebraic sets [Sch], do not readily yield constructive formulas for representing measures (see also [BeMa] for related results). By contrast, in the present paper, for the *quadratic moment problem* ($n = 1$), we develop concrete algorithms for constructing $\operatorname{rank} M(1)$-atomic (minimal) representing measures supported in the unit disk or unit circle. These algorithms are expressed solely in terms of the data $\gamma^{(2)}$ and are independent of Theorem 1.2.

To illustrate our viewpoint, consider $K = \bar{\mathbf{D}}$. In the case when $\operatorname{rank} M(1) = 2$, the necessary conditions $M(1) \geq 0$ and $1 = \gamma_{00} > \gamma_{01}$ entail $\eta := \gamma_{02} - \gamma_{01}^2 \neq 0$, and we let w denote a fixed square root of η. In Theorem 4.2 we compute a closed interval $I \subseteq (0, +\infty)$ such that the 2-atomic (minimal) representing measures for $\gamma^{(2)}$ supported in $\bar{\mathbf{D}}$ are precisely those of the form $\mu_t \equiv \frac{1}{1+t^2}\delta_{\gamma_{01}+tw} + \frac{t^2}{1+t^2}\delta_{\gamma_{01}-\frac{1}{t}w}$ $(t \in I)$, where δ_z denotes the unit mass at z. In Sections 4 and 5, we use this result and a "rank reduction" construction to analyze the case when $M(1)$ is positive and invertible. In Theorem 4.11 we compute an open region $\Omega \subseteq \mathbf{D}$ such that a point $z_0 \in \bar{\mathbf{D}}$ is an atom of a 3-atomic (minimal) representing measure μ for $\gamma^{(2)}$ supported in $\bar{\mathbf{D}}$ if and only if $z_0 \in \bar{\mathbf{D}} \setminus \Omega$. In Section 5 we identify $\mathcal{E} := \partial\Omega$ as an ellipse and show, further, that the two remaining atoms of μ can be chosen on the unit circle if and only if $z_0 \in \mathcal{E}$ (Theorem 5.5).

The preceding viewpoint (working with the atoms of a minimal representing measure) was motivated by the difficulty we encountered when we tried to directly construct flat extensions satisfying the conditions of Theorem 1.2 for the case $n = 1$, $q(z, \bar{z}) := 1 - \bar{z}z$, and $k = 1$. (By virtue of Theorem 1.8, we know that such extensions do exist.) For the quadratic moment problem with $K = \mathbf{T}$, we again bypass Theorem 1.2, but in a different way. We prove Theorem 3.1 by constructing flat extensions $M(2)$ in which there is a column relation $\bar{Z}Z = \mathit{1}$; for $n = k = 1$, this is equivalent to the condition $M_{1-\bar{z}z}(2) = 0$ [CuFi5, Proposition 3.9].

In this paper, we focus almost exclusively on the quadratic moment problem ($n = 1$). For $n > 1$, in [CuFi4, Theorem 1.5] we presented an abstract solution to the truncated complex moment problem, but a concrete analysis exists only for special cases. Indeed, even for the *quartic moment problem* ($n = 2$), the existence of positive $M(2)$ which do not admit representing measures [CuFi6] considerably complicates the theory. On the other hand, some of our techniques, including the "rotation" technique of Section 2 and the "rank reduction" technique of Sections 2 and 4, are clearly applicable to general truncated complex moment problems. We plan to study the implications of these techniques for $n > 1$ elsewhere.

In order to further describe our results, we require some terminology and results concerning moment matrices. Recall that for $n \geq 1$ and $m \equiv m(n) := (n+1)(n+2)/2$, the $m \times m$ *moment matrix* $M(n) \equiv M(n)(\gamma)$ is defined as follows. For $0 \leq i, j \leq n$, the $(i+1) \times (j+1)$ matrix $M[i,j]$ has as entries the moments of order $i + j$:

$$M[i,j] := \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \gamma_{i+j-1,1} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{0,i+j} & \gamma_{1,i+j-1} & & \gamma_{j,i} \end{pmatrix}, \tag{1.1}$$

and $M(n) := (M[i,j])_{0 \leq i,j \leq n}$.

For $n = 1$, the *quadratic moment problem* with data $\gamma \equiv \gamma^{(2)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ corresponds to

$$M(1) = \begin{pmatrix} M[0,0] & M[0,1] \\ M[1,0] & M[1,1] \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}.$$

In [CuFi2, Theorem 6.1] we established the following result.

Theorem 1.3 *Let $r := \operatorname{rank} M(1)$. The following statements are equivalent for $\gamma \equiv \gamma^{(2)}$.*
(i) *γ has a representing measure;*
(ii) *γ has an r-atomic representing measure;*
(iii) $M(1) \geq 0$.

In this case,
(a) *if* $r = 1$*, the unique representing measure is* $\gamma_{00}\delta_{\gamma_{01}/\gamma_{00}}$*;*
(b) *if* $r = 2$*, the* 2*-atomic representing measures are parameterized by a complex line* $\bar{z} = \alpha + \beta z$ $(z \neq \gamma_{01})$*;*
(c) *if* $r = 3$*, the* 3*-atomic representing measures contain a sub-parameterization by a circle.*

Recall that $\mathcal{P}_n \subseteq \mathbf{C}[z, \bar{z}]$ denotes the polynomials in z and $\bar{z}$ of total degree at most n. For $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \le i+j \le n} a_{ij} \bar{z}^i z^j$, let $\hat{p} := (a_{00}, a_{01}, a_{10}, ..., a_{0n}, ..., a_{n0})^T \in \mathbf{C}^{m(n)}$. For $q \in \mathcal{P}_{2n}$, $q(z, \bar{z}) \equiv \sum_{0 \le i+j \le 2n} b_{ij} \bar{z}^i z^j$, let $\Lambda(q) := \sum_{0 \le i+j \le 2n} b_{ij} \gamma_{ij}$. $M(n)$ is the unique matrix in $M_{m(n)}(\mathbf{C})$ satisfying $\Lambda(p\bar{q}) = \langle p, q \rangle_{M(n)} := (M(n)\hat{p}, \hat{q})$ $(p, q \in \mathcal{P}_n)$. The basic connection between the moment matrix $M(n)$ and any representing measure μ is provided by the identity $\int p\bar{q}\, d\mu = \langle p, q \rangle_{M(n)}$ $(p, q \in \mathcal{P}_n)$ [CuFi2]; in particular, $(M(n)\hat{p}, \hat{p}) = \int |p|^2\, d\mu \ge 0$, so

$$\text{if } \gamma \text{ admits a representing measure, then } M(n) \ge 0. \tag{1.2}$$

For $p \in \mathcal{P}_n$, $p \equiv \sum a_{ij} \bar{z}^i z^j$, let $p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j \in \mathcal{C}_{M(n)}$ and let $\mathcal{Z}(p) := \{z \in \mathbf{C} : p(z, \bar{z}) = 0\}$. For $0 \le r + s \le n$, let $e_{s,r} \equiv (0, ..., 0, 1, 0, ..., 0)^T$, with a 1 in the $\bar{Z}^s Z^r$ position. Note that for $0 \le r + s \le n$, the (r, s) element of $\mathbf{v} := p(Z, \bar{Z})$ is equal to

$$\begin{aligned} \mathbf{v}_{r,s} &= \langle \mathbf{v}, e_{s,r} \rangle \\ &= \langle p, \bar{z}^s z^r \rangle_{M(n)} \\ &= \textstyle\sum_{0 \le i+j \le n} a_{ij} \langle \bar{z}^i z^j, \bar{z}^s z^r \rangle_{M(n)} \\ &= \textstyle\sum_{0 \le i+j \le n} a_{ij} \gamma_{i+r, j+s}. \end{aligned}$$

We list below four fundamental results needed for our purposes here.

Proposition 1.4 ([CuFi2, Proposition 3.1]) *Suppose* μ *is a representing measure for* γ*. For* $p \in \mathcal{P}_n$*,*

$$\operatorname{supp} \mu \subseteq \mathcal{Z}(p) \Leftrightarrow p(Z, \bar{Z}) = \mathbf{0}.$$

Proposition 1.5 ([CuFi2, Corollary 3.7]) *Let* μ *be a representing measure for* γ*. Then*

$$\operatorname{card} \operatorname{supp} \mu \ge \operatorname{rank} M(n).$$

Proposition 1.6 ([CuFi2, Remark 3.15(ii)]) *Suppose* $M(n)$ *admits a positive moment matrix extension* $M(n+1)$*. Then* $M(n)$ *is recursively generated, i.e., if* $f, g, fg \in \mathcal{P}_n$ *and* $f(Z, \bar{Z}) = \mathbf{0}$ *in* $\mathcal{C}_{M(n)}$*, then* $(fg)(Z, \bar{Z}) = \mathbf{0}$ *in* $\mathcal{C}_{M(n)}$*.*

Proposition 1.7 ([CuFi2, Lemma 3.10]) *Let* $M(n)$ *be a moment matrix, and let* $p \in \mathcal{P}_n$*. If* $p(Z, \bar{Z}) = \mathbf{0}$ *then* $\bar{p}(Z, \bar{Z}) = \mathbf{0}$*.*

Let us now consider the circle and disk quadratic moment problems for $\gamma^{(2)}$: $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$; in view of (1.2) we may assume $M(1) \ge 0$. If one insists that a representing measure for $\gamma^{(2)}$ have support in the unit circle or unit disk, additional necessary conditions beyond positivity of $M(1)$ must be imposed, to reflect the fact that, in the support set, $\bar{z}z = 1$ or $\bar{z}z \leq 1$, respectively. This leads to consideration of a relation between γ_{00} and γ_{11}. For, if μ is a representing measure for γ, $\gamma_{00} = \int 1 \, d\mu$ and $\gamma_{11} = \int \bar{z}z \, d\mu$, so the above mentioned conditions on $z\bar{z}$ easily imply that $\gamma_{11} = \gamma_{00}$ or $\gamma_{11} \leq \gamma_{00}$, respectively. This observation also shows that if μ is supported in the closed unit disk, then $\gamma_{11} = \gamma_{00}$ implies that μ is actually supported in the unit circle.

Note that γ_{00}, the total mass, has no effect on the size or location of the support of a representing measure. In order to simplify certain calculations, we generally assume, without loss of generality, that $\gamma_{00} = 1$. Moreover, one can focus on the cases $\operatorname{rank} M(1) = 2$ and $\operatorname{rank} M(1) = 3$. For, it is easy to see that

$$\operatorname{rank} M(1) = 1 \Leftrightarrow \gamma_{11} = |\gamma_{01}|^2 \text{ and } \gamma_{02} = \gamma_{01}^2 \tag{1.3}$$

(in which case $M(1) \geq 0$). It then follows that when $\operatorname{rank} M(1) = 1$, $\mu := \delta_{\gamma_{01}}$ is the unique representing measure for γ. Since $\operatorname{supp} \mu = \{\gamma_{01}\}$, we see that $\operatorname{supp} \mu \subseteq \mathbf{T}$ (the unit circle) or $\operatorname{supp} \mu \subseteq \bar{\mathbf{D}}$ (the closed unit disk), depending upon whether $|\gamma_{01}| = 1$ or $|\gamma_{01}| \leq 1$. By contrast, we show that γ_{01} is never in the support of a minimal representing measure when $\operatorname{rank} M(1) = 2$ or $\operatorname{rank} M(1) = 3$ (Corollary 2.7).

We next briefly introduce the case when $\operatorname{rank} M(1) = 2$. Proposition 1.7 allows us to assume that $\{\mathit{1}, Z\}$ is linearly independent and that

$$\bar{Z} = \alpha \mathit{1} + \beta Z, \tag{1.4}$$

where $\alpha, \beta \in \mathbf{C}$. Moreover, (1.4) implies at once that

$$\begin{cases} \alpha + \bar{\alpha}\beta &= 0 \\ |\beta| &= 1 \end{cases} \tag{1.5}$$

(cf. [CuFi3, (2.2)]). In Section 4, we use (1.4) and (1.5) to solve the rank 2 disk problem; in particular, we prove that the support of a representing measure lies in the line $\bar{z} = \alpha + \beta z$. Also in Section 4, we reduce the case when $\operatorname{rank} M(1) = 3$ to an equivalent problem with $\operatorname{rank} M(1) = 2$.

Let us recall the notion of *localizing matrix* introduced in [CuFi5]. Let $q \in \mathcal{P}_{2n}$, $q \not\equiv 0$, and define k by $\deg q = 2k$ or $\deg q = 2k - 1$. There exists a unique *localizing matrix* $M_q(n)$ (of size $\frac{(n-k+1)(n-k+2)}{2}$) such that

$$\left\langle M_q(n)\hat{f}, \hat{g} \right\rangle = \Lambda(qf\bar{g}) \;\; (f, g \in \mathcal{P}_{n-k}).$$

Thus, if a representing measure μ for γ is supported in K_q, then for $f \in \mathcal{P}_{n-k}$,

$$\left\langle M_q(n)\hat{f}, \hat{f} \right\rangle = \Lambda(q\,|f|^2) = \int q\,|f|^2 \, d\mu \geq 0,$$

whence $M_q(n) \geq 0$.

For $q(z, \bar{z}) := 1 - \bar{z}z$, [CuFi5, (1.6)] shows that

$$M_{1-\bar{z}z}(2) := \begin{pmatrix} 1 - \gamma_{11} & \gamma_{01} - \gamma_{12} & \gamma_{10} - \gamma_{21} \\ \gamma_{10} - \gamma_{21} & \gamma_{11} - \gamma_{22} & \gamma_{20} - \gamma_{31} \\ \gamma_{01} - \gamma_{12} & \gamma_{02} - \gamma_{13} & \gamma_{11} - \gamma_{22} \end{pmatrix}.$$

Theorem 1.2 implies that $\gamma^{(2)}$ has a rank $M(1)$-atomic representing measure supported in $\bar{\mathbf{D}}$ if and only if $M(1) \geq 0$ and $M_{1-\bar{z}z}(2) \geq 0$ for some flat extension $M(2)$ of $M(1)$. Notice that the condition $\gamma_{11} \leq \gamma_{00}(=1)$ is obviously necessary if one wants to ensure that $M_{1-\bar{z}z}(2) \geq 0$. Our main existence result, which follows, shows that $M(1) \geq 0$ and $\gamma_{11} \leq \gamma_{00}$ are indeed *sufficient* to guarantee the existence of a rank $M(1)$-atomic representing measure supported in $\bar{\mathbf{D}}$.

Theorem 1.8 *The following statements are equivalent for* $\gamma \equiv \gamma^{(2)}$.
(i) *There exists a representing measure supported in* $\bar{\mathbf{D}}$;
(ii) *There exists a* rank $M(1)$*-atomic representing measure supported in* $\bar{\mathbf{D}}$;
(iii) *There exists a flat extension* $M(2)$ *for which the associated measure* $\nu[M(2)]$ (*cf.* *Theorem* 1.1) *is supported in* $\bar{\mathbf{D}}$, *with* rank $M(1)$ − rank $M_{1-\bar{z}z}(2)$ *atoms in* $\mathbf{T}$;
(iv) $M(1) \geq 0$ *and* $\gamma_{11} \leq \gamma_{00}$.

