AN ANALOGUE OF THE RIESZ-HAVILAND THEOREM FOR THE TRUNCATED MOMENT PROBLEM

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Abstract. Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{|i|\le 2n}$ denote a d-dimensional real multisequence, let K denote a closed subset of $\mathbb{R}^d$, and let $\mathcal{P}_{2n} := \{p \in \mathbb{R}[x_1,\ldots,x_d] : \deg\ p \le 2n\}$. Corresponding to β, the *Riesz functional* $L \equiv L_\beta : \mathcal{P}_{2n} \longrightarrow \mathbb{R}$ is defined by $L(\sum a_i x^i) := \sum a_i \beta_i$. We say that L is *K-positive* if whenever $p \in \mathcal{P}_{2n}$ and $p|_K \ge 0$, then $L(p) \ge 0$. We prove that β admits a K-representing measure if and only if L_β admits a K-positive linear extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$. This provides a generalization (from the full moment problem to the truncated moment problem) of the Riesz-Haviland Theorem. We also show that a semialgebraic set solves the truncated moment problem in terms of natural "degree-bounded" positivity conditions if and only if each polynomial strictly positive on that set admits a degree-bounded weighted sum-of-squares representation.

1. Introduction

Let $\beta \equiv \beta^{(\infty)} = \{\beta_i\}_{i\in\mathbb{Z}_+^d}$ denote a d-dimensional real multisequence and let K denote a closed subset of $\mathbb{R}^d$. The *full K-moment problem* asks for conditions on β such that there exists a positive Borel measure μ, with supp $\mu \subseteq K$, satisfying $\beta_i = \int x^i \, d\mu$ $(i \in \mathbb{Z}_+^d)$ (here $x^i := x_1^{i_1}\cdots x_d^{i_d}$, for $x \equiv (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $i \equiv (i_1,\ldots,i_d) \in \mathbb{Z}_+^d$). In the *truncated K-moment problem* of degree m (where $1 \le m < \infty$), the data are restricted to $\beta \equiv \beta^{(m)} = \{\beta_i\}_{i\in\mathbb{Z}_+^d, |i|\le m}$. A theorem of J. Stochel [St2, Theorem 4] shows that $\beta^{(\infty)}$ has a *K-representing measure* μ (as above) if and only if for each m, $\beta^{(m)}$ has a K-representing measure. In this sense, the truncated moment problem is more general than the full moment problem, and several results from [F3, Section 4] and [CF11, Section 6] illustrate how the truncated moment problem can be used to solve the full moment problem in special cases (cf. Remark 2.15(ii)). By contrast, existence theorems for the full moment problem cannot simply be "truncated" to give valid results for the truncated moment problem (cf. Example 2.1). In this note we study analogues for the truncated moment problem of known existence theorems for representing measures in the full moment problem.

Let $\mathcal{P} := \mathbb{R}[x_1,\ldots,x_d]$ and for $p \equiv \sum a_i x^i \in \mathcal{P}$, let $\hat{p} \equiv (a_i)$ denote the coefficient vector of p with respect to the basis for $\mathcal{P}$ consisting of the monomials in degree-lexicographic order. Corresponding to $\beta \equiv \beta^{(\infty)}$, the *Riesz functional* $L \equiv L_\beta : \mathcal{P} \longrightarrow \mathbb{R}$ is defined by $L(\sum a_i x^i) := \sum a_i \beta_i$. We say that L is *K-positive* if whenever $p \in \mathcal{P}$ and $p|_K \ge 0$, then $L(p) \ge 0$; if L is K-positive for $K = \mathbb{R}^d$, we say simply that L is *positive.* K-positivity is a necessary condition for the existence of a K-representing measure μ, since if $p \in \mathcal{P}$ satisfies $p|_K \ge 0$, then $p|_{\text{supp }\mu} \ge 0$, whence $L(p) = \int p \, d\mu \ge 0$. Conversely, the classical theorem of M. Riesz [Ri] $(d = 1)$ and Haviland [H] $(d > 1)$ provides a fundamental existence criterion for K-representing measures.