Section 2 is devoted to preliminary reductions concerning the quadratic moment problem for the closed unit disk. In Section 3 we give a detailed (constructive) analysis of the quadratic moment problem for the unit circle. In Theorem 3.1 we prove that there exists a rank $M(1)$-atomic representing measure supported in the unit circle if and only if $M(1) \geq 0$ and $\gamma_{11} = \gamma_{00}$. Minimal representing measures are unique in the rank 1 and rank 2 cases (Remarks 3.2 and 3.4), and are parameterized by the points of $\mathbf{T}$ in the rank 3 case (Corollary 4.12(ii)). We present the proof of Theorem 1.8 in Section 4, partially as a consequence of our analysis of the $\mathbf{T}$-moment problem in Section 3; our proof of (iv) ⇒ (ii) is constructive, and independent of the flat extension technique. In particular, when rank $M(1) = 2$, Theorem 4.2 gives a complete description of the 2-atomic (minimal) representing measures supported in $\bar{\mathbf{D}}$. For the case when rank $M(1) = 3$, in Theorem 4.11 and Algorithm 4.14 we give a complete (constructive) description of the 3-atomic representing measures supported in $\bar{\mathbf{D}}$. Section 5 is devoted to a refined analysis of the rank 3 case; in Theorem 5.5 we identify an ellipse $\mathcal{E} \subseteq \mathbf{D}$ such that $z_0 \in \mathcal{E}$ if and only if z_0 is an atom of a 3-atomic representing measure supported in $\bar{\mathbf{D}}$ with the remaining two atoms in the unit circle. More generally, $z_0 \in \bar{\mathbf{D}}$ is an atom of a 3-atomic representing measure supported in $\bar{\mathbf{D}}$ if and only z_0 is not in the interior region of $\mathcal{E}$ (Theorem 5.7). For such a point z_0, Theorem 5.7 completely describes the minimal representing measures supported in $\bar{\mathbf{D}}$ for which z_0 is an atom. Finally, as an application, in Section 6 we identify classes of cubic polynomials having three distinct roots in the unit circle; these include, for example, the polynomials $z^3 + w\bar{\beta}z^2 - \beta z - w$, where $|w| = 1$, $|\beta| < 1$.

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2 Preliminaries

This section contains some useful reductions in the quadratic moment problem for the unit disk. Thus, we shall always assume $M(1) \geq 0$ and $\gamma_{11} \leq \gamma_{00} = 1$. We begin by showing that one can always assume $\gamma_{01} \geq 0$ (Corollary 2.2 below).

Lemma 2.1 *Let $M(n)$ be the moment matrix for $\gamma \equiv \gamma^{(2n)}$. For $0 \neq \lambda \in \mathbf{C}$, define D_λ as the $m(n) \times m(n)$ diagonal matrix with $\bar{\lambda}^i \lambda^j$ as the entry corresponding to row and column $\bar{Z}^i Z^j$, that is, $D_\lambda := diag(1, \lambda, \bar{\lambda}, \lambda^2, \bar{\lambda}\lambda, \bar{\lambda}^2, ...)$. Let $M(n)\tilde{} := D^* M(n) D$. Then*
(i) *$M(n)\tilde{}$ is a moment matrix, namely the moment matrix associated to $\tilde{\gamma}_{ij} := \bar{\lambda}^i \lambda^j \gamma_{ij}$ $(0 \leq i + j \leq 2n)$;*
(ii) *$M(n)\tilde{} \geq 0$ if and only if $M(n) \geq 0$;*
(iii) *$\operatorname{rank} M(n)\tilde{} = \operatorname{rank} M(n)$;*
(iv) *γ admits a finitely atomic representing measure $\mu \equiv \sum \rho_k \delta_{z_k}$ if and only if $\tilde{\gamma}$ admits a finitely atomic representing measure $\tilde{\mu} \equiv \sum \rho_k \delta_{\tilde{z}_k}$, with $\tilde{z}_k := \lambda z_k$.*
In brief, $\gamma^{(2n)}$ and $\tilde{\gamma}^{(2n)}$ give rise to equivalent truncated moment problems, whose representing measures satisfy the relation $\operatorname{supp} \tilde{\mu} = \lambda \operatorname{supp} \mu$.

Proof. Straightforward from the definition of $M(n)\tilde{}$. ∎

Corollary 2.2 *Let $M(n)$ be the moment matrix for $\gamma^{(2n)}$, with $\gamma_{01} \neq 0$, and define $\lambda := \dfrac{\gamma_{10}}{|\gamma_{01}|}$ and $\tilde{\gamma}_{ij} := \bar{\lambda}^i \lambda^j \gamma_{ij}$ $(0 \leq i + j \leq 2n)$. Then the equivalent family $\tilde{\gamma}^{(2n)}$ satisfies $\tilde{\gamma}_{01} > 0$ and $\tilde{\gamma}_{ii} = \gamma_{ii}$ $(0 \leq i \leq n)$.*

Remark 2.3 *Observe that λ in Corollary 2.2 is of modulus 1, so multiplication by λ is a rotation. It follows that representing measures μ and $\tilde{\mu}$ for γ and $\tilde{\gamma}$, respectively, will simultaneously share any rotation-invariant properties (e.g., the number of atoms located in $\mathbf{D}$ or $\mathbf{T}$). This will be important for the qualitative analysis of the support to be carried out in Algorithm 4.14.* ∎

Lemma 2.4 *Suppose $M(1)$ is positive and invertible, with $\gamma_{11} \leq \gamma_{00} = 1$ and $\gamma_{01} \in \mathbf{R}$. Then $|\gamma_{01}| < 1$ and $|\gamma_{02}| < \gamma_{11}$.*

Proof. For a square matrix A, let $A_{\{i_1,...,i_k\}}$ denote the compression of A to rows and columns indexed by $i_1, ..., i_k$; if A is positive and invertible, so is $A_{\{i_1,...,i_k\}}$. Now $\det M(1)_{\{1,2\}} = \gamma_{11} - \gamma_{01}^2 > 0$, so $|\gamma_{01}| < \sqrt{\gamma_{11}} \leq 1$; also, $\det M(1)_{\{2,3\}} = \gamma_{11}^2 - |\gamma_{02}|^2 > 0$, so $|\gamma_{02}| < \gamma_{11}$. ∎

Our next result will allow us to restrict attention to the case when $\operatorname{rank} M(1) = 2$.

Lemma 2.5 *Assume that $M(1) \geq 0$, $\operatorname{rank} M(1) = 3$, $\gamma_{11} \leq \gamma_{00} = 1$ and $\gamma_{01} \in \mathbf{R}$. For $\rho \neq 1$ let*

$$M(1)(\rho) := \begin{pmatrix} 1 & \frac{\gamma_{01}-\rho}{1-\rho} & \frac{\gamma_{01}-\rho}{1-\rho} \\ \frac{\gamma_{01}-\rho}{1-\rho} & \frac{\gamma_{11}-\rho}{1-\rho} & \frac{\gamma_{20}-\rho}{1-\rho} \\ \frac{\gamma_{01}-\rho}{1-\rho} & \frac{\gamma_{02}-\rho}{1-\rho} & \frac{\gamma_{11}-\rho}{1-\rho} \end{pmatrix}. \tag{2.1}$$

Then there exists a unique ρ_0 $(0 < \rho_0 < 1)$ such that $M(1)(\rho_0) \geq 0$ and $\operatorname{rank} M(1)(\rho_0) = 2$.

Proof. Observe that for $\rho \neq 1$,

$$\det M(1)(\rho) = \frac{\det M(1) - P\rho}{(1-\rho)^3},$$

where

$$P := \det M(1) + 2(1-\gamma_{01})^2(\gamma_{11} - \text{Re}\ \gamma_{02}) > \det M(1) > 0 \tag{2.2}$$

(by Lemma 2.4). It follows that $\det M(1)(\rho) = 0$ if and only if

$$\rho = \frac{\det M(1)}{P}. \tag{2.3}$$

Thus, we let $\rho_0 := \frac{\det M(1)}{P}$. Observe that $0 < \rho_0 < 1$. Moreover, $M(1)(\rho)$ is positive and has rank 2 if and only if

$$\det \begin{pmatrix} 1 & \dfrac{\gamma_{01}-\rho}{1-\rho} \\ \dfrac{\gamma_{01}-\rho}{1-\rho} & \dfrac{\gamma_{11}-\rho}{1-\rho} \end{pmatrix} > 0. \tag{2.4}$$

Now, (2.4) holds if and only if $(\gamma_{01}-\rho)^2 < (1-\rho)(\gamma_{11}-\rho)$, or equivalently,

$$\rho < \frac{\gamma_{11} - \gamma_{01}^2}{1 + \gamma_{11} - 2\gamma_{01}}. \tag{2.5}$$

(Observe that $1 + \gamma_{11} - 2\gamma_{01} = (1-\gamma_{01})^2 + (\gamma_{11} - \gamma_{01}^2) > 0$.) By combining (2.3) and (2.5), we see that $M(1)(\rho_0) \geq 0$ and $\operatorname{rank} M(1)(\rho_0) = 2$ if and only if

$$\rho_0 \equiv \frac{\det M(1)}{P} < \frac{\gamma_{11} - \gamma_{01}^2}{1 + \gamma_{11} - 2\gamma_{01}}. \tag{2.6}$$

Now,

$$\begin{aligned} &(\gamma_{11} - \gamma_{01}^2)P - (1 + \gamma_{11} - 2\gamma_{01}) \det M(1) \\ &= (\gamma_{11} - \gamma_{01}^2)[\det M(1) + (1 - \gamma_{01}^2)(2\gamma_{11} - \gamma_{02} - \gamma_{20})] - [(1 - \gamma_{01}^2) + (\gamma_{11} - \gamma_{01}^2)] \det M(1) \\ &= (\gamma_{11} - \gamma_{01}^2)(1 - \gamma_{01}^2)(2\gamma_{11} - \gamma_{02} - \gamma_{20}) - (1 - \gamma_{01}^2) \det M(1) \\ &= (1 - \gamma_{01}^2)[(\gamma_{11} - \gamma_{01}^2)(2\gamma_{11} - \gamma_{02} - \gamma_{20}) - \det M(1)] \\ &= (1 - \gamma_{01}^2)\, |\gamma_{11} - \gamma_{02}|^2 > 0 \text{ (by Lemma 2.4)}, \end{aligned}$$

so (2.6) holds. ∎

Corollary 2.6 *Given a positive and invertible $M(1)$, with $\gamma_{11} \leq \gamma_{00} = 1$ and $\gamma_{01} \in \mathbf{R}$, let $\rho_0 := \dfrac{\det M(1)}{P}$, where $P := \det M(1) + 2(1-\gamma_{01})^2(\gamma_{11} - \text{Re}\ \gamma_{02})$. Let $M(1)(\rho_0)$ be given by (2.1) (so that $\operatorname{rank} M(1)(\rho_0) = 2$). The representing measures for $\gamma^{(2)}$ having an atom at 1 are of the form $\mu := (1-\rho_0)\nu + \rho_0\delta_1$, where ν is a representing measure for $M(1)(\rho_0)$. The measure μ is 3-atomic (minimal) if and only if ν is 2-atomic (minimal); moreover, $\operatorname{supp}\nu \subseteq \bar{\mathbf{D}}$ (resp. $\mathbf{T}$) if and only if $\operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$ (resp. $\mathbf{T}$).*

Proof. Observe that $\gamma_{ij}(\mu) = (1-\rho_0)\gamma_{ij}(\nu) + \rho_0 \ (0 \le i+j \le 2)$. We claim that $M(1)(\rho_0)$ cannot admit a minimal (i.e., 2-atomic) representing measure ν whose support includes an atom at 1. For, if that were the case, μ would be a 2-atomic representing measure for $M(1)$, a contradiction to Proposition 1.5. ∎

Corollary 2.7 *Let μ be a minimal finitely atomic representing measure for $\gamma^{(2)}$, with $\gamma_{00}[\mu] = 1$. Then $\gamma_{01} \in \operatorname{supp}\mu$ if and only if $\mu = \delta_{\gamma_{01}}$ and* $\operatorname{rank} M(1) = 1$.

Proof. Assume first that $\operatorname{rank} M(1) = 3$ and $\gamma_{01} \in \operatorname{supp}\mu$, so $\mu = \rho\delta_{\gamma_{01}} + (1-\rho)\nu$, with $0 < \rho < 1$ and ν 2-atomic. By writing the moment identities,

$$\left\{ \begin{array}{lcl} \gamma_{01} & = & \rho\gamma_{01} + (1-\rho)\gamma_{01}[\nu] \\ \gamma_{02} & = & \rho\gamma_{01}^2 + (1-\rho)\gamma_{02}[\nu] \\ \gamma_{11} & = & \rho\left|\gamma_{01}\right|^2 + (1-\rho)\gamma_{11}[\nu] \end{array} \right. . \tag{2.7}$$

It follows easily from (2.7) that $\gamma_{01}[\nu] = \gamma_{01}$, $\gamma_{02}[\nu] = \frac{\gamma_{02}-\rho\gamma_{01}^2}{1-\rho}$ and $\gamma_{11}[\nu] = \frac{\gamma_{11}-\rho|\gamma_{01}|^2}{1-\rho}$. Using these data we now form $M(1)[\nu]$, which must be of rank 2. However, a straightforward calculation shows that $\det M(1)[\nu] = \frac{\det M(1)}{(1-\rho)^2} > 0$, a contradiction. If instead $\operatorname{rank} M(1) = 2$, μ must be of the form $\rho\delta_{\gamma_{01}} + (1-\rho)\delta_z$ for some $z \neq \gamma_{01}$, with $0 < \rho < 1$. As before, $\gamma_{01} = \rho\gamma_{01} + (1-\rho)z$, which implies that $z = \gamma_{01}$, again a contradiction. We conclude that the only instance in which γ_{01} lies in the support of a minimal representing measure is when $\operatorname{rank} M(1) = 1$. ∎

Example 2.8 *Let $\gamma_{00} := 1$, $\gamma_{01} := (1+i)/2$, $\gamma_{02} := 1/8 + i/2$ and $\gamma_{11} := 3/4$. The associated moment matrix $M(1)$ is positive and invertible, with* $\det M(1) = 3/64$. *Let $\lambda := \gamma_{10}/\left|\gamma_{01}\right| = (1-i)/\sqrt{2}$. By Corollary 2.2, the family $\tilde{\gamma}_{00} := 1$, $\tilde{\gamma}_{01} := \lambda\gamma_{01} = 1/\sqrt{2}$, $\tilde{\gamma}_{02} := \lambda^2\gamma_{02} = 1/2 - i/8$ and $\tilde{\gamma}_{11} := |\lambda|^2\gamma_{11} = 3/4$ is equivalent to the family $\{\gamma_{ij}\}$, and $\tilde{\gamma}_{01} > 0$. Lemma 2.5 says that with $\tilde{P} := \det\widetilde{M(1)} + 2(1-\tilde{\gamma}_{01})(\tilde{\gamma}_{11} - \operatorname{Re}\tilde{\gamma}_{02}) = 51/64 - 1/\sqrt{2}$ (cf. (2.2)) and $\tilde{\rho} := \det\widetilde{M(1)}/\tilde{P} = 3/(51-32\sqrt{2})$ (cf. (2.6)), the matrix $\widetilde{M(1)(\tilde{\rho})}$ given as in (2.1) has rank 2. Moreover, if $\tilde{\nu}$ is a 2-atomic representing measure for $\widetilde{M(1)(\tilde{\rho})}$, then $\tilde{\mu} := (1-\tilde{\rho})\tilde{\nu} + \tilde{\rho}\delta_1$ is a 3-atomic representing measure for $\widetilde{M(1)}$. By rotating the support by $\bar{\lambda}$, the measure $\tilde{\mu}$ gives rise to a 3-atomic representing measure μ for $M(1)$.* ∎

We shall revisit Example 2.8 in Sections 4 and 5.

3 Existence of Representing Measures Supported on the Unit Circle

In this section we solve the quadratic moment problem for the unit circle; as usual, we assume $\gamma_{00} = 1$.

Theorem 3.1 *Suppose $M(1)$ is positive. The following statements are equivalent for $\gamma^{(2)}$.*
(i) *There exists a representing measure supported in $\mathbf{T}$;*
(ii) *There exists a* $\operatorname{rank} M(1)$*-atomic representing measure supported in $\mathbf{T}$;*
(iii) $\gamma_{11} = \gamma_{00}$.

Remark 3.2 *The proof of Theorem 3.1 will show that when (iii) holds, the flat extensions* $M(2)$ *corresponding to* $\operatorname{rank} M(1)$*-atomic representing measures supported in* $\mathbf{T}$ *are completely determined by appropriate choices of* γ_{03}*. The choice is unique when* $\operatorname{rank} M(1) = 1$ *or* $\operatorname{rank} M(1) = 2$*, so in these cases there exist unique* $\operatorname{rank} M(1)$*-atomic representing measures supported in* $\mathbf{T}$*. When* $\operatorname{rank} M(1) = 3$*,* γ_{03} *may be chosen from a circle of positive radius, so in this case there exist infinitely many* 3*-atomic representing measures supported in* $\mathbf{T}$*. The case when* $\operatorname{rank} M(1) = 1$ *corresponds to* $\det M(1)_{\{1,2\}} = 0$*, i.e.,* $|\gamma_{01}| = \gamma_{11} = \gamma_{00} = 1$*; in this case the unique representing measure,* $\mu \equiv \delta_{\gamma_{01}}$ *(cf. Theorem 1.3), is clearly supported in* $\mathbf{T}$*, and the unique flat extension corresponds to* $\gamma_{03} := \gamma_{01}\gamma_{02}$*. In the sequel we prove Theorem 3.1 in the cases when* $\operatorname{rank} M(1) > 1$*.* ∎

Lemma 3.3 ([Smu], [CuFi3]) *Assume* $M(1) \geq 0$*. A moment matrix extension*

$$M(2) \equiv \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix} \tag{3.1}$$

of $M(1)$ *is flat (i.e., rank-preserving) if and only if there exists a* 3×3 *matrix* W *such that* $B = M(1)W$ *and* $C = W^*M(1)W$*.*

Proof of Theorem 3.1. Observe that in order for a moment matrix extension $M(2)$ to give rise to a representing measure μ supported on the unit circle $\mathbf{T}$, it is necessary that the moments satisfy

$$\gamma_{i+1,j+1} = \int_{\mathbf{T}} \bar{z}^{i+1} z^{j+1} \, d\mu = \int_{\mathbf{T}} |z|^2 \, \bar{z}^i z^j \, d\mu = \gamma_{ij}; \tag{3.2}$$

thus, $\gamma_{11} = \gamma_{00}$, $\gamma_{22} = \gamma_{11}$, $\gamma_{12} = \gamma_{01}$, $\gamma_{13} = \gamma_{02}$, etc. It follows from Lemma 3.3 that, irrespective of $\operatorname{rank} M(1)$, a flat moment matrix extension $M(2)$ is completely determined by the value assigned to γ_{03} (see (3.3) below). Indeed, it is always the case that the B block is determined by its leftmost column, in this case $(\gamma_{02}, \gamma_{12}, \gamma_{03})^T$ $(= (\gamma_{02}, \gamma_{01}, \gamma_{03})^T)$. If, in addition, $M(2)$ is a flat extension, then the C block must be of the form W^*B, where $B = M(1)W$. Therefore, any flat extension $M(2)$ must be of the form

$$M(2) = \begin{pmatrix} 1 & \gamma_{01} & \gamma_{10} & \gamma_{02} & 1 & \gamma_{20} \\ \gamma_{10} & 1 & \gamma_{20} & \gamma_{01} & \gamma_{10} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & 1 & \gamma_{03} & \gamma_{01} & \gamma_{10} \\ \gamma_{20} & \gamma_{10} & \gamma_{30} & 1 & \gamma_{20} & \gamma_{40} \\ 1 & \gamma_{01} & \gamma_{10} & \gamma_{02} & 1 & \gamma_{20} \\ \gamma_{02} & \gamma_{03} & \gamma_{01} & \gamma_{04} & \gamma_{02} & 1 \end{pmatrix}, \tag{3.3}$$

where γ_{04} depends upon the value assigned to γ_{03}. (Note that $\bar{Z}Z = \mathit{1}$ in $\mathcal{C}_{M(2)}$, as required by Proposition 1.4).

When $\operatorname{rank} M(1) = 2$, we can assume, without loss of generality, that

$$\bar{Z} = \alpha \mathit{1} + \beta Z, \tag{3.4}$$

whence $\alpha + \bar{\alpha}\beta = 0$ and $|\beta| = 1$ (cf. (1.5)). Moreover, since $M(1)$ is recursively generated, [CuFi4, Proposition 1.7(i)] implies that any flat extension $M(2)$ is recursively generated, so in $\mathcal{C}_{M(2)}$, $1 = Z\bar{Z} = \alpha Z + \beta Z^2$, which establishes at once a linear dependence between the column Z^2 and the columns 1 and Z, namely

$$Z^2 = \frac{1}{\beta}1 - \frac{\alpha}{\beta}Z = \bar{\beta}1 - \alpha\bar{\beta}Z = \bar{\beta}1 + \bar{\alpha}Z. \tag{3.5}$$

By Proposition 1.7, this also shows that $\bar{Z}^2$ must then be a linear combination of 1 and $\bar{Z}$, namely $\bar{Z}^2 = \beta 1 + \alpha\bar{Z}$. Therefore, the construction of $M(2)$ really depends on building the column Z^2, which in turn depends on finding suitable complex numbers γ_{03} and γ_{04} fitting (3.5). From (3.3), the unique solution is obtained by specifying

$$\gamma_{03} := \bar{\beta}\gamma_{01} + \bar{\alpha}\gamma_{02}. \tag{3.6}$$

It follows from Lemma 3.3 that the corresponding flat extension is a moment matrix $M(2)$ if the block $C := W^*M(1)W$ is Toeplitz. By [CuFi2, Proposition 2.3], it suffices to check that $C_{11} = C_{22}$. Now, using (3.5) and the relation $\bar{Z}Z = 1$ in $\mathcal{C}_{M(2)}$, we have

$$C_{11} = \bar{\beta}\gamma_{20} + \bar{\alpha}\gamma_{10} = \overline{\alpha\gamma_{01} + \beta\gamma_{02}} = 1 = C_{22},$$

by (3.4), as desired. Since $\bar{Z}Z = 1$ in $M(2)$, Theorem 1.1 and Proposition 1.4 imply that the 2-atomic representing measure corresponding to $M(2)$ is supported in $\mathbf{T}$.

We now consider the case when rank $M(1) = 3$ and $\gamma_{11} = \gamma_{00} = 1$. Thus $\det M(1) > 0$ and $1 - |\gamma_{01}|^2 = \det M(1)_{\{1,2\}} > 0$. Write $\gamma_{01} \equiv a + ib$, $\gamma_{02} \equiv c + id$, $\gamma_{03} \equiv r + is$. A calculation of $C := B^*M(1)^{-1}B$ shows that $C_{11} = C_{22}$ (i.e., C is Toeplitz) if and only if (r, s) satisfies

$$(r - \frac{c_r}{1 - |\gamma_{01}|^2})^2 + (s - \frac{c_s}{1 - |\gamma_{01}|^2})^2 = (\frac{\det M(1)}{1 - |\gamma_{01}|^2})^2,$$

where

$$c_r := -a^3 + 3ab^2 + 2ac - ac^2 - 2bd - 2bcd + ad^2$$

and

$$c_s := -3a^2b + b^3 + 2bc + bc^2 + 2ad - 2acd - bd^2.$$

For each such $\gamma_{03} \equiv r + is$, the corresponding 3-atomic representing measure is supported in $\mathbf{T}$ since $\bar{Z}Z = 1$ (Theorem 1.1 and Proposition 1.4). ∎

Remark 3.4 *(i) In the rank* 2 *case of Theorem 3.1(iii) ⇒ (ii), we can give a direct argument to show that the representing measure we constructed is supported in* $\mathbf{T}$. *Indeed, the zeros of the generating polynomial* $g(z) := z^2 - (\bar{\beta} + \bar{\alpha}z)$ *are* $z_0 := \frac{\bar{\alpha} - \sqrt{\bar{\alpha}^2 + 4\bar{\beta}}}{2}$ *and* $z_1 := \frac{\bar{\alpha} + \sqrt{\bar{\alpha}^2 + 4\bar{\beta}}}{2}$, *both on the unit circle. Moreover, we know that* card supp $\mu \geq$ rank $M(1) = 2$, *so* $z_0 \neq z_1$, *i.e.,* $\alpha^2 + 4\beta \neq 0$. *(Alternatively, if* $\alpha^2 + 4\beta = 0$ *then* $\alpha(\alpha\bar{\alpha} - 4) = \alpha^2\bar{\alpha} + 4\beta\bar{\alpha} = 0$ *(using (1.5)), so* $|\alpha| = 2$. *However, (3.4) implies* $\alpha + \beta\gamma_{01} = \gamma_{10}$, *so* $|\alpha| \leq |\gamma_{10} - \beta\gamma_{01}| \leq 2|\gamma_{01}| < 2\gamma_{11} = 2$, *a contradiction.)*
(ii) For the rank 3 *case of Theorem 3.1(iii) ⇒ (ii), we shall see later on (Corollary 4.12 (ii)) that a point* z_0 *is in the support of a* 3*-atomic representing measure* μ *with* supp $\mu \subseteq \bar{\mathbf{D}}$ *if and only if* $z_0 \in \mathbf{T}$, *in which case* supp $\mu \subseteq \mathbf{T}$.

(iii) In Theorem 3.1, as an alternative check that $\nu[M(2)]$ is supported in $\mathbf{T}$, we can apply Theorem 1.2 and verify that $M_{1-\bar{z}z}(2)=0$ in this case. For, using [CuFi5, (1.6)],

$$M_{1-\bar{z}z}(2)=M_1(2)-M_{\bar{z}z}(2)=\begin{pmatrix}1&\gamma_{01}&\gamma_{10}\\\gamma_{10}&\gamma_{11}&\gamma_{20}\\\gamma_{01}&\gamma_{02}&\gamma_{11}\end{pmatrix}-\begin{pmatrix}\gamma_{11}&\gamma_{12}&\gamma_{21}\\\gamma_{21}&\gamma_{22}&\gamma_{31}\\\gamma_{12}&\gamma_{13}&\gamma_{22}\end{pmatrix}$$
$$=M(1)-M(1)=0,$$

by (3.2). ∎

The preceding arguments readily lead to a more general result for truncated $\mathbf{T}$-moment problems, which we now state without proof.

Theorem 3.5 *Let $M(n)$ be a positive moment matrix satisfying $\gamma_{i+1,j+1}=\gamma_{ij}$ for every (i,j) such that $0\le i+j\le 2n-2$. Then one can always choose a complex number $\gamma_{0,2n+1}$ in such a way that the restriction of the column Z^{n+1} to the first $m(n)$ rows becomes a linear combination of the columns in $M(n)$. Therefore, $M(n)$ admits a flat extension $M(n+1)$, which also satisfies the condition $M_{1-\bar{z}z}(n+1)=0$, thus giving rise to a* rank $M(n)$*-atomic (minimal) representing measure supported in $\mathbf{T}$.*

Corollary 3.6 (*cf.* [AhKr, Theorem I.I.12], [CuFi1, Theorem 6.12], [Ioh, p. 211]) *The classical truncated trigonometric moment problem*

$$\int z^k\,d\mu(z)=\gamma_k\ \ (-n\le k\le n)$$

($\gamma_0>0$ and $\gamma_{-k}=\bar{\gamma}_k$ for $k=1,...,n$), admits a solution if and only if the Toeplitz matrix $T_\gamma:=(\gamma_{i-j})_{i,j=0}^n$ is positive.

Example 3.7 *We conclude this section by examining how the results of Section 2 and the present section apply to the case when $M(1)=I$, i.e., $\gamma_{00}=\gamma_{11}=1$ and $\gamma_{01}=\gamma_{10}=\gamma_{02}=\gamma_{20}=0$. Now* rank $M(1)=3$*, $P=3$, and $\rho_0=\frac{1}{3}$, so Corollary 2.6 reduces the existence problem for representing measures supported at $z_0=1$ to the study of the rank 2 matrix*

$$M(1)\tilde{}\equiv M(1)(\rho_0)=\begin{pmatrix}1&-\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&1&-\frac{1}{2}\\-\frac{1}{2}&-\frac{1}{2}&1\end{pmatrix}.$$

Further, $\bar{Z}=-1-Z$ in $\mathcal{C}_{M(1)\tilde{}}$, so the unique flat extension with representing measure supported on the unit circle is determined by $Z^2=-1-Z$ (cf. (3.5)). This leads to the generating polynomial $g(z)=z^2+z+1$, whose zeros are the two cube roots of unity $\omega=\frac{-1+\sqrt{3}i}{2}$ and $\omega^2=\frac{-1-\sqrt{3}i}{2}$. We can then use the Vandermonde equation $V(\omega,\omega^2)(\tilde{\rho}_0,\tilde{\rho}_1)^T=(\tilde{\gamma}_{00},\tilde{\gamma}_{01})=(1,-\frac{1}{2})^T$ to compute the associated densities $\tilde{\rho}_0=\tilde{\rho}_1=\frac{1}{2}$ (cf. Theorem 1.1), so $M(1)\tilde{}$ admits the 2-atomic representing measure $\nu=\frac{1}{2}(\delta_\omega+\delta_{\omega^2})$. By Corollary 2.6, $M(1)$ then admits the 3-atomic representing measure $\mu=\frac{1}{3}(\delta_1+\delta_\omega+\delta_{\omega^2})$ (and $M(2)\equiv M(2)[\mu]$ is given as in [CuFi2, Example 3.2]); μ is thus the unique 3-atomic (minimal) representing measure for $\gamma^{(2)}$ supported in $\mathbf{T}$ with $1\in$ supp μ. However, we pointed out earlier (Theorem

1.3) that the set of 3-atomic measures for $\gamma^{(2)}$ contains a sub-parameterization by a circle, so there are infinitely many such measures. Indeed, any choice of γ_{03} would lead to a flat moment matrix extension $M(2)$, by letting $\gamma_{12} \equiv \gamma_{21} := 0$, $\gamma_{30} := \bar{\gamma}_{03}$, $\gamma_{04} := 0$, $\gamma_{13} \equiv \gamma_{31} := 0$, and $\gamma_{22} := |\gamma_{03}|^2$. If one insists, however, that the support be contained in the unit circle, then $|\gamma_{03}| = 1$, and the generating polynomial is $g(z) \equiv z^3 - \gamma_{03}$, whose zeros are the cube roots of γ_{03}. Since the flat extensions of $M(1)$ are completely determined by choices for γ_{03} (cf. Remark 3.2), we see that the 3-atomic representing measures for $M(1)$ that are supported in $\mathbf{T}$ are completely parameterized by the points γ_{03} of $\mathbf{T}$. (In Theorem 4.11 we will show that for any rank 3 circle problem, the minimal representing measures supported in $\mathbf{T}$ are naturally parameterized by the points of $\mathbf{T}$.) ∎

4 Existence of Representing Measures Supported in the Unit Disk

The first goal of this section is to give a complete constructive description of the 2-atomic (minimal) representing measures in the rank 2 disk problem (Theorem 4.2). In Corollary 4.3, we use Theorem 4.2 and the rank reduction tools of Section 2 to prove the existence of a 3-atomic (minimal) representing measure in the rank 3 disk problem. These results are then used to prove Theorem 1.8. In Theorem 4.11 we continue our analysis of the rank 3 disk problem by providing a computational test to determine whether a prescribed point $z \in \bar{\mathbf{D}}$ is an atom of some minimal representing measure μ with $\operatorname{supp} \mu \subseteq \bar{\mathbf{D}}$. By combining this result with Theorem 4.2, in Algorithm 4.14 we give a constructive parameterization of all 3-atomic (minimal) representing measures having z as an atom.