Theorem 1.1. *(Riesz-Haviland Theorem) $\beta \equiv \beta^{(\infty)}$ admits a representing measure supported in the closed set $K \subseteq \mathbb{R}^d$ if and only if L_β is K-positive.*

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In the truncated moment problem for $\beta^{(2n)}$, it follows as above that the existence of a K-representing measure implies that the Riesz functional $L_{\beta^{(2n)}} : \mathcal{P}_{2n} \longrightarrow \mathbb{R}$ is K-positive, i.e., $p \in \mathcal{P}_{2n}$, $p|_K \geq 0 \Longrightarrow L_{\beta^{(2n)}}(p) \geq 0$. A result of V. Tchakaloff [T, Théorème II, p. 129] implies that the converse is true in case K is compact. However, we show in Section 2 that in general it is not true that $\beta^{(2n)}$ has a K-representing measure whenever $L_{\beta^{(2n)}}$ is K-positive, so the most direct analogue of Theorem 1.1 for the truncated moment problem is false (cf. Example 2.1). Instead, the appropriate analogue of the Riesz-Haviland Theorem for the truncated K-moment problem, which is our main result, assumes the following form.

Theorem 1.2. *$\beta \equiv \beta^{(2n)}$ admits a K-representing measure if and only if L_β admits a K-positive linear extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$.*

Note that Theorem 1.2 actually implies Theorem 1.1. Indeed, given $\beta \equiv \beta^{(\infty)}$, if L_β is K-positive, then for each n, $L_{\beta^{(2n+2)}}$ is a K-positive extension of $L_{\beta^{(2n)}}$. Theorem 1.2 then implies that for each n, $\beta^{(2n)}$ has a K-representing measure, whence Stochel's theorem implies that β has a K-representing measure. We note also that Theorem 1.2 remains true if $\beta^{(2n)}$ is replaced by $\beta^{(2n+1)}$; this is clear from the proof of Theorem 1.2 in Section 2 (cf. Theorem 2.4). In the sequel, we focus on $\beta^{(2n)}$ rather than on $\beta^{(2n+1)}$, primarily because the data for $\beta^{(2n)}$ define a complete *real moment matrix* $\mathcal{M}(n)$ (as described below), so it is notationally more convenient to treat $\beta^{(2n)}$.

Let $\mathcal{Q} \equiv \{q_0 := 1, q_1, \cdots, q_m\} \subseteq \mathcal{P}$ and let $K_{\mathcal{Q}}$ denote the semialgebraic set $\{x \in \mathbb{R}^d : q_i(x) \geq 0 \ (1 \leq i \leq m)\}$. Our main application of Theorem 1.2 is Theorem 1.6, which shows that if each polynomial that is strictly positive on $K_{\mathcal{Q}}$ admits a "degree-bounded" sum-of-squares representation, then the truncated moment problem on $K_{\mathcal{Q}}$ can be solved in terms of positivity for the localizing matrices associated to each q_i. In Section 2 we present some concrete conditions for representing measures related to the *algebraic variety* $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n))$ associated to $\beta^{(2n)}$. In particular, $\mathcal{V}$-positivity for $L_{\beta^{(2n)}}$ implies the existence of $\mathcal{V}$-representing measures for $\beta^{(2n)}$ when $d = 1$ (Proposition 2.14); when $d = 2$, $n \geq 2$ and $\mathcal{V}$ is a subset of a planar curve $p(x, y) = 0$ with deg $p \leq 2$ (Proposition 2.16); when $\mathcal{V}$ is compact (Proposition 2.17); or when card $\mathcal{V} =$ rank $\mathcal{M}(n)$ (Proposition 2.18).