Let us initially assume that $M(1) \geq 0$, $\gamma_{11} \leq \gamma_{00} = 1$, and $\operatorname{rank} M(1) = 2$ (that is, $|\gamma_{01}|^2 < \gamma_{11}$ and $\bar{Z} = \alpha 1 + \beta Z$). Direct calculation shows that

$$\alpha = \frac{\gamma_{10}\gamma_{11} - \gamma_{01}\gamma_{20}}{\delta}$$

and

$$\beta = \frac{\gamma_{20} - \gamma_{10}^2}{\delta}, \tag{4.1}$$

where $\delta := \gamma_{11} - |\gamma_{01}|^2$ (> 0; cf. [CuFi2, (6.1)]); observe that (4.1) leads to

$$\bar{\beta}\delta = \gamma_{02} - \gamma_{01}^2. \tag{4.2}$$

To find a 2-atomic representing measure $\mu \equiv \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1}$ supported in $\bar{\mathbf{D}}$, we shall solve the system of equations

$$\begin{cases} \rho_0 + \rho_1 & = & 1 \\ \rho_0 z_0 + \rho_1 z_1 & = & \gamma_{01} \\ \rho_0 z_0^2 + \rho_1 z_1^2 & = & \gamma_{02} \\ \rho_0 |z_0|^2 + \rho_1 |z_1|^2 & = & \gamma_{11} \end{cases}, \tag{4.3}$$

for $\rho_0, \rho_1 > 0$ and $z_0, z_1 \in \bar{\mathbf{D}}$. Let $\rho := \rho_0$, so $\rho_1 = 1 - \rho$. It follows that $\rho z_0 + (1-\rho) z_1 = \gamma_{01}$, from which we obtain

$$z_1 = \frac{\gamma_{01} - \rho z_0}{1 - \rho}. \tag{4.4}$$

The third equation in (4.3) now leads to

$$\rho z_0^2 + (1-\rho)\left(\frac{\gamma_{01}-\rho z_0}{1-\rho}\right)^2 = \gamma_{02},$$

which is equivalent to

$$\rho(z_0-\gamma_{01})^2 - (1-\rho)(\gamma_{02}-\gamma_{01}^2) = 0,$$

or

$$(z_0-\gamma_{01})^2 = \frac{(1-\rho)(\gamma_{02}-\gamma_{01}^2)}{\rho},$$

so

$$z_0 = \gamma_{01} \pm \sqrt{\frac{(1-\rho)\left|\gamma_{02}-\gamma_{01}^2\right|}{\rho}}e^{\frac{1}{2}iArg\left(\gamma_{02}-\gamma_{01}^2\right)}.$$

Let $t := \sqrt{\dfrac{1-\rho}{\rho}}$; since $0 < \rho < 1$, we see that $0 < t < +\infty$. (Note that we can recover ρ from t, that is,

$$\rho = \frac{1}{1+t^2}.) \tag{4.5}$$

Now, let

$$z_0 := \gamma_{01} + \sqrt{\frac{(1-\rho)\left|\gamma_{02}-\gamma_{01}^2\right|}{\rho}}e^{\frac{1}{2}iArg\left(\gamma_{02}-\gamma_{01}^2\right)} = \gamma_{01} + t\sqrt{\bar{\beta}\delta}, \tag{4.6}$$

(using (4.2)), where we agree to denote $\sqrt{\left|\gamma_{02}-\gamma_{01}^2\right|}e^{\frac{1}{2}iArg\left(\gamma_{02}-\gamma_{01}^2\right)}$ by $\sqrt{\bar{\beta}\delta}$. A straightforward calculation using (4.4), (4.5) and (4.6) then leads to

$$z_1 = \gamma_{01} - \frac{1}{t}\sqrt{\bar{\beta}\delta}. \tag{4.7}$$

From (4.6), we then have

$$\left|z_0\right|^2 = \left|\gamma_{01}\right|^2 + 2\text{Re }(t\bar{\gamma}_{01}\sqrt{\bar{\beta}\delta}) + t^2\delta = \left|\gamma_{01}\right|^2 + 2t\text{Re }(\bar{\gamma}_{01}\sqrt{\bar{\beta}\delta}) + t^2\delta,$$

and our goal is to make $|z_0| \le 1$. Similarly, (4.7) implies

$$\left|z_1\right|^2 = \left|\gamma_{01}\right|^2 - \frac{2}{t}\text{Re }(\bar{\gamma}_{01}\sqrt{\bar{\beta}\delta}) + \frac{1}{t^2}\delta,$$

and we wish to make $|z_1| \le 1$. Let $A :=\text{Re }(\bar{\gamma}_{01}\sqrt{\bar{\beta}\delta})$ and

$$C := 1 - \left|\gamma_{01}\right|^2 \ge \gamma_{11} - \left|\gamma_{01}\right|^2 = \delta > 0; \tag{4.8}$$

we are naturally led to the system of inequalities

$$\begin{cases} \delta t^2 + 2tA - C & \le & 0 \\ Ct^2 + 2tA - \delta & \ge & 0 \end{cases} \tag{4.9}$$

(representing the conditions $|z_0| \le 1$ and $|z_1| \le 1$, respectively). With $D := \sqrt{A^2 + C\delta} > |A|$, we obtain $\delta t^2 + 2tA - C = \delta(t - t_{11})(t - t_{12})$ and $Ct^2 + 2tA - \delta = C(t - t_{21})(t - t_{22})$, where

$$t_{11} := \frac{-A - D}{\delta} < 0,$$

$$t_{12} := \frac{-A + D}{\delta} > 0,$$

$$t_{21} := \frac{-A - D}{C} < 0$$

and

$$t_{22} := \frac{-A + D}{C} > 0.$$

Since $C \ge \delta$ (by (4.8)), we have $t_{22} \le t_{12}$, so the solution set for (4.9) is $[t_{22}, t_{12}]$.

With a value of t in the interval $[t_{22}, t_{12}]$, we define ρ, z_0 and z_1 using (4.5), (4.6), and (4.7) (or 4.4); calculations using (4.3) show that

$$\mu \equiv \mu_t := \rho\delta_{z_0} + (1 - \rho)\delta_{z_1} \tag{4.10}$$

is a representing measure for $M(1)$ supported in $\bar{\mathbf{D}}$.

Remark 4.1 *(i) Observe that if $t = t_{12}$, the first inequality in (4.9) becomes an equality, i.e., $|z_0| = 1$; if $t = t_{22}$ the second inequality in (4.9) becomes an equality, i.e., $|z_1| = 1$. To get both z_0 and z_1 in the unit circle we must have $t_{22} = t_{12}$, or $C = \delta$, that is, $\gamma_{11} = 1$ (in accord with Section 3).*
(ii) To obtain μ_t in (4.10) we chose one particular value of the square root of $\bar{\beta}\delta$, call it w; that is, $z_0(t) := \gamma_{01} + tw$ and $z_1(t) := \gamma_{01} - \frac{w}{t}$ $(t_{22} \le t \le t_{12})$. Similarly, we could have chosen $\tilde{w} := -w$ as the square root of $\bar{\beta}\delta$, and this would have led instead to $\tilde{z}_0(t) := \gamma_{01} + t\tilde{w}$ and $\tilde{z}_1(t) := \gamma_{01} - \frac{\tilde{w}}{t}$. If we let $\tilde{t} := \frac{1}{t}$ $(\frac{1}{t_{12}} \le \tilde{t} \le \frac{1}{t_{22}})$, we see at once that $\tilde{z}_0(\tilde{t}) = \gamma_{01} + \tilde{t}\tilde{w} = \gamma_{01} - \frac{w}{t} = z_1(t)$ and $\tilde{z}_1(\tilde{t}) = \gamma_{01} + tw = z_0(t)$. In particular, $\left|\tilde{z}_0(\frac{1}{t_{22}})\right| = |z_1(t_{22})| = 1$ and $\left|\tilde{z}_1(\frac{1}{t_{12}})\right| = |z_0(t_{12})| = 1$, so that $\tilde{t}_{22} = \frac{1}{t_{12}}$ and $\tilde{t}_{12} = \frac{1}{t_{22}}$. It follows that an atom of a 2-atomic representing measure can be chosen in either the segment $[z_0(t_{12}), z_0(t_{22})]$ or the segment $[z_1(t_{22}), z_1(t_{12})]$. We now claim that these two segments do not overlap, and that $z_0(t_{22}) \ne z_1(t_{12})$. As a matter of fact,

$$\frac{z_1(t_{12}) - z_0(t_{22})}{z_1(t_{22}) - z_0(t_{12})} = \frac{(\gamma_{01} - \frac{w}{t_{12}}) - (\gamma_{01} + t_{22}w)}{(\gamma_{01} - \frac{w}{t_{22}}) - (\gamma_{01} + t_{12}w)} = \frac{-\frac{w}{t_{12}} - t_{22}w}{-\frac{w}{t_{22}} - t_{12}w} = \frac{t_{22}}{t_{12}} > 0.$$

This shows that the segment $[z_0(t_{22}), z_1(t_{12})]$, which represents the "exclusion zone" for atoms of representing measures, has positive length. Incidentally, notice that γ_{01} is in that zone, as anticipated by Corollary 2.7. (For an illustration of the "exclusion zone," see Figure 1.) ∎

We summarize the preceding discussion in the following result.

Theorem 4.2 *Let $M(1)$ be positive, with $\gamma_{11} \leq \gamma_{00} = 1$ and* $\operatorname{rank} M(1) = 2$. *The 2-atomic (minimal) representing measures for $\gamma^{(2)}$ that are supported in $\bar{\mathbf{D}}$ are the measures $\mu_t := \frac{1}{1+t^2}\delta_{\gamma_{01}+tw} + \frac{t^2}{1+t^2}\delta_{\gamma_{01}-\frac{1}{t}w}$ $(t_{22} \leq t \leq t_{12})$, where w is a square root of $\bar{\beta}\delta$ (cf. (4.10)). More precisely, for $t_{22} < t_{12}$ and $t \in (t_{22}, t_{12})$, the associated representing measure μ_t has both atoms inside the open unit disk, while $\mu_{t_{22}}$ and $\mu_{t_{12}}$ each has one atom inside and one atom on the circle. When $t_{22} = t_{12}$ (which requires $\gamma_{11} = 1$), $\mu_{t_{22}}$ has both atoms on the unit circle. Finally, a point z belongs to the support of a 2-atomic (minimal) representing measure supported in the unit disk if and only if z belongs to the line segment $\{\gamma_{01}+tw : t_{22} \leq t \leq t_{12}\}$ or to the line segment $\{\gamma_{01} - \frac{w}{t} : t_{22} \leq t \leq t_{12}\}$.*

Corollary 4.3 *Let $M(1)$ be positive and invertible, with $\gamma_{11} \leq \gamma_{00} = 1$. Choose $\lambda \in \mathbf{T}$ so that the rotated system $\tilde{\gamma}_{ij}$ corresponding to λ via Lemma 2.1 satisfies $\tilde{\gamma}_{01} \in \mathbf{R}$ (e.g., let $\lambda := 1$ if $\gamma_{01} = 0$ and $\lambda := \frac{\gamma_{10}}{|\gamma_{01}|}$ if $\gamma_{01} \neq 0$* (*cf. Corollary* 2.2)). *Let $\widetilde{M(1)}$ denote the moment matrix associated with $\tilde{\gamma}_{ij}$ and let $\widetilde{M(1)}(\rho_0)$ denote its reduction* (*via Lemma* 2.5) *to a rank 2 disk problem. Let $\{\mu_t\}_{t\in[t_{12},t_{22}]}$ denote the 2-atomic representing measures for $\widetilde{M(1)}(\rho_0)$ that are supported in $\bar{\mathbf{D}}$* (*cf. Theorem* 4.2). *Then the 3-atomic representing measures for $\gamma^{(2)}$ that are supported in $\bar{\mathbf{D}}$, with an atom at $\bar{\lambda}$, are the measures $\nu_t := (1-\rho_0)\omega_t + \rho_0\delta_{\bar{\lambda}}$ $(t_{22} \leq t \leq t_{12})$, where ω_t is obtained from μ_t by a $\bar{\lambda}$-rotation of the support. The measure ν_t has exactly two atoms on the unit circle* (*including* $\bar{\lambda}$) *if and only if $\gamma_{11} < 1$ and $t = t_{22}$ or $t = t_{12}$; ν_t has three atoms on the unit circle if and only if $\gamma_{11} = 1$, in which case $t = t_{12} = t_{22}$.*

Proof. Combine Theorem 4.2 with Corollary 2.6 and Remark 2.3. ∎

Proof of Theorem 1.8. Clearly (i) ⇒ (iv) and (ii) ⇒ (i), and we have just finished proving that (iv) ⇒ (ii). Therefore (i), (ii) and (iv) are all equivalent. Theorems 1.1 and 1.2 provide the equivalence (ii) ⇔ (iii). ∎

Example 4.4 ***(Example 2.8 Revisited)*** *Consider*

$$M(1) := \begin{pmatrix} 1 & \frac{70-51\sqrt{2}}{32(-3+2\sqrt{2})} & \frac{70-51\sqrt{2}}{32(-3+2\sqrt{2})} \\ \frac{70-51\sqrt{2}}{32(-3+2\sqrt{2})} & \frac{3(-47+32\sqrt{2})}{64(-3+2\sqrt{2})} & \frac{(180+51i)-(128+32i)\sqrt{2}}{384-256\sqrt{2}} \\ \frac{70-51\sqrt{2}}{32(-3+2\sqrt{2})} & \frac{(180-51i)-(128-32i)\sqrt{2}}{384-256\sqrt{2}} & \frac{3(-47+32\sqrt{2})}{64(-3+2\sqrt{2})} \end{pmatrix}.$$

$M(1)$ is the moment matrix obtained from Example 2.8 after rotating the original data $\gamma^{(2)}$ by $\lambda := \frac{1-i}{\sqrt{2}}$ and applying Lemma 2.5 to bring the rank down to 2 (i.e., $M(1)$ coincides with $\widetilde{M(1)}(\tilde{\rho})$ of Example 2.8). Using the notation leading up to the statement of Theorem 4.2, we see that $\alpha = (2-i)(-39866+28191\sqrt{2})/(65400-46240\sqrt{2})$, $\beta = (-3+4i)/5$, $\delta = 5(281 - 198\sqrt{2})/[512(3-2\sqrt{2})^2] \cong 0.327005$, $\sqrt{\bar{\beta}\delta} = \sqrt{(3/512+i/128)(-281+198\sqrt{2})/(3-2\sqrt{2})}$, $A \cong 0.0989761$, $C \cong 0.850212$, and $D \cong 0.536488$. It follows that $t_{22} \cong 0.514592$ and $t_{12} \cong 1.33794$. By taking $t = t_{12}$, we obtain $\mu_{t_{12}}$, with $z_0 \cong 0.729183 - 0.684318i$ and $z_1 \cong 0.195882 + 0.382284i$, where $|z_0| = 1$ and $|z_1| \cong 0.429547$. If instead, $t = t_{22}$, then $\mu_{t_{22}}$ has $z_0 \cong 0.518624 - 0.263199i$ and $z_1 \cong -0.109945 + 0.993938i$, with $|z_0| \cong 0.581588$ and

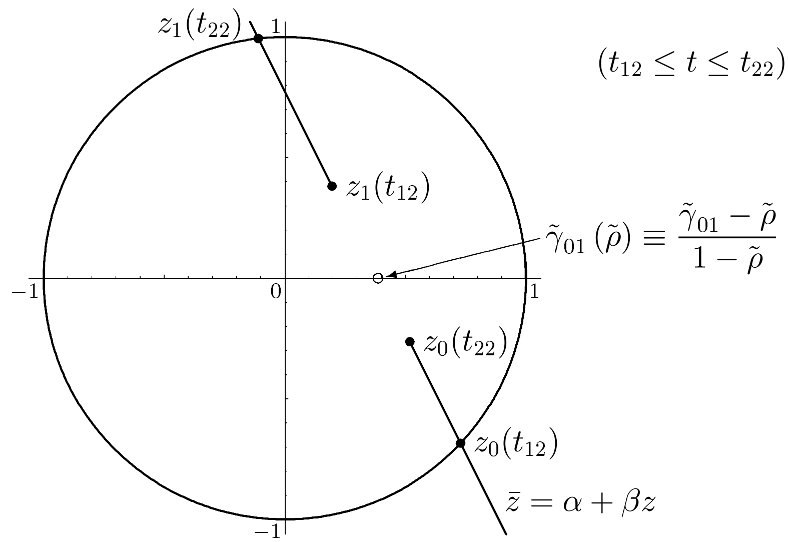

Figure 1: Location of $z_0(t_{12})$, $z_0(t_{22})$, $z_1(t_{12})$ and $z_1(t_{22})$ in Example 4.4

$|z_1| = 1$. *The latter case leads (via Corollary 4.3) to a* 3*-atomic representing measure* μ *(for the original data* $\gamma^{(2)}$*) with* $\operatorname{supp}\mu = \{w_0, w_1, w_2\} \subseteq \bar{\mathbf{D}}$*, where* $w_0 \cong 0.552833 + 0.180612i$*,* $w_1 \cong -0.780563 + 0.625078$ *and* $w_2 \equiv \bar{\lambda} = (1+i)/\sqrt{2}$ *(both* w_1 *and* w_2 *belong to the unit circle); the corresponding densities are* $\rho_0 \cong 0.377783$*,* $\rho_1 \cong 0.100039$ *and* $\rho_2 \cong 0.522178$*. Figure 1 shows the unit disk, the line* $\bar{z} = \alpha + \beta z$*, and the "exclusion zone" for atoms of* 2*-atomic representing measures.* ▮

For the rank 3 case of the disk problem, we can use Theorem 4.2 to develop a constructive algorithm which completely parameterizes the 3-atomic representing measures supported in $\bar{\mathbf{D}}$. The main step, which follows, provides a criterion for a prescribed point of $\bar{\mathbf{D}}$ to belong to the support of such a measure.

Suppose now that $M(1)$ is positive and invertible, with $\gamma_{11} \le \gamma_{00} = 1$. By analogy with the rank reduction result of Section 2 (Lemma 2.5), consider the moment matrix

$$M(1)[\rho; z] := \begin{pmatrix} 1 & \frac{\gamma_{01}-\rho z}{1-\rho} & \frac{\gamma_{10}-\rho\bar{z}}{1-\rho} \\ \frac{\gamma_{10}-\rho\bar{z}}{1-\rho} & \frac{\gamma_{11}-\rho|z|^2}{1-\rho} & \frac{\gamma_{20}-\rho\bar{z}^2}{1-\rho} \\ \frac{\gamma_{01}-\rho z}{1-\rho} & \frac{\gamma_{02}-\rho z^2}{1-\rho} & \frac{\gamma_{11}-\rho|z|^2}{1-\rho} \end{pmatrix} \quad (\rho \ne 1, z \in \mathbf{C}). \tag{4.11}$$

Observe that

$$M(1)[\rho; z] = \frac{1}{1-\rho}[M(1) - \rho \mathbf{z}\mathbf{z}^*] = \frac{1}{1-\rho} M(1)[I - \rho M(1)^{-1}\mathbf{z}\mathbf{z}^*], \tag{4.12}$$

where $\mathbf{z} := \begin{pmatrix} 1 & \bar{z} & z \end{pmatrix}^t$. To analyze $\det M(1)[\rho; z]$, we need some auxiliary results; for completeness, we include the proofs.

Lemma 4.5 *For* $\mathbf{x}, \mathbf{y} \in \mathbf{C}^m$ *let* $\mathbf{x} \bigotimes \mathbf{y}$ *denote the operator on* $\mathbf{C}^m$ *given by* $\mathbf{x} \bigotimes \mathbf{y}(\mathbf{u}) := \langle \mathbf{u}, \mathbf{y} \rangle \mathbf{x}$*. Then* $\mathbf{x}\mathbf{y}^*$ *is the matrix associated with* $\mathbf{x} \bigotimes \mathbf{y}$ *with respect to the canonical basis for* $\mathbf{C}^m$*.*

Proof. $\left\langle (\mathbf{x} \bigotimes \mathbf{y})\mathbf{e}_j, \mathbf{e}_i \right\rangle = \langle \mathbf{e}_j, \mathbf{y} \rangle \langle \mathbf{x}, \mathbf{e}_i \rangle = x_i \bar{y}_j = (\mathbf{x}\mathbf{y}^*)_{ij} \;\; (1 \le i, j \le m).$ ∎

Lemma 4.6 *Let* $A \equiv \mathbf{x} \bigotimes \mathbf{y}$ *be a rank-one matrix. Then*

$$\det(I + A) = 1 + \operatorname{Tr} A = 1 + \langle \mathbf{x}, \mathbf{y} \rangle \,, \tag{4.13}$$

where Tr *denotes the canonical trace.*

Proof. Choose an orthonormal basis $\{\mathbf{f}_i\}_{i=1}^m$ such that $\mathbf{f}_1 := \frac{\mathbf{x}}{\|\mathbf{x}\|}$. The matrix of $I + A$ with respect to this basis is

$$\begin{pmatrix} 1 + \langle \mathbf{x}, \mathbf{y} \rangle & \langle \mathbf{f}_2, \mathbf{y} \rangle \|\mathbf{x}\| & \cdots & \langle \mathbf{f}_m, \mathbf{y} \rangle \|\mathbf{x}\| \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

so $\det(I + A) = 1 + \langle \mathbf{x}, \mathbf{y} \rangle$. Moreover,

$$\operatorname{Tr} A = \operatorname{Tr}(\mathbf{x}\mathbf{y}^*) = \sum_{i=1}^m x_i \bar{y}_i = \langle \mathbf{x}, \mathbf{y} \rangle \,. \tag{4.14}$$

∎

Corollary 4.7 *For* $M(1)[\rho; z]$ *given by (4.11),*

$$\det M(1)[\rho; z] = \frac{\det M(1)}{(1-\rho)^3}(1 - \rho \left\langle M(1)^{-1}\mathbf{z}, \mathbf{z} \right\rangle). \tag{4.15}$$

Proof. By (4.12),

$$\begin{aligned} \det M(1)[\rho; z] &= \frac{1}{(1-\rho)^3} \det M(1) \det(I - \rho M(1)^{-1}\mathbf{z}\mathbf{z}^*) \\ &= \frac{\det M(1)}{(1-\rho)^3}(1 - \rho \operatorname{Tr}(M(1)^{-1}\mathbf{z}\mathbf{z}^*) \\ &= \frac{\det M(1)}{(1-\rho)^3}(1 - \rho \operatorname{Tr}(M(1)^{-1}\mathbf{z} \bigotimes \mathbf{z})) \\ &\quad \text{(by (4.13) and Lemma 4.5, respectively)} \\ &= \frac{\det M(1)}{(1-\rho)^3}(1 - \rho \left\langle M(1)^{-1}\mathbf{z}, \mathbf{z} \right\rangle) \end{aligned}$$

(by (4.14)), as desired. ∎

Lemma 4.8

$$\begin{aligned} \left\langle M(1)^{-1}\mathbf{z}, \mathbf{z} \right\rangle = 1 + \frac{1}{\gamma_{11} - |\gamma_{01}|^2} (|\gamma_{01} - z|^2 \\ + \frac{\left|\gamma_{02}\gamma_{10} - \gamma_{01}\gamma_{11} - |\gamma_{01}|^2 z + \gamma_{11} z + \gamma_{01}^2 \bar{z} - \gamma_{02}\bar{z}\right|^2}{\det M(1)}). \end{aligned} \tag{4.16}$$

In particular, $\langle M(1)^{-1}\mathbf{z}, \mathbf{z} \rangle \ge 1$ *for all* $z \in \mathbf{C}$*, with equality holding precisely when* $z = \gamma_{01}$*.*

Proof. Straightforward symbolic manipulation establishes (4.16); the last statement follows easily from (4.16). ∎

Lemma 4.8 allows us to make the following

Definition 4.9 *For $z \in \mathbf{C}$, let* $\rho_z := \frac{1}{\langle M(1)^{-1}\mathbf{z},\mathbf{z}\rangle}$.

Proposition 4.10 *Let $z \in \mathbf{C}$.*
(i) $0 < \rho_z \le 1$;
(ii) $\rho_z \le \frac{\gamma_{11}-|\gamma_{01}|^2}{\gamma_{11}-|\gamma_{01}|^2+|\gamma_{01}-z|^2}$;
(iii) *equality holds in* (i) *or* (ii) *if and only if* $z = \gamma_{01}$;
(iv) *if $M(1)$ is positive and invertible, and if $z \neq \gamma_{01}$, then ρ_z is the unique solution of the equation* $\det M(1)[\rho; z] = 0$.

Proof. By (4.16),

$$\left\langle M(1)^{-1}\mathbf{z},\mathbf{z}\right\rangle \ge 1 + \frac{1}{\gamma_{11}-|\gamma_{01}|^2}|z-\gamma_{01}|^2 = \frac{\gamma_{11}-|\gamma_{01}|^2+|\gamma_{01}-z|^2}{\gamma_{11}-|\gamma_{01}|^2},$$

from which (i), (ii) and (iii) follow at once. Finally, notice that when $z \neq \gamma_{01}$, $\rho_z \neq 1$ (by Lemma 4.8), so $M(1)[\rho_z; z]$ is well defined, and by (4.15)

$$\det M(1)[\rho_z; z] = \frac{\det M(1)}{(1-\rho_z)^3}(1-\rho_z\left\langle M(1)^{-1}\mathbf{z},\mathbf{z}\right\rangle) = 0.$$

Since (4.15) is linear in ρ, it follows that ρ_z is the unique solution of the equation $\det M(1)[\rho; z] = 0$. ∎

We are now ready to give a criterion for the existence of a 3-atomic representing measure for $M(1)$ with z_0 as one of its atoms; recall that $\mathbf{z}_0 := \begin{pmatrix} 1 & \bar{z}_0 & z_0 \end{pmatrix}^t$.

Theorem 4.11 *Suppose $M(1)$ is positive and invertible, with $\gamma_{11} \le \gamma_{00} = 1$, and let $z_0 \in \bar{\mathbf{D}}$. Then there exists a 3-atomic representing measure μ with $z_0 \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$ if and only if*
(i) $\gamma_{11} < \gamma_{00}$, $z_0 \neq \gamma_{01}$, *and*

$$\rho_{z_0} \le \frac{1-\gamma_{11}}{1-|z_0|^2} \tag{4.17}$$

(*this condition is vacuous if* $|z_0| = 1$); *or*
(ii) $\gamma_{11} = \gamma_{00}$ *and* $z_0 \in \mathbf{T}$, *in which case μ is uniquely determined, and* $\operatorname{supp}\mu \subseteq \mathbf{T}$.

Proof. Since $M(1)$ is positive and invertible, $\det[M(1)]_2 > 0$, so $1 = \gamma_{11} > |\gamma_{01}|^2$. Thus, if $z_0 \in \mathbf{T}$, then $|z_0| = 1 > |\gamma_{01}|$. We may thus assume in case (ii), as well as in case (i), that $z_0 \neq \gamma_{01}$. Proposition 4.10(iii) now implies that $0 < \rho_{z_0} < 1$ and

$$\rho_{z_0} < \frac{\gamma_{11}-|\gamma_{01}|^2}{\gamma_{11}-|\gamma_{01}|^2+|\gamma_{01}-z_0|^2}. \tag{4.18}$$

Now (4.18) is equivalent to

$$\begin{aligned}
&\rho_{z_0}\left|\gamma_{01}-z_0\right|^2<(\gamma_{11}-\rho_{z_0}\left|z_0\right|^2)(1-\rho_{z_0})+(\rho_{z_0}\left|z_0\right|^2-\left|\gamma_{01}\right|^2)(1-\rho_{z_0})\\
\Leftrightarrow\ &\rho_{z_0}(\left|\gamma_{01}\right|^2-2\ \mathrm{Re}\ \gamma_{01}\bar{z}_0+\left|z_0\right|^2)\\
&\quad<(\gamma_{11}-\rho_{z_0}\left|z_0\right|^2)(1-\rho_{z_0})+\rho_{z_0}\left|z_0\right|^2-\left|\gamma_{01}\right|^2-\rho_{z_0}^2\left|z_0\right|^2+\rho_{z_0}\left|\gamma_{01}\right|^2\\
\Leftrightarrow\ &\rho_{z_0}^2\left|z_0\right|^2-2\rho_{z_0}\ \mathrm{Re}\ \gamma_{01}\bar{z}_0+\left|\gamma_{01}\right|^2<(\gamma_{11}-\rho_{z_0}\left|z_0\right|^2)(1-\rho_{z_0})\\
\Leftrightarrow\ &\frac{\left|\gamma_{01}-\rho_{z_0}z_0\right|^2}{(1-\rho_{z_0})^2}<\frac{\gamma_{11}-\rho_{z_0}\left|z_0\right|^2}{1-\rho_{z_0}}\\
\Leftrightarrow\ &\det[M(1)[\rho_{z_0};z_0]]_2>0.
\end{aligned}$$

The last inequality and Proposition 4.10(iv) imply that $M\equiv M(1)[\rho_{z_0};z_0]$ is positive and of rank 2. Thus M represents a rank 2 disk problem if and only if $M_{22}\le M_{11}$, i.e.,

$$\frac{\gamma_{11}-\rho_{z_0}\left|z_0\right|^2}{1-\rho_{z_0}}\le 1. \tag{4.19}$$

We now turn to case (i). Since $\gamma_{11}<1$ and $|z_0|\le 1$, (4.17) is equivalent to (4.19); thus M represents a rank 2 disk problem. By Theorem 4.2, M admits a 2-atomic representing measure ν supported in $\bar{\mathbf{D}}$. It is easy to see that $\mu:=(1-\rho_{z_0})\nu+\rho_{z_0}\delta_{z_0}$ is a 3-atomic representing measure for $M(1)$ supported in $\bar{\mathbf{D}}$ with an atom at z_0; indeed, $z_0\in\operatorname{supp}\nu$ implies $\operatorname{card}\operatorname{supp}\mu=2$, which is impossible by Proposition 1.5.

For the converse in case (i), let μ be a 3-atomic representing measure for $M(1)$ with $z_0\in\operatorname{supp}\mu\subseteq\bar{\mathbf{D}}$, and let $\hat{\rho}_0:=\mu(\{z_0\})$. Write $\mu\equiv\hat{\rho}_0\delta_{z_0}+\rho_1\delta_{z_1}+\rho_2\delta_{z_2}$, and observe that

$$M(1)[\hat{\rho}_0;z_0]=\begin{pmatrix}1&\frac{\rho_1z_1+\rho_2z_2}{1-\hat{\rho}_0}&\frac{\rho_1\bar{z}_1+\rho_2\bar{z}_2}{1-\hat{\rho}_0}\\ \frac{\rho_1\bar{z}_1+\rho_2\bar{z}_2}{1-\hat{\rho}_0}&\frac{\rho_1|z_1|^2+\rho_2|z_2|^2}{1-\hat{\rho}_0}&\frac{\rho_1\bar{z}_1^2+\rho_2\bar{z}_2^2}{1-\hat{\rho}_0}\\ \frac{\rho_1z_1+\rho_2z_2}{1-\hat{\rho}_0}&\frac{\rho_1z_1^2+\rho_2z_2^2}{1-\hat{\rho}_0}&\frac{\rho_1|z_1|^2+\rho_2|z_2|^2}{1-\hat{\rho}_0}\end{pmatrix}.$$

A straightforward computation now shows that

$$\det M(1)[\hat{\rho}_0;z_0]=\frac{\rho_1\rho_2(\rho_1+\rho_2+\hat{\rho}_0-1)(z_1\bar{z}_2-\bar{z}_1z_2)}{(1-\hat{\rho}_0)^3}=0,$$

so $\hat{\rho}_0=\rho_{z_0}$ (by Proposition 4.10(iv)) and, moreover,

$$\det M(1)[\rho_{z_0};z_0]_{\{1,2\}}=\det M(1)[\hat{\rho}_0;z_0]_{\{1,2\}}=\frac{\rho_1\rho_2\left|z_1-z_2\right|^2}{(1-\hat{\rho}_0)^2}>0,$$

so $\operatorname{rank} M(1)[\rho_{z_0};z_0]=2$. Finally, observe that (4.17) holds, since

$$M(1)[\rho_{z_0};z_0]_{22}=\frac{\rho_1\left|z_1\right|^2+\rho_2\left|z_2\right|^2}{1-\hat{\rho}_0}\le\frac{\rho_1+\rho_2}{1-\hat{\rho}_0}=\frac{1-\rho_{z_0}}{1-\hat{\rho}_0}=1.$$

For case (ii), assume $\gamma_{11}=\gamma_{00}(=1)$ and $z_0\in\mathbf{T}$. M defines a rank 2 circle problem since $\frac{\gamma_{11}-\rho_{z_0}|z_0|^2}{1-\rho_{z_0}}=\frac{1-\rho_{z_0}}{1-\rho_{z_0}}=1$. By Theorem 3.1, M admits a unique 2-atomic representing measure ν supported in $\mathbf{T}$, whence $\mu := (1-\rho_{z_0})\nu + \rho_{z_0}\delta_{z_0}$ is a representing measure for $M(1)$ supported in $\mathbf{T}$. As in case (i), it follows that μ is 3-atomic with an atom at z_0. For uniqueness, suppose η is a 3-atomic representing measure for $M(1)$ with $z_0 \in \operatorname{supp}\eta \subseteq \bar{\mathbf{D}}$ (equivalently, $\operatorname{supp}\eta \subseteq \mathbf{T}$). Write $\eta \equiv \check{\rho}_{z_0}\delta_{z_0}+\omega$, where $\check{\rho}_{z_0} := \eta(\{z_0\})$; exactly as in the proof of the converse direction of case (i), we see that $\check{\rho}_{z_0} = \rho_{z_0}$, whence $M = M(1)[\omega] = M(1)[\nu]$. Thus, ω and ν are representing measures for M supported in $\mathbf{T}$, so Remark 3.2 (uniqueness in the rank 2 circle problem) implies $\omega = \nu$, whence $\eta = \mu$.