We note that positivity for $\tilde{L} := L_{\beta^{(2n+2)}}$ is in general a much stronger condition than positivity for the corresponding moment matrix $\mathcal{M}(n+1)$. For this reason, in general it is quite difficult to directly verify that an extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$ is K-positive. One approach to establishing K-positivity or the existence of representing measures is through extensions of moment matrices. For simplicity, consider the case $K = \mathbb{R}^d$ (the case of a general semialgebraic set is discussed following Theorem 1.4). [CF10] implies that β admits a finitely atomic representing measure if and only if for some $k \geq 0$, $\mathcal{M}(n)$ has a positive semidefinite extension $\mathcal{M}(n+k)$, which in turn admits a *flat* (i.e., rank-preserving) extension $\mathcal{M}(n+k+1)$. The generalization of this result to measures with finite moments up to at least degree $2n+1$ follows from [Pu2] or [CF9], and the extension to general measures follows from a recent result of [BT] (cf. Section 2). When the extension $\mathcal{M}(n+k)$ (as above) exists, we may always take $k \leq \min\{\text{card } \mathcal{V} - \text{rank } \mathcal{M}(n), \dim \mathcal{P}_{2n} - \text{rank } \mathcal{M}(n)\}$ [F4, Proposition 2.3], and examples of [F2] and [F4] illustrate cases where $k > 0$ is required. Corresponding to the flat extension $\mathcal{M}(n+k+1)$ is a computable rank $\mathcal{M}(n+k)$-atomic representing measure μ for $\beta^{(2n)}$, so in this approach we circumvent K-positivity (though the Riesz functional associated to $\mathcal{M}(n+1)[\mu]$ is clearly a positive extension of L_β). Various sufficient conditions for flat extensions appear in [CF3], [CF4], [CF8], [CF11], [CFM] and [Moe]; perhaps the basic condition is $\mathcal{M}(n) \geq 0$ and rank $\mathcal{M}(n) =$ rank $\mathcal{M}(n-1)$ [CF2]. Theorem 1.2 shows that, in principle, the existence of an extension $\mathcal{M}(n+k)$ (as above) is completely determined by a choice of $\mathcal{M}(n+1)$ for which the corresponding Riesz functional $L^{(2n+2)}$ is positive.

A second approach to positivity for an extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$ concerns the structure of positive polynomials. Note that the main difficulty associated with Theorem 1.1 is that for a general closed set $K \subseteq \mathbb{R}^d$ there is no concrete representation theorem for polynomials that are nonnegative on K, so there may be no practical test to check whether L_β is K-positive. In this sense, Theorem 1.1 is an "abstract" solution to the moment problem, and, similarly, Theorem 1.2 is an abstract solution to the truncated moment problem. In the case when K is a compact semialgebraic set, a celebrated theorem of K. Schmüdgen [Sm1, Theorem 1] provides a concrete test for the K-positivity of $L_{\beta(\infty)}$ (cf. Theorem 1.3 below). In Section 3 we will derive certain analogues of Schmüdgen's results for the truncated moment problem.

Let $\mathcal{Q} = \{q_0, q_1, \ldots, q_m\} \subseteq \mathcal{P}$ (with $q_0 \equiv 1$) and consider the semialgebraic set

$$K_\mathcal{Q} = \{x \in \mathbb{R}^d : q_i(x) \geq 0 \ (1 \leq i \leq m)\}.$$

Moreover, let $\mathcal{Q}^\pi$ denote the set of products of distinct polynomials in $\mathcal{Q}$, that is,

$$\mathcal{Q}^\pi := \{q_{i_1} \cdots q_{i_s} : q_{i_j} \in \mathcal{Q}, 0 \leq i_1 < \cdots < i_s \leq m, 1 \leq s \leq m+1\};$$

observe that $\mathcal{Q} \subseteq \mathcal{Q}^\pi$ and that $K_\mathcal{Q} = K_{\mathcal{Q}^\pi}$. Let $\tilde{\mathcal{Q}}$ denote any set satisfying $\mathcal{Q} \subseteq \tilde{\mathcal{Q}} \subseteq \mathcal{Q}^\pi$, so that $K_\mathcal{Q} = K_{\tilde{\mathcal{Q}}}$. In our applications, we will specify $\tilde{\mathcal{Q}} = \mathcal{Q}$ or $\tilde{\mathcal{Q}} = \mathcal{Q}^\pi$ as needed.
Recall from [CF4] and [CF10] the *moment matrix* $\mathcal{M} \equiv \mathcal{M}(\infty)(\beta)$ associated with $\beta \equiv \beta^{(\infty)}$, defined by $\langle \mathcal{M}\hat{f}, \hat{g}\rangle := L_\beta(fg)$ $(f, g \in \mathcal{P})$. For $p \in \mathcal{P}$, the *localizing matrix* $\mathcal{M}_p \equiv \mathcal{M}_p(\infty)$ is defined by $\langle \mathcal{M}_p\hat{f}, \hat{g}\rangle := L_\beta(fgp)$ $(f, g \in \mathcal{P})$; observe that $\mathcal{M}_1 = \mathcal{M}$. If μ is a representing measure for β, then for $f \in \mathcal{P}$, $\langle \mathcal{M}\hat{f}, \hat{f}\rangle = L_\beta(f^2) = \int f^2 \, d\mu \geq 0$, so $\mathcal{M}$ is positive semidefinite ($\mathcal{M} \geq 0$). Similarly, if μ is a representing measure supported in $K_\mathcal{Q}$ and $r \equiv q_{i_1} \cdots q_{i_s} \in \mathcal{Q}^\pi$, then $\langle \mathcal{M}_r\hat{f}, \hat{f}\rangle = L_\beta(rf^2) = \int q_{i_1} \cdots q_{i_s} f^2 \, d\mu \geq 0$ (since $q_{i_j}|_{\text{supp } \mu} \geq 0$), whence $\mathcal{M}_r \geq 0$. The results in [Sm1] are presented in terms of positive multisequences; here we give an equivalent reformulation in terms of moment matrices.

Theorem 1.3. *(K. Schmüdgen* [Sm1]*) Suppose $K_\mathcal{Q}$ is compact and $\tilde{\mathcal{Q}} = \mathcal{Q}^\pi$. The sequence $\beta \equiv \beta^{(\infty)}$ has a representing measure supported in $K_\mathcal{Q}$ if and only if $\mathcal{M}_r \geq 0$ for each polynomial $r \in \tilde{\mathcal{Q}}$.*

The conclusion of Theorem 1.3 also holds for certain semialgebraic sets that are not compact. Consider the following property for a semialgebraic set $K_\mathcal{Q}$ and an associated $\tilde{\mathcal{Q}}$:

$$(\mathbf{S}) \quad \begin{cases} \beta \equiv \beta^{(\infty)} \text{ has a representing measure supported in } K_\mathcal{Q} \\ \quad \text{if and only if } \mathcal{M}_r \geq 0 \text{ for each polynomial } r \in \tilde{\mathcal{Q}}. \end{cases}$$

Hamburger's Theorem for the real line $\mathbb{R}$ is equivalent to the assertion that $(\mathbf{S})$ holds with $d = 1$, $\mathcal{Q} = \tilde{\mathcal{Q}} = \{1\}$, and Stieltjes' Theorem for the half-line $[0, +\infty)$ is equivalent to the statement that $(\mathbf{S})$ holds when $d = 1$, $\mathcal{Q} = \tilde{\mathcal{Q}} = \{1, x\}$ (cf. [A], [ShT]). Moreover, a theorem of J. Stochel [St1] is equivalent to the statement that if $d = 2$ and deg $p \leq 2$, then $(\mathbf{S})$ holds for the algebraic set $K_\mathcal{Q}$, where $\mathcal{Q} = \tilde{\mathcal{Q}} = \{1, p, -p\}$. Other algebraic sets satisfying $(\mathbf{S})$ are described in [St1], [SZ], [Se1], [PoSc] and [Sm2].

In [Sm1], for the case when $K_\mathcal{Q}$ is compact, Schmüdgen used Theorem 1.3 to establish a structure theorem for polynomials that are strictly positive on $K_\mathcal{Q}$. Consider the convex cones in $\mathcal{P}$ defined by

$$(1.1) \qquad \Sigma_{\tilde{\mathcal{Q}}} := \{p \in \mathcal{P} : p = \sum_j f_j^2 + \sum_k r_k \sum_j g_{kj}^2 : f_j, g_{kj} \in \mathcal{P}, \ r_k \in \tilde{\mathcal{Q}}\}.$$