For the converse in case (ii), assume that μ is a 3-atomic representing measure with $z_0 \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$. Since $\gamma_{11} = \gamma_{00}$, it follows that $\operatorname{supp}\mu \subseteq T$ (cf. remarks following Proposition 1.7), whence $z_0 \in \mathbf{T}$. ∎

Corollary 4.12 (i) *Let $M(1)$ be positive and invertible, with $\gamma_{11} < \gamma_{00} = 1$, and let $\mathcal{M}_3 := \{z \in \bar{\mathbf{D}} : z \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$, card $\operatorname{supp}\mu = 3$, μ a representing measure for $M(1)\}$. $\mathcal{M}_3$ is a proper closed subset of $\bar{\mathbf{D}}$ which contains $\mathbf{T}$.*
(ii) *Let $M(1)$ be positive and invertible, with $\gamma_{11} = \gamma_{00} = 1$. Then z_0 is in the support of a rank 3 representing measure μ with $\operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$ if and only if $z_0 \in \mathbf{T}$, in which case $\operatorname{supp}\mu \subseteq \mathbf{T}$.*

Proof. (i) By Theorem 4.11, $\mathcal{M}_3$ is clearly closed. We also know that $\gamma_{01} \notin \mathcal{M}_3$. Finally, Theorem 4.11 implies that $\mathbf{T} \subseteq \mathcal{M}_3$.

(ii) The result follows immediately from Theorem 4.11(ii). ∎

In Section 5 we will see that the set $\mathcal{M}_3$ in Corollary 4.12 coincides with the complement in $\bar{\mathbf{D}}$ of the interior region of a nondegenerate ellipse that is contained in $\mathbf{D}$.

Remark 4.13 *We have already seen that the point z_0 in Theorem 4.11 cannot be chosen arbitrarily if one seeks a 3-atomic representing measure. Concretely, if $z_0 = \gamma_{01}$, it is easy to see that $\det M(1)[\rho; z_0] = \frac{\det M(1)}{(1-\rho)^2} > 0$ for all $\rho \neq 1$; this implies that $M(1)[\rho; z_0]$ cannot admit a 2-atomic representing measure (by Proposition 1.5), and a fortiori, $\gamma^{(2)}$ cannot admit a 3-atomic representing measure supported in $\bar{\mathbf{D}}$ with an atom at z_0 (cf. Corollary 2.7).* ∎

Theorem 4.11 establishes an algorithm to parameterize the space of 3-atomic solutions for a given rank 3 $\bar{\mathbf{D}}$-moment problem, as follows.

Algorithm 4.14 (Parameterization of 3-atomic (minimal) representing measures supported in $\bar{\mathbf{D}}$)

Let $M(1)$ be positive and invertible, with $\gamma_{11} < \gamma_{00} = 1$. Assume a point $z_0 \in \bar{\mathbf{D}}$ is proposed as an atom of a 3-atomic representing measure supported in $\bar{\mathbf{D}}$; as before, let $\mathbf{z}_0 := \begin{pmatrix} 1 & \bar{z}_0 & z_0 \end{pmatrix}^t$ and let $\rho_{z_0} := \frac{1}{\langle M(1)^{-1}\mathbf{z}_0, \mathbf{z}_0\rangle}$.

Step (a): If $\rho_{z_0} \geq 1$, then $M(1)$ admits no 3-atomic representing measure μ with $z_0 \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$; otherwise proceed to Step (b).

Step (b): By Proposition 4.10(i), we know that ρ_{z_0} is well defined and $0 < \rho_{z_0} < 1$. If $\rho_{z_0} > \frac{1-\gamma_{11}}{1-|z_0|^2}$, then Theorem 4.11 says that $M(1)$ admits no 3-atomic representing measure μ with $z_0 \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$; otherwise $M(1)$ does admit a 3-atomic representing measure μ with $z_0 \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$. To construct such a measure, proceed to Step (c).

Step (c): Assuming that $z_0 \in \bar{\mathbf{D}}$ is an atom for a 3-atomic representing measure, then two cases arise when Theorem 4.2 is applied to the rank 2 disk problem for $M(1)[\rho_{z_0}; z_0]$:

(c$_1$): $t_{12} = t_{22}$ (this happens when $\rho_{z_0} = \frac{1-\gamma_{11}}{1-|z_0|^2}$), in which case the two remaining atoms will be on the unit circle; and

(c$_2$): $t_{12} < t_{22}$, which leads to one of the two remaining atoms inside the disk and one on the unit circle (choose $t = t_{12}$ or $t = t_{22}$), or the two remaining atoms inside the unit disk (choose $t_{12} < t < t_{22}$). ∎

Consider the points described by Case (c$_1$) of Algorithm 4.14. The main result of Section 5 below says that the locus of z_0's that qualify as atoms in $\mathbf{D}$ for 3-atomic representing measures with the two remaining atoms in $\mathbf{T}$ is an ellipse, whose location is completely determined by the initial data.

We conclude this section by presenting a concrete application of Algorithm 4.14.

Example 4.15 *Let*

$$M(1) := \begin{pmatrix} 1 & \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{3}{4} & \frac{1}{8}-\frac{i}{2} \\ \frac{1+i}{2} & \frac{1}{8}+\frac{i}{2} & \frac{3}{4} \end{pmatrix}.$$

As we have seen before (Example 2.8), $M(1) \geq 0$ *and* $\det M(1) = \frac{3}{64} > 0$. *Now,* $\langle M(1)^{-1}\mathbf{z}, \mathbf{z}\rangle = \frac{1}{3}(19 - 16x + 16x^2 - 48y + 48y^2)$, *so* $\rho \equiv \rho_z = \frac{3}{19-16x+16x^2-48y+48y^2}$, *where* $z \equiv x + iy$. *Observe that*

$$19 - 16x + 16x^2 - 48y + 48y^2 = 16(x - \frac{1}{2})^2 + 48(y - \frac{1}{2})^2 + 3, \tag{4.20}$$

so ρ *is always positive (as predicted by Proposition 4.10). Fix* $z_0 \equiv x_0 + iy_0 \in \bar{\mathbf{D}}$. *To check Step (a) of Algorithm 4.14, we compute*

$$\begin{aligned} 1 - \langle M(1)^{-1}\mathbf{z}, \mathbf{z}\rangle &= 1 - \frac{1}{3}[16(x - \frac{1}{2})^2 + 48(y - \frac{1}{2})^2 + 3] \\ &= -\frac{16}{3}[(x - \frac{1}{2})^2 + 3(y - \frac{1}{2})^2] \leq 0 \end{aligned}$$

It follows that

$$\begin{aligned} \rho_{z_0} \geq 1 &\Leftrightarrow (x_0 - \tfrac{1}{2})^2 + 3(y_0 - \tfrac{1}{2})^2 = 0 \\ &\Leftrightarrow z_0 = \tfrac{1}{2} + \tfrac{1}{2}i. \end{aligned}$$

According to Step (a) of Algorithm 4.14, for $z_0 = \frac{1}{2} + \frac{1}{2}i$, $M(1)$ *admits no* 3*-atomic representing measure* μ *with* $z_0 \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$ *(observe that* $z_0 = \gamma_{01}$*, as anticipated by Proposition*

4.10; see also Corollary 2.7 and Remark 4.13). We next focus on $z_0 \neq \frac{1}{2} + \frac{1}{2}i$. *A simple calculation shows that* $\rho_{z_0} - \frac{1-\gamma_{11}}{1-|z_0|^2} = -\frac{7-16x_0+28x_0^2-48y_0+60y_0^2}{4(1-|z_0|^2)(19-16x_0+16x_0^2-48y_0+48y_0^2)}$, *so by (4.20)*

$$\begin{gathered}\rho_0 > \tfrac{1-\gamma_{11}}{1-|z_0|^2} \Leftrightarrow 7 - 16x_0 + 28x_0^2 - 48y_0 + 60y_0^2 = \\ 28(x_0 - \tfrac{2}{7})^2 + 60(y_0 - \tfrac{2}{5})^2 - \tfrac{171}{35} < 0 \\ \Leftrightarrow z_0 \in \Omega \equiv E((\tfrac{2}{7}, \tfrac{2}{5}), \{\tfrac{3}{14}\sqrt{\tfrac{19}{5}}, \tfrac{1}{10}\sqrt{\tfrac{57}{7}}\}),\end{gathered}$$

where $E((p,q),\{a,b\})$ *denotes the interior of the ellipse given by* $\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$. *($\Omega$ represents, therefore, the "exclusion zone." Incidentally, observe that* $\frac{1}{2} + \frac{1}{2}i \in \Omega$, *as expected.) Note that* $\rho_{z_0} = \frac{1-\gamma_{11}}{1-|z_0|^2}$ *precisely when* $7 - 16x_0 + 28x_0^2 - 48y_0 + 60y_0^2 = 0$, *that is, when* z_0 *is on the ellipse* $\mathcal{E}_0 := \partial\Omega$:

$$\rho_{z_0} = \frac{1-\gamma_{11}}{1-|z_0|^2} \Leftrightarrow z_0 \in \mathcal{E}_0. \tag{4.21}$$

By Step (b) of Algorithm 4.14, $M(1)$ *admits no* 3*-atomic representing measure* μ *with* $z_0 \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$ *whenever* $z_0 \in \Omega$. *Moreover,* $M(1)$ *admits a* 3*-atomic representing measure* μ *with* $z_0 \in \operatorname{supp}\mu \subseteq \bar{\mathbf{D}}$ *if and only if* $z_0 \in \bar{\mathbf{D}} \setminus \Omega$ *(cf. Figure 2). When* $z_0 \in \bar{\mathbf{D}} \setminus \Omega$, *the associated* 3*-atomic representing measure* μ *has the two remaining atoms in the unit circle precisely when* $z_0 \in \mathcal{E}_0$*; otherwise,* μ *has at most one other atom on the unit circle. Consider the case when* $z_0 = 0 \in \bar{\mathbf{D}} \setminus \Omega$. *Here,* $P_0 = \frac{19}{64}$ *and* $\rho_0 = \frac{3}{19}$, *so the associated moment matrix is*

$$M(1)[\rho_0; 0] = \begin{pmatrix} 1 & \frac{19}{32}(1+i) & \frac{19}{32}(1-i) \\ \frac{19}{32}(1-i) & \frac{57}{64} & \frac{19}{16}(\frac{1}{8} - \frac{i}{2}) \\ \frac{19}{32}(1+i) & \frac{19}{16}(\frac{1}{8} + \frac{i}{2}) & \frac{57}{64} \end{pmatrix},$$

and $\delta := \gamma_{11} - |\gamma_{01}|^2 = \frac{95}{512}$. *The coefficients* α *and* β *are* $-\frac{19}{40}(1-3i)$ *and* $\frac{4+3i}{5}$, *respectively, and* $\bar{\beta}\delta = \frac{19}{512}(4-3i)$ *(cf. (4.2)). Then* $A = Re\ (\gamma_{10}\sqrt{\bar{\beta}\delta}) = Re\ [\frac{19}{32}(1-i)\sqrt{\frac{19}{512}(4-3i)}]$, $C = 1 - |\gamma_{01}|^2 = \frac{151}{512}$ *and* $D = \sqrt{A^2 + C\delta} = \frac{3\sqrt{589}}{256}$. *It follows that* $t_{22} \cong 0.415873$ *and* $t_{12} \cong 0.66102$. *We now choose* $t \in [t_{22}, t_{12}]$, *say* $t = \frac{1}{2}$. *By (4.5), (4.6) and (4.7), we obtain* $z_1 \cong 0.798073 + 0.525642i$ *and* $z_2 \cong -0.223544 + 0.866181i$. *Note that* $|z_1|^2 \cong 0.955626$ *and* $|z_2|^2 \cong 0.894562$, *so* $z_1, z_2 \in \mathbf{D}$, *in agreement with Theorem 4.2. The associated densities are* $\rho_1 = \frac{64}{95}$ *and* $\rho_2 = \frac{16}{95}$. *Therefore,* $\mu = \frac{3}{19}\delta_0 + \frac{64}{95}\delta_{z_1} + \frac{16}{95}\delta_{z_2}$. ∎

We shall revisit Example 4.15 at the end of Section 5.

5 Description of the Space of Solutions

In this section we complete our parameterization of minimal representing measures in the rank 3 disk problem. Assume that $M(1)$ positive and invertible, with $\gamma_{11} < \gamma_{00} = 1$, and let $z_2 \in \mathbf{D}$. Using Algorithm 4.14, Step (c$_1$), we can determine whether z_2 is a support point of some 3-atomic representing measure having the other two atoms, z_0 and z_1, in $\mathbf{T}$; such a point z_2 will be called *admissible*. In the present section we show that the admissible points trace out a computable ellipse $\mathcal{E}$ inside $\mathbf{D}$. Further, this ellipse actually coincides with the

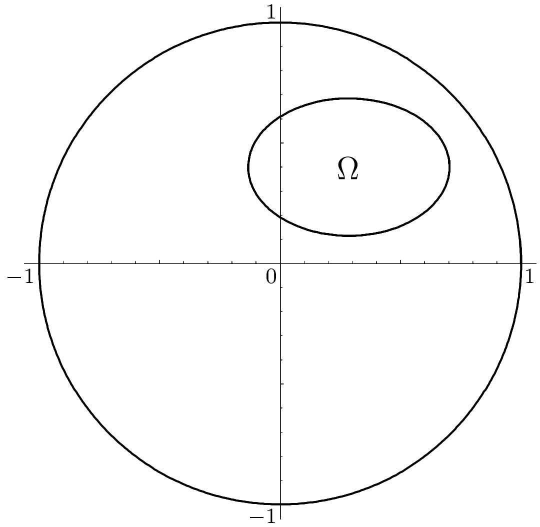

Figure 2: $M(1)$ in Example 4.15 admits no 3-atomic representing measures having a support point inside Ω.

boundary of the region $\bar{\mathbf{D}} \setminus \mathcal{M}_3$ (cf. Corollary 4.12). Thus, a point $z \in \bar{\mathbf{D}}$ is in the support of a (minimal) 3-atomic representing measure μ with $\operatorname{supp} \mu \subseteq \bar{\mathbf{D}}$ if and only if z is not in the interior region determined by $\mathcal{E}$. Since the conclusions that we seek are invariant under rotation, we shall assume that $\gamma_{01} \in \mathbf{R}$ (by Lemmas 2.1 and 2.2 and Remark 2.3).