For $\tilde{\mathcal{Q}} = \mathcal{Q}$ (resp. $\mathcal{Q}^\pi$) we denote $\Sigma_{\tilde{\mathcal{Q}}}$ by $\Sigma_\mathcal{Q}$ (resp. $\Sigma_{\mathcal{Q}^\pi}$). In [Sm1, Corollary 3], Schmüdgen proved that if $K_\mathcal{Q}$ is compact, then each polynomial that is strictly positive on $K_\mathcal{Q}$ belongs to $\Sigma_{\mathcal{Q}^\pi}$. It is straightforward to check that Schmüdgen's proof holds more generally whenever $K_\mathcal{Q}$ is compact and satisfies $(\mathbf{S})$ with $\tilde{\mathcal{Q}}$. The converse of [Sm1, Corollary 3] is also true, and holds for general $K_\mathcal{Q}$. For suppose that each polynomial that is strictly positive on $K_\mathcal{Q}$ belongs to $\Sigma_{\tilde{\mathcal{Q}}}$.

To show that $K_{\mathcal{Q}}$ and $\tilde{\mathcal{Q}}$ satisfy (**S**), let $\beta \equiv \beta^{(\infty)}$ be given and assume that $\mathcal{M}_r \geq 0$ for each $r \in \tilde{\mathcal{Q}}$. To prove that β has a $K_{\mathcal{Q}}$-representing measure, we will verify that $L \equiv L_\beta$ is $K_{\mathcal{Q}}$-positive. For $p \in \mathcal{P}$ with $p|_{K_{\mathcal{Q}}} \geq 0$, and for $\varepsilon > 0$, we have $p + \varepsilon > 0$, so $p + \varepsilon$ belongs to $\Sigma_{\tilde{\mathcal{Q}}}$. Thus, $p + \varepsilon = \sum_j f_j^2 + \sum_k r_k \sum_j g_{kj}^2$ (as in (1.1)), whence

$$
\begin{aligned}
L(p) + \varepsilon L(1) &= L(p+\varepsilon) = L(\sum_j f_j^2 + \sum_k r_k \sum_j g_{kj}^2) \\
&= \sum_j \langle \mathcal{M}\hat{f}_j, \hat{f}_j\rangle + \sum_k \sum_j \langle \mathcal{M}_{r_k} \hat{g_{kj}}, \hat{g_{kj}}\rangle \geq 0. \qquad (1.2)
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, it follows that $L(p) \geq 0$. Thus, L is $K_{\mathcal{Q}}$-positive, so the existence of a $K_{\mathcal{Q}}$-representing measure follows from Theorem 1.1. Since the converse implication in (**S**) is always true, $K_{\mathcal{Q}}$ and $\tilde{\mathcal{Q}}$ satisfy (**S**). We thus have the following result for the full moment problem on semialgebraic sets.

Theorem 1.4. *(cf.* [Sm1, Corollary 3], [PoSc, Corollary 3.1]*) If $K_{\mathcal{Q}}$ is compact and satisfies (***S***) with $\tilde{\mathcal{Q}}$, then each polynomial that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\tilde{\mathcal{Q}}}$. The converse holds for arbitrary $K_{\mathcal{Q}}$.*

Consider a (necessarily compact) $K_{\mathcal{Q}}$ with the following property:

(**P**) There exists $R > 0$ such that $R - (x_1^2 + \cdots + x_d^2) \in \Sigma_{\mathcal{Q}}$.

In [Pu1], M. Putinar proved that if $K_{\mathcal{Q}}$ satisfies (**P**), then $K_{\mathcal{Q}}$ satisfies (**S**) with $\tilde{\mathcal{Q}} = \mathcal{Q}$, and that each polynomial that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q}}$. For other results related to [Sm1], see [KM], [PoSc], [Se1], [Se2], and the references cited therein.

In Section 3 we establish analogues of Theorem 1.4 for the truncated moment problem. To motivate these results, we first recall a general existence criterion for representing measures. Let $\beta \equiv \beta^{(2n)}$ and define the associated moment matrix $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ by $\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle := L_\beta(pq)$ $(p, q \in \mathcal{P}_n)$ [CF3], [CF10]; $\mathcal{M}(n)$ is a square matrix, of size dim $\mathcal{P}_n$. Let $K_{\mathcal{Q}}$ be as above and let deg $q_i = 2k_i$ or $2k_i - 1$ $(1 \leq i \leq m)$. For $s \in \mathcal{P}_{2n}$ with deg $s = 2k$ or $2k-1$, recall from [CF10] the localizing matrix $\mathcal{M}_s(n) \equiv \mathcal{M}_s(n)(\beta)$, defined by $\langle \mathcal{M}_s(n)\hat{p}, \hat{q}\rangle := L_\beta(spq)$ $(p, q \in \mathcal{P}_{n-k})$; the size of $\mathcal{M}_s(n)$ is dim $\mathcal{P}_{n-k}$. Also recall from [CF10] that if $\mathcal{M}(n)$ (≥ 0) admits a flat extension $\mathcal{M}(n+1)$, then $\mathcal{M}(n+1)$ admits unique successive flat extensions $\mathcal{M}(n+2)$, $\mathcal{M}(n+3)$, $\cdots$. The existence criterion of [CF10, Theorem 1.1] states that β has a finitely atomic $K_{\mathcal{Q}}$-representing measure if and only if $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+k)$, which in turn admits a flat extension $\mathcal{M}(n+k+1)$ such that $\mathcal{M}_{q_i}(n+k+k_i) \geq 0$ $(1 \leq i \leq m)$. (An estimate for k and an extension to general measures follow as above for the case $K_{\mathcal{Q}} = \mathbb{R}^d$.) In Section 3 we study cases where the conditions of [CF10] can be relaxed, as we next describe.

Let $K_{\mathcal{Q}}$ be as above and choose n so that $2n \geq$ deg q_i for $i = 1, \cdots, m$. For $k \geq 0$, consider the following properties for $K_{\mathcal{Q}}$:

$(\mathbf{S}_{n,k})$ $\begin{cases} \beta^{(2n)} \text{ has a } K_{\mathcal{Q}}\text{-representing measure if and only if} \\ \mathcal{M}(n) \text{ admits a positive extension } \mathcal{M}(n+k) \text{ such that} \\ \mathcal{M}_{q_i}(n+k) \geq 0 \text{ for } i = 1, \cdots, m \end{cases}$

and

$(\mathbf{R}_{n,k})$ $\begin{cases} \beta^{(2n)} \text{ has a } K_{\mathcal{Q}}\text{-representing measure if and only if} \\ \mathcal{M}(n) \text{ admits a positive, recursively generated extension } \mathcal{M}(n+k) \\ \text{such that } \mathcal{M}_{q_i}(n+k) \geq 0 \text{ for } i = 1, \cdots, m. \end{cases}$

Observe that for fixed n and k, $(\mathbf{S}_{n,k})$ implies $(\mathbf{R}_{n,k})$. Note also that $(\mathbf{R}_{n,k})$ implies $(\mathbf{S}_{n,k+1})$; this follows from the fact that if $\mathcal{M}(n+k+1)$ is positive, then $\mathcal{M}(n+k)$ is positive and recursively generated [CF2, Theorem 3.14]. In verifying property $(\mathbf{S}_{n,k})$ or $(\mathbf{R}_{n,k})$, one direction is always true. For, suppose $\beta \equiv \beta^{(2n)}$ has a $K_{\mathcal{Q}}$-representing measure μ. [BT] then implies that β admits a finitely atomic $K_{\mathcal{Q}}$-representing measure ν (cf. Section 2). Since ν has moments of all orders, for each k, $\mathcal{M}(n+k)[\nu]$ is a positive and recursively generated extension of $\mathcal{M}(n)$, and since supp $\nu \subseteq K_{\mathcal{Q}}$, then $\mathcal{M}_{q_i}(n+k)[\nu] \geq 0$ $(1 \leq i \leq m)$.