Proposition 5.1 *Let $M(1)$ be a positive and invertible moment matrix, assume $\gamma_{11} < 1$ $(= \gamma_{00})$, and let $\mu \equiv \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1} + \rho_2 \delta_{z_2}$ be a representing measure for $M(1)$ with $z_0, z_1 \in \mathbf{T}$ and $z_2 \in \mathbf{D}$ (such a measure exists by Corollary 4.3). Then $z_2 \in D_1 := \{z \in \mathbf{D} : |z| < \sqrt{\gamma_{11}}\}$.*

Proof. Since μ is a representing measure for $M(1)$, we know that $\rho_0 + \rho_1 + \rho_2 = \gamma_{00} = 1$ and $\rho_0 |z_0|^2 + \rho_1 |z_1|^2 + \rho_2 |z_2|^2 = \gamma_{11}$, from which it follows that $1 - \rho_2 = \gamma_{11} - \rho_2 |z_2|^2$; therefore,

$$\rho_2 = \frac{1 - \gamma_{11}}{1 - |z_2|^2}. \tag{5.1}$$

Since $0 < \rho_2 < 1$, (5.1) implies that $1 - \gamma_{11} < 1 - |z_2|^2$, so $|z_2| < \sqrt{\gamma_{11}}$. ∎

Proposition 5.1 not only says that $z_2 \in D_1$, it also establishes a formula for ρ_2, given by (5.1). For arbitrary but fixed $z \in D_1$ we now let $\rho := \frac{1-\gamma_{11}}{1-|z|^2}$ and we build a quadratic moment problem $\widetilde{M(1)} \equiv M(1)[z]$ with data $\tilde{\gamma}_{00} := 1 - \rho$, $\tilde{\gamma}_{01} := \gamma_{01} - \rho z$, $\tilde{\gamma}_{02} := \gamma_{02} - \rho z^2$ and $\tilde{\gamma}_{11} := 1 - \rho$.

Proposition 5.2 *$M(1)$ admits a 3-atomic representing measure μ with one atom at $z \in D_1$ and the two remaining atoms on $\mathbf{T}$ if and only if $\widetilde{M(1)}$ admits a 2-atomic representing measure ν supported in $\mathbf{T}$.*

Proof. Observe that μ and ν are related by the equation $\mu = \nu + \rho\delta_z$. ∎

In light of Proposition 5.2, we now focus on the moment matrix $\widetilde{M(1)}$. Theorem 3.1 implies that $\widetilde{M(1)}$ admits a 2-atomic representing measure supported in $\mathbf{T}$ if $\widetilde{M(1)} \geq 0$ and rank $\widetilde{M(1)} = 2$. Conversely, if $\widetilde{M(1)}$ admits a 2-atomic representing measure, then $2 \geq \text{rank}\, \widetilde{M(1)}$ (by Proposition 1.5), and if rank $\widetilde{M(1)} = 1$, then the unique representing measure is 1-atomic (Proposition 1.4), a contradiction. Thus, $\widetilde{M(1)}$ admits a 2-atomic representing measure supported in $\mathbf{T}$ if and only if $\widetilde{M(1)} \geq 0$ and rank $\widetilde{M(1)} = 2$.

To guarantee that $\widetilde{M(1)}$ is both positive and of rank 2, two conditions are required: (i) $\det[\widetilde{M(1)}]_2 > 0$ (here $[\,\cdot\,]_2$ denotes compression to the first two rows and columns); and (ii) $\det \widetilde{M(1)} = 0$.

Proposition 5.3 $\det[\widetilde{M(1)}]_2 > 0$ *if and only if* z *belongs to the disk*

$$D_2 := \{w \in \mathbf{D} : \left| w - \frac{(1-\gamma_{11})\gamma_{01}}{1-\gamma_{01}^2} \right| < \frac{\gamma_{11} - \gamma_{01}^2}{1-\gamma_{01}^2}\}.$$

Proof. Observe first that

$$\tilde{\gamma}_{01} = \gamma_{01} - \rho z = \gamma_{01} - \frac{1-\gamma_{11}}{1-|z|^2} \cdot z,$$

$$\tilde{\gamma}_{02} = \gamma_{02} - \rho z^2 = \gamma_{02} - \frac{1-\gamma_{11}}{1-|z|^2} \cdot z^2,$$

and

$$\tilde{\gamma}_{00} = \tilde{\gamma}_{11} = 1 - \rho = 1 - \frac{1-\gamma_{11}}{1-|z|^2} = \frac{\gamma_{11} - |z|^2}{1-|z|^2}.$$

Thus

$$\det[\widetilde{M(1)}]_2 = \det \begin{pmatrix} \tilde{\gamma}_{00} & \tilde{\gamma}_{01} \\ \tilde{\gamma}_{10} & \tilde{\gamma}_{00} \end{pmatrix} > 0$$

if and only if $|\tilde{\gamma}_{01}| < \tilde{\gamma}_{00}$, or

$$\left| \gamma_{01} - \frac{1-\gamma_{11}}{1-|z|^2} \cdot z \right| < \frac{\gamma_{11} - |z|^2}{1-|z|^2}. \tag{5.2}$$

If we let $\gamma_{01} := a$, $\gamma_{11} := e$ and $z := x + iy$, a calculation (squaring both sides) shows that (5.2) is equivalent to

$$\frac{a^2 - e^2 - 2ax + 2aex + x^2 - a^2x^2 + y^2 - a^2y^2}{1 - x^2 - y^2} < 0,$$

which in turn is equivalent to

$$\frac{a^2 - e^2 - 2ax + 2aex + x^2 - a^2x^2 + y^2 - a^2y^2}{1 - a^2} < 0 \tag{5.3}$$

(because both denominators are positive: $z \in D_1$ by construction, and $|a| < 1$ by Lemma 2.4). Observe that the numerator of (5.3) can be rewritten as

$$(1-a^2)[(x - \frac{a(1-e)}{1-a^2})^2 + y^2] - \frac{a^4 - 2a^2e + e^2}{1-a^2},$$

so (5.3) is equivalent to

$$(x - \frac{a(1-e)}{1-a^2})^2 + y^2 - \frac{(e-a^2)^2}{(1-a^2)^2} < 0.$$

The latter inequality is equivalent to

$$\left| z - \frac{(1-\gamma_{11})\gamma_{01}}{1-\gamma_{01}^2} \right| < \frac{\gamma_{11} - \gamma_{01}^2}{1-\gamma_{01}^2},$$

which proves that $\det[M(1)[z]]_2 > 0$ if and only if $z \in D_2$. ∎

We now show that the disk D_2 is actually contained properly in D_1.

Corollary 5.4 *With the notation as above, $\bar{D}_2 \subseteq D_1$.*

Proof. It suffices to show that the sum of the absolute value of the center of D_2 and the radius of D_2 is less than the radius of D_1 (recall that D_1 is centered at the origin), that is

$$\left| \frac{(1-\gamma_{11})\gamma_{01}}{1-\gamma_{01}^2} \right| + \frac{\gamma_{11} - \gamma_{01}^2}{1-\gamma_{01}^2} < \sqrt{\gamma_{11}}.$$

Recall that $\gamma_{00} = 1 > \gamma_{11} > \gamma_{01}^2$, $\gamma_{01} \geq 0$, and observe that

$$\begin{aligned} \left| \tfrac{(1-\gamma_{11})\gamma_{01}}{1-\gamma_{01}^2} \right| + \tfrac{\gamma_{11}-\gamma_{01}^2}{1-\gamma_{01}^2} &= \tfrac{(1-\gamma_{11})\gamma_{01}}{1-\gamma_{01}^2} + \tfrac{\gamma_{11}-\gamma_{01}^2}{1-\gamma_{01}^2} \\ = \tfrac{\gamma_{01}+\gamma_{11}}{1+\gamma_{01}} &< \tfrac{\sqrt{\gamma_{11}}+\gamma_{11}}{1+\sqrt{\gamma_{11}}} = \sqrt{\gamma_{11}}, \end{aligned}$$

the latter inequality being a consequence of the fact that the function $f_{a,b}(t) := \frac{t+a}{b+t}$ is strictly increasing on the interval $[0, +\infty)$ whenever $0 < a < b$. ∎

We are now ready to discuss the second condition, $\det \widetilde{M(1)} = 0$. A calculation shows that $\det \widetilde{M(1)} = 0$ precisely when

$$\left| \tilde{\gamma}_{02} - \tilde{\gamma}_{01}^2 \right|^2 + (1 - \tilde{\gamma}_{11})(\tilde{\gamma}_{11}^2 - |\tilde{\gamma}_{02}|^2) = (\tilde{\gamma}_{11} - |\tilde{\gamma}_{01}|^2)^2. \tag{5.4}$$

If we let $\gamma_{01} = a$, $\gamma_{02} = c + id$, $\gamma_{11} = e$, and $z \equiv x + iy$, it follows that (5.4) is equivalent to

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \tag{5.5}$$

where

$$\begin{gathered} A := \det M(1) + 2(1-e)(e-c) \\ B := -2d(1-e) \\ C := \det M(1) + 2(1-e)(e+c-2a^2) \\ D := -2a(1-e)(e-c) \\ E := 2ad(1-e) \\ F := 2a^2(e-c) - e(e^2 - c^2 - d^2). \end{gathered}$$

Now observe that $0 < \det M(1) = (e-c)(e+c-2a^2) - d^2$ and $e - c > 0$ by Lemma 2.4, so $e + c - 2a^2 > \dfrac{d^2}{e-c} \geq 0$. It follows that $A + C > 0$. Moreover,

$$\begin{aligned} AC - B^2 &= \det M(1)^2 + 2\det M(1)(1-e)(2e-2a^2) \\ &\quad +4(1-e)^2(e-c)(e+c-2a^2) - 4d^2(1-e)^2 \\ &= \det M(1)^2 + 4\det M(1)(1-e)(e-a^2) + 4\det M(1)(1-e)^2 \\ &= \det M(1)[\det M(1) + 4(1-e)(1-a^2)] > 0. \end{aligned}$$

We thus see that (5.5) is an ellipse, which we will denote by $\mathcal{E}$. A priori, it is not obvious whether this ellipse is real or imaginary, and the usual rotation and translation transformations to bring it to its canonical form do not reveal immediately whether any points inside the unit disk belong to the ellipse. Here is where Corollary 4.3 is central: we know that there exists a 3-atomic representing measure for $M(1)$, $\mu \equiv \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2}$, such that $z_0, z_1 \in \mathbf{T}$ and $z_2 \in \mathbf{D}$. Thus $M(1)[z_2]$ is positive and of rank 2 (by Proposition 5.2 and the subsequent remarks), so $\det M(1)[z] = 0$ for $z = z_2$; that is, $z_2 \in \mathcal{E}$ (the ellipse is real). Our goal is to show that the entire ellipse $\mathcal{E}$ is inside D_2!

Theorem 5.5 *Let $M(1)$ be a positive and invertible, with $\gamma_{11} < \gamma_{00} = 1$. Then $\mathcal{E} := \{z \equiv x + iy \in \mathbf{D} : \det M(1)[z] = 0\}$ is an ellipse, and $\mathcal{E} \subseteq D_2$. A 3-atomic representing measure for $\gamma^{(2)}$ is of the form $\mu \equiv \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2}$, with $z_2 \in \mathbf{D}$ and $z_0, z_1 \in \mathbf{T}$, if and only if $z_2 \in \mathcal{E}$, $\rho_2 = \frac{1-\gamma_{11}}{1-|z_2|^2}$, and $\rho_0\delta_{z_0} + \rho_1\delta_{z_1}$ is the unique 2-atomic representing measure for $M(1)[z_2]$ supported in $\mathbf{T}$ (cf. Theorem 3.1). In particular, μ is uniquely determined by z_2.*

Proof. We know from the preceding discussion that there exists $z_2 \in \mathcal{E} \cap D_2$. Suppose $\mathcal{E} \nsubseteq D_2$. By connectedness, $\mathcal{E}$ must intersect the boundary of D_2, say at point w, nearest to z_2. Since $\bar{D}_2 \subseteq D_1 \subseteq \mathbf{D}$ (by Corollary 5.4), we can form $M(1)[w]$. While remaining on the ellipse and inside D_2, let $t \mapsto \varphi(t)$ be the geodesic path from z_2 to w, with $\varphi(0) = z_2$ and $\varphi(1) = w$. Since $\varphi(t) \in \mathcal{E}$ for every $t \in [0, 1)$, we have $M(1)[\varphi(t)] \geq 0$, so by upper semicontinuity of matrix positivity we must also have $M(1)[w] \geq 0$. However, w is in the boundary of D_2, so the upper left 2×2 corner of $M(1)[w]$ is singular. Therefore, $Z = \alpha \mathit{1}$ in the column space of $M(1)[w]$, which implies $\bar{Z} = \bar{\alpha}\mathit{1}$. Then $M(1)[w]$ is a rank 1 positive moment matrix, so by (1.3) and the ensuing discussion, the unique representing measure ν for $M(1)[w]$ is 1-atomic, with $\operatorname{supp}\nu \subseteq \mathbf{T}$. It follows that $\mu_w := \nu + \rho_w\delta_w$ is a 2-atomic representing measure for $M(1)$, so by Proposition 1.5, $\operatorname{rank} M(1) \leq \operatorname{card}\operatorname{supp}\mu_w = 2$, a contradiction. We must therefore have $\mathcal{E} \subseteq D_2$.

For $z_2 \in \mathcal{E}$, let $\rho_2 := \frac{1-\gamma_{11}}{1-|z_2|^2}$; Proposition 5.3, Corollary 5.4, and the ensuing discussion imply that $M(1)[z_2]$ is positive and has rank 2. Theorem 3.1 implies that $M(1)[z_2]$ admits a unique 2-atomic representing measure ν supported in $\mathbf{T}$, so Proposition 5.2 implies that $\mu := \nu + \rho_2\delta_{z_2}$ is a 3-atomic representing measure for $M(1)$ with an atom at z_2 and two atoms in the unit circle. Conversely, if $M(1)$ admits a 3-atomic representing measure $\rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2}$ with $z_2 \in \mathbf{D}$ and $z_0, z_1 \in \mathbf{T}$, then the preceding results imply that $z_2 \in \mathcal{E}$ $(\subset D_1)$, $\rho_2 = \frac{1-\gamma_{11}}{1-|z_2|^2}$ (by (5.1)), and that $\rho_0\delta_{z_0} + \rho_1\delta_{z_1}$ is the unique representing measure for the rank 2 circle problem corresponding to $M(1)[z_2]$. ∎

We are now in a position to complete the parameterization of minimal representing measures in the rank 3 disk problem (cf. Section 4); we begin with the case when $\gamma_{01} \in \mathbf{R}$.

Proposition 5.6 *Let $M(1)$ be a given positive and invertible moment matrix, with $\gamma_{11} < \gamma_{00} = 1$ and $\gamma_{01} \in \mathbf{R}$.*
(i) *The curve $\rho_z = \frac{1-\gamma_{11}}{1-|z|^2}$ of Theorem 4.11 is precisely the ellipse $\mathcal{E} := \{z \equiv x + iy \in \mathbf{D} : \det M(1)[z] = 0\}$ described in Theorem 5.5. Thus, $M(1)$ admits a 3-atomic representing measure μ with $z \in \operatorname{supp} \mu \subseteq \bar{\mathbf{D}}$ if and only if $z \in \bar{\mathbf{D}} \backslash \Omega$, where Ω is the interior region of $\mathcal{E}$.*
(ii) *For each w in the interior region of $\mathcal{E}$, there exists a (minimal) 4-atomic representing measure μ with $w \in \operatorname{supp} \mu \subseteq \bar{\mathbf{D}}$.*

Proof. (i) Consider the curve $\rho_z = \frac{1-\gamma_{11}}{1-|z|^2}$ of Theorem 4.11. Replace ρ_z by $\frac{1}{\langle M(1)^{-1}\mathbf{z},\mathbf{z}\rangle}$ using (4.16), write $z \equiv x+iy$, and use symbolic manipulation to express the curve in the equivalent form

$$A'x^2 + 2B'xy + C'y^2 + 2D'x + 2E'y + F' = 0. \tag{5.6}$$

It is straightforward to check that (5.6) is equivalent to (5.5), so the curve coincides with the ellipse $\mathcal{E}$ of Theorem 5.5.