Whereas Schmüdgen works with the cone $\Sigma_{\mathcal{Q}^\pi} \bigcap \mathcal{P}_{2n}$, we focus on the sub-cone $\Sigma_{\mathcal{Q},n}$, defined by:

$$\Sigma_{\mathcal{Q},n} := \{p \in \mathcal{P}_{2n} : p = \sum_j f_{0j}^2 + q_1 \sum_j f_{1j}^2 + ... + q_m \sum_j f_{mj}^2,\ q_i f_{ij}^2 \in \mathcal{P}_{2n}\ (0 \leq i \leq m)\}.$$

It follows from an application of Carathéodory's Theorem described in [T, pp. 126-127] that the total number of terms $q_i f_{ij}^2$ in such a representation of p can always be taken to be at most dim $\mathcal{P}_{2n}$ (cf. Lemma 3.2). In the sequel we consider the property for $K_{\mathcal{Q}}$ that for some n and k, each $p \in \mathcal{P}_{2n}$ with $p|_{K_{\mathcal{Q}}} > 0$ admits a *degree-bounded* sum-of-squares representation, in the sense that $p \in \Sigma_{\mathcal{Q},n+k}$. In Section 3 we obtain the following analogue of one direction of Theorem 1.4 for the truncated moment problem.

Theorem 1.5. *(i) Assume that $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$ for some n and k. Then every polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k}$.*
(ii) Assume that $K_{\mathcal{Q}}$ satisfies $(\mathbf{R}_{n,k})$ for some n and k. Then each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k+1}$.

Theorem 1.5 provides a sufficient condition for finite convergence in the polynomial optimization method of J. Lasserre [Las]; indeed, if the conditions of Theorem 1.5(i) hold, then for $f \in \mathcal{P}_{2n}$, the optimal value $f^* := \inf \{f(x) : x \in K_{\mathcal{Q}}\}$ is realized at the $(n+k)$-th Lasserre relaxation (cf. Section 3). In Theorem 3.16 we show that Theorem 1.5 can be extended to nonnegative polynomials in those cases where the cone $\Sigma_{\mathcal{Q},n+k}$ is closed in $\mathcal{P}_{2(n+k)}$. In Section 3 we also establish the following converse of Theorem 1.5, an analogue to the converse direction in Theorem 1.4.

Theorem 1.6. *(i) If $k \geq 1$ and each polynomial in $\mathcal{P}_{2n+2}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k}$, then $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$.*
(ii) If $k = 0$, $K_{\mathcal{Q}}$ is compact, and each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n}$, then $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,0})$.

Example 2.1 (below) shows that the compactness hypothesis cannot be dropped in Theorem 1.6(ii). Given $\tilde{\mathcal{Q}}$ such that $Q \subseteq \tilde{\mathcal{Q}} \subseteq \mathcal{Q}^\pi$, consider the convex cone $\Sigma_{\tilde{\mathcal{Q}},n} := \{p \in \mathcal{P}_{2n} : p = \sum_i r_i \sum_j f_{ij}^2, r_i \in \tilde{\mathcal{Q}}, r_i f_{ij}^2 \in \mathcal{P}_{2n}\}$. Let $(\tilde{\mathbf{S}}_{n,k})$ be the property that $\beta^{(2n)}$ has a $K_{\mathcal{Q}}$-representing measure if and only if $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+k)$ such that $\mathcal{M}_r(n+k) \geq 0$ for every $r \in \tilde{\mathcal{Q}}$. It is straightforward to modify the proofs of Theorems 1.5 and 1.6 so as to obtain analogues where $(\mathbf{S}_{n,k})$ is replaced by $(\tilde{\mathbf{S}}_{n,k})$ and $\Sigma_{\mathcal{Q},n+k}$ is replaced by $\Sigma_{\tilde{\mathcal{Q}},n+k}$.