(ii) Following the rank-reduction strategy used in Section 2, for $0 < \rho < 1$, consider the associated system $\tilde{\gamma}^{(2)}$: $\tilde{\gamma}_{00} := 1 - \rho$, $\tilde{\gamma}_{01} := \gamma_{01} - \rho w$, $\tilde{\gamma}_{02} := \gamma_{02} - \rho w^2$ and $\tilde{\gamma}_{11} := \gamma_{11} - \rho |w|^2$; let $\widetilde{M(1)}$ denote the corresponding moment matrix. In order for $\tilde{\gamma}^{(2)}$ to give rise to a disk problem, we need $\tilde{\gamma}_{11} \leq \tilde{\gamma}_{00}$, that is $\gamma_{11} - \rho |w|^2 \leq 1 - \rho$. This means that $\rho \leq \frac{1-\gamma_{11}}{1-|w|^2}$. Since the ellipse $\mathcal{E}$ is strictly contained in the disk $D_1 = D((0,0), \sqrt{\gamma_{11}})$ (by Theorem 5.5 and Corollary 5.4), we have $|w|^2 < \gamma_{11}$, so $\frac{1-\gamma_{11}}{1-|w|^2} < 1$. It follows that we can let $\rho := \frac{1-\gamma_{11}}{1-|w|^2}$ and have $0 < \rho < 1$ and $\tilde{\gamma}_{11} \leq \tilde{\gamma}_{00}$. By the results in Section 4, we know that there exists a rank $\widetilde{M(1)}$-atomic representing measure $\tilde{\mu}$ for $\tilde{\gamma}^{(2)}$ with $\operatorname{supp} \tilde{\mu} \subseteq \bar{\mathbf{D}}$. Now let $\mu := \rho\delta_w + (1-\rho)\tilde{\mu}$; clearly, μ is a representing measure for $\gamma^{(2)}$. Since w is in the interior region of $\mathcal{E}$, Theorem 4.11 implies card supp $\mu \geq 4$. Now

$$4 \leq \text{card supp}\, \mu \leq 1 + \text{card supp}\, \tilde{\mu} \leq 1 + 3 = 4,$$

so μ is a 4-atomic minimal representing measure for $\gamma^{(2)}$. ∎

We can now state the complete parameterization of minimal representing measures in the rank 3 disk problem.

Theorem 5.7 *Let $M(1)$ be positive and invertible, with $\gamma_{11} < \gamma_{00} = 1$, and let $z \in \bar{\mathbf{D}}$. Choose $\lambda \in \mathbf{T}$ such that the rotated system $\tilde{\gamma}^{(2)}$ corresponding to λ via Lemma 2.1 satisfies $\tilde{\gamma}_{01} \in \mathbf{R}$. Let $\tilde{\mathcal{E}} \subseteq \mathbf{D}$ denote the ellipse corresponding to $\tilde{\gamma}^{(2)}$ via Proposition 5.6, and let $\tilde{\Omega}$ denote the interior region of $\tilde{\mathcal{E}}$. Then $M(1)$ admits a 3-atomic (minimal) representing measure μ with $z \in \operatorname{supp} \mu \subseteq \bar{\mathbf{D}}$ if and only if $z \in \bar{\mathbf{D}} \backslash \Omega$, where $\Omega := \bar{\lambda}\tilde{\Omega}$ is the interior region of the ellipse $\mathcal{E} := \bar{\lambda}\tilde{\mathcal{E}}$. The measure μ is uniquely determined by z, and the remaining two atoms of μ belong to $\mathbf{T}$, precisely when $z \in \mathcal{E}$ (cf. Theorem 5.5). For $z \in \bar{\mathbf{D}} \setminus \bar{\Omega}$, μ is not uniquely determined by z, and the remaining two atoms may be chosen using Algorithm 4.14, Step (c). Finally, for $z \in \Omega$, there exists a (minimal) 4-atomic representing measure μ with $z \in \operatorname{supp} \mu \subseteq \bar{\mathbf{D}}$.*

Example 5.8 ***(Example 4.15 Revisited)*** *For the matrix*

$$M(1) := \begin{pmatrix} 1 & \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{3}{4} & \frac{1}{8}-\frac{i}{2} \\ \frac{1+i}{2} & \frac{1}{8}+\frac{i}{2} & \frac{3}{4} \end{pmatrix}$$

of Example 4.15, we have already applied Algorithm 4.14 to determine the location of a first atom z_2 for a 3-atomic representing measure. (Note that in Example 4.15 we called the first atom z_0.) Here we will refine this calculation to exhibit the precise location of z_2 so that the remaining two atoms belong to the unit circle $\mathbf{T}$. To this end, we need to describe explicitly the disks D_1 and D_2 as well as the ellipse $\mathcal{E}$ described above. First recall, however, that for the calculation of D_1, D_2, and $\mathcal{E}$, a basic assumption was that $\gamma_{01} \in \mathbf{R}$. In the case at hand, this means that we must first "rotate" the matrix $M(1)$ to obtain

$$\widetilde{M(1)} = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{3}{4} & \frac{1}{2}+\frac{i}{8} \\ \frac{1}{\sqrt{2}} & \frac{1}{2}-\frac{i}{8} & \frac{3}{4} \end{pmatrix}$$

(cf. Example 2.8). For this matrix, the associated disks are $\tilde{D}_1 = D((0,0), \sqrt{3}/2)$ and $\tilde{D}_2 = D(1/(2\sqrt{2}), 0), 1/2)$. Moreover, $a = \frac{1}{\sqrt{2}}$, $c = \frac{1}{2}$, $d = -\frac{1}{8}$, and $e = \frac{3}{4}$. It follows that $A = \det \widetilde{M(1)} + 2(1-e)(e-c) = \frac{11}{64}$, $B = -2d(1-e) = \frac{1}{16}$, $C = \det \widetilde{M(1)} + 2(1-e)(e+c-2a^2) = \frac{11}{64}$, $D = -2a(1-e)(e-c) = -\frac{1}{8\sqrt{2}}$, $E = 2ad(1-e) = -\frac{1}{16\sqrt{2}}$, and $F = 2a^2(e-c) - e(e^2-c^2-d^2) = \frac{7}{256}$. Thus, in homogeneous coordinates, $\mathcal{E}^{\tilde{}}$ is described by the 3×3 matrix

$$\begin{pmatrix} \frac{11}{64} & \frac{1}{16} & -\frac{1}{8\sqrt{2}} \\ \frac{1}{16} & \frac{11}{64} & -\frac{1}{16\sqrt{2}} \\ -\frac{1}{8\sqrt{2}} & -\frac{1}{16\sqrt{2}} & \frac{7}{256} \end{pmatrix}.$$

For the original moment problem, therefore, we must rotate $\tilde{D}_1$, $\tilde{D}_2$, and $\mathcal{E}^{\tilde{}}$ to obtain $D_1 = \tilde{D}_1$, $D_2 = D((\frac{1}{4}, \frac{1}{4}), \frac{1}{2})$, and $\mathcal{E}$ the ellipse with associated matrix

$$\begin{pmatrix} \frac{7}{64} & 0 & -\frac{1}{32} \\ 0 & \frac{15}{64} & -\frac{3}{32} \\ -\frac{1}{32} & -\frac{3}{32} & \frac{7}{256} \end{pmatrix}.$$

It is straightforward to verify that $\mathcal{E} = \mathcal{E}_0 := \partial\Omega$, where $\Omega \equiv E((\frac{2}{7}, \frac{2}{5}), \{\frac{3}{14}\sqrt{\frac{19}{5}}, \frac{1}{10}\sqrt{\frac{57}{7}}\})$ is the interior of the ellipse described in Example 4.15. We graph D_1, D_2 and $\mathcal{E}$ in Figure 3. ∎

6 An Application: Location of the Zeros of Some Cubic Polynomials

In this section we give a parameterization (in terms of the coefficients) of certain analytic monic cubic polynomials having three distinct roots on the unit circle. To this end, we

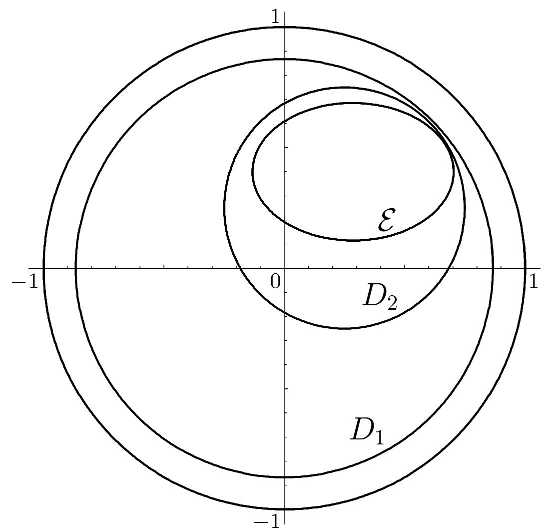

Figure 3: Relative position of D_1, D_2 and $\mathcal{E}$ inside the unit disk

restrict attention to rank 3 moment matrices $M(1)$ with representing measures supported in $\mathbf{T}$; we may thus assume $0 \le \gamma_{01} < \gamma_{11} = \gamma_{00} = 1$ and $|\gamma_{02}| < 1$.

As we have seen before (cf. (3.6) and the discussion immediately preceding it), to find a flat extension $M(2) \equiv \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix}$ of $M(1)$, it suffices to specify an appropriate value for γ_{03}. Now, the first column of B, $\mathbf{v} \equiv \begin{pmatrix} \gamma_{02} & \gamma_{12} & \gamma_{03} \end{pmatrix}^T$, must be a linear combination of the columns in $M(1)$, say

$$Z^2 = \alpha_1 \mathit{1} + \alpha_2 Z + \alpha_3 \bar{Z}, \tag{6.1}$$

where $\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^T = M(1)^{-1}\mathbf{v}$. In $\mathcal{C}_{M(2)}$ the same relation must hold and, moreover,

$$\bar{Z}Z = \mathit{1}, \tag{6.2}$$

so $\gamma_{12} = \gamma_{01}$. Further, the unique flat extension $M(3)$ of $M(2)$ is determined by $Z^3 = \alpha_1 Z + \alpha_2 Z^2 + \alpha_3 \bar{Z}Z = \alpha_1 Z + \alpha_2 Z^2 + \alpha_3 \mathit{1}$. It follows that the generating polynomial

$$g(z) \equiv z^3 - \alpha_2 z^2 - \alpha_1 z - \alpha_3 \tag{6.3}$$

has three distinct roots on the unit circle. Conversely, if z_0, z_1, z_2 are three distinct points of $\mathbf{T}$ then the measure $\mu \equiv \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1} + \rho_2 \delta_{z_2}$ has a moment matrix $M(1)$ whose generating polynomial is $(z - z_0)(z - z_1)(z - z_2)$.

We seek to determine the appropriate choices of γ_{03}. Let $\gamma_{01} \equiv a$, $\gamma_{02} \equiv c + id$, and $\gamma_{03} \equiv x + iy$. Looking at the fourth row of $M(2)$, it follows from (6.1) that

$$\alpha_1 \gamma_{20} + \alpha_2 \gamma_{21} + \alpha_3 \gamma_{30} = \gamma_{22},$$

so (6.2) implies

$$\alpha_1 (c - id) + \alpha_2 a + \alpha_3 (x - iy) = 1. \tag{6.4}$$

A calculation shows that (6.4) is equivalent to

$$(x-h)^2+(y-k)^2=r^2, \tag{6.5}$$

where

$$\begin{array}{r} h := \frac{a(2c+d^2-a^2-c^2)}{1-a^2} \\ k := \frac{2ad(1-c)}{1-a^2} \\ r := \frac{\left|1-2a^2+2a^2c-c^2-d^2\right|}{1-a^2} \end{array}. \tag{6.6}$$

In special cases, the preceding parameterization of the generating polynomial in (6.3) in terms of the moment data leads to some easily recognizable cubic polynomials having three roots in the unit circle.

Theorem 6.1 *Let $|\beta|<1$, $|w|=1$. The cubic polynomial $g(z)\equiv z^3+w\bar{\beta}z^2-\beta-w$ has three distinct roots in the unit circle.*

Proof. Define the rank 3 circle moment problem $M(1)$ in which $\gamma_{00}\equiv\gamma_{11}:=1$, $\gamma_{01}:=0$ and $\gamma_{02}:=\beta$. By direct computation,

$$\begin{array}{c} \alpha_1=\gamma_{02}=\beta, \\ \alpha_2=-\frac{\gamma_{03}\gamma_{20}}{1-|\gamma_{02}|^2}=-\frac{\bar{\beta}\gamma_{03}}{1-|\beta|^2}, \text{ and} \\ \alpha_3=\frac{\gamma_{03}}{1-|\gamma_{02}|^2}=\frac{\gamma_{03}}{1-|\beta|^2}. \end{array}$$

By choosing $\gamma_{03}:=w(1-|\beta|^2)$, we obtain at once that the generating polynomial is $g(z)\equiv z^3+w\bar{\beta}z^2-\beta-w$. ∎

Remark 6.2 *In principle, the above procedure allows us to characterize all cubic polynomials with three distinct roots in the unit circle. For instance, a similar technique as that employed in the proof of Theorem 6.1 shows that all polynomials of the form $p_t(z)\equiv z^3-tz^2+tz-1$ $(0\le t\le 1+\sqrt{2})$ have three distinct roots in the unit circle (using $\gamma_{02}=0$ instead of $\gamma_{01}=0$).* ∎

References

[AhKr] N.I. Ahiezer and M. Krein, "Some Questions in the Theory of Moments," Transl. Math. Monographs, vol. 2, American Math. Soc., Providence, 1962.

[Akh] N.I. Akhiezer, "The Classical Moment Problem," Hafner Publ. Co., New York, 1965.

[Atz] A. Atzmon, A moment problem for positive measures on the unit disc, *Pacific J. Math.* 59(1975), 317-325.

[BeMa] C. Berg and P.H. Maserick, Polynomially positive definite sequences, *Math. Ann.* 259(1982), 487-495.

[Cur] R. Curto, An operator-theoretic approach to truncated moment problems, in "Linear Operators," Banach Center Publ., vol. 38, 1997, pp. 75-104.

[CuFi1] R. Curto and L. Fialkow, Recursiveness, positivity, and truncated moment problems, *Houston J. Math.* 17(1991), 603-635.

[CuFi2] R. Curto and L. Fialkow, Solution of the truncated complex moment problem for flat data, *Mem. Amer. Math. Soc.* 119(1996), no. 568.

[CuFi3] R. Curto and L. Fialkow, Flat extensions of positive moment matrices: Relations in analytic or conjugate terms, *Operator Th.: Adv. Appl.* 104(1998), 59-82.

[CuFi4] R. Curto and L. Fialkow, Flat extensions of positive moment matrices: Recursively generated relations, *Memoirs Amer. Math. Soc.* 136(1998), no. 648.

[CuFi5] R. Curto and L. Fialkow, The truncated K-moment problem, *Trans. Amer. Math. Soc.*, to appear.

[CuFi6] R. Curto and L. Fialkow, Solution of the singular quartic moment problem, preprint 2000.

[Fi1] L. Fialkow, Positivity, extensions and the truncated complex moment problem, *Contemp. Math.*, vol. 185, 1995, pp. 133-150.

[Fi2] L. Fialkow, Multivariable quadrature and extensions of moment matrices, preprint 1996.

[Fi3] L. Fialkow, Minimal representing measures arising from rank-increasing moment matrix extensions, *J. Operator Theory 42(1999), 425-436.*

[Ioh] I.S. Iohvidov, "Hankel and Toeplitz Matrices and Forms: Algebraic Theory," Birkhäuser Verlag, Boston, 1982.

[KrNu] M.G. Krein and A.A. Nudel'man, "The Markov Moment Problem and Extremal Problems," Transl. Math. Monographs, vol. 50, American Math. Soc., Providence, 1977.

[Put] M. Putinar, A two-dimensional moment problem, *J. Funct. Anal.* 80(1988), 1-8.

[Sch] K. Schmüdgen, The K-moment problem for semi-algebraic sets, *Math. Ann.* 289(1991), 203-206.

[ShTa] J.A. Shohat and J.D. Tamarkin, "The Problem of Moments," Math. Surveys I, American Math. Soc., Providence, 1943.

[Smu] Ju.L. Smul'jan, An operator Hellinger integral (Russian), *Mat. Sb.* (N.S.) 49(1959), 381-430.

[Wol] Wolfram Research, Inc., Mathematica, Version 3.0, Wolfram Research, Inc., Champaign, IL, 1996.

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