In the classical literature, the truncated moment problem was solved concretely only for the interval $[a,b]$, the circle, and for some cases of the line $\mathbb{R}$ and the half-line $[0,+\infty)$ (cf. [A], [KN], [ShT]). These results were proved using the representations of positive polynomials for these sets, and thus illustrate Theorem 1.6 (cf. Section 3). An alternate approach to truncated moment problems is through extensions of moment matrices. In [CF1] we used *recursiveness* of moment matrices to complete the one-dimensional results for the line and half-line, and also in [CF3], [CF6] and [CF11] to solve the truncated moment problem for lines, circles, and ellipses. In [CF8] and [CF11] we used recursiveness and the *variety condition* rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\mathcal{M}(n))$ to solve the truncated moment problem for the other conics (parabolas and hyperbolas). These results, together with Theorem 1.5, yield "degree-bounded" representations for polynomials that are positive on these sets (cf. Section 3). These representations are apparently new for parabolas and hyperbolas (and possibly for lines and ellipses). Despite the preceding results, the applicability of weighted sums of squares to truncated moment problems (as in Theorems 1.5 and 1.6) seems limited. Indeed, by contrast with Schmüdgen's results for the full moment problem (cf. Theorems 1.3 and 1.4), [Se3, Example 5.1] implies that if $d \geq 2$ and $K_{\mathcal{Q}}$ has nonempty interior, then for every

$n \geq 3$ and every $k \geq 0$, $K_{\mathcal{Q}}$ fails to satisfy $(\mathbf{S}_{n,k})$. In the case of the closed unit disk, we show that $(\mathbf{S}_{1,0})$ is satisfied (Proposition 3.17), but whether the disk satisfies $(\mathbf{S}_{2,k})$ for some k appears to be open.

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2. An analogue of the Riesz-Haviland Theorem for the truncated moment problem

We begin with an example which shows that the truncated moment problem does not admit the most direct analogue of the Riesz-Haviland Theorem.

Example 2.1. For $d = 1$ and $K = \mathbb{R}$ (the real line), we will exhibit $\beta \equiv \beta^{(4)}$ for which $L \equiv L_\beta$ is K-positive, but β admits no representing measure. Define $\beta^{(4)}$ by $\beta_0 \equiv \beta_1 \equiv \beta_2 \equiv \beta_3 := 1$ and $\beta_4 := 2$, so that $\mathcal{M}(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. A calculation shows that $\mathcal{M}(2) \geq 0$. Indeed, a partitioned real symmetric matrix $\mathcal{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is positive semidefinite if and only if $A \geq 0$ and there exists a matrix W such that $B = AW$ and $C \geq W^T AW$ (cf. [Smu], [CF4, Proposition 2.2]). In the present case, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 \end{pmatrix}$, and we may take $W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In this case, L_β is defined by $L(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) := a_0 + a_1 + a_2 + a_3 + 2a_4$. To see that L is $\mathbb{R}$-positive, recall that if $p \in \mathcal{P}_4$ satisfies $p|_{\mathbb{R}} \geq 0$, then there exist $f, g \in \mathcal{P}_2$ such that $p = f^2 + g^2$ [PS, Solution 44; p. 259]. Now $L(p) = L(f^2 + g^2) = \langle \mathcal{M}(2)\hat{f}, \hat{f} \rangle + \langle \mathcal{M}(2)\hat{g}, \hat{g} \rangle \geq 0$; thus, L is $\mathbb{R}$-positive. Assume that μ is a representing measure for β. Since $\int (x-1)^2 \, d\mu = L(x^2 - 2x + 1) = \beta_2 - 2\beta_1 + \beta_0 = 0$, it follows that $(x-1)|_{\text{supp } \mu} \equiv 0$. We thus have $(x-1)x^3|_{\text{supp } \mu} \equiv 0$, so $0 = \int (x-1)x^3 \, d\mu = L(x^4 - x^3) = \beta_4 - \beta_3 = 1$, a contradiction. Thus L is K-positive, but β has no representing measure. $\square$

We will return to Example 2.1 in the sequel. We now turn to the proof of Theorem 1.2, which we restate for ease of reference.

Theorem 2.2. *$\beta \equiv \beta^{(2n)}$ admits a K-representing measure if and only if L_β admits a K-positive extension $L : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$.*

We require some preliminary results and notation. Let X be a locally compact Hausdorff space. A continuous function $f : X \longrightarrow \mathbb{R}$ *vanishes at infinity* if, for each $\varepsilon > 0$, there is a compact set $C_\varepsilon \subseteq X$ such that $X \setminus C_\varepsilon \subseteq \{x \in X : |f(x)| < \varepsilon\}$. Let $C_0(X)$ denote the Banach space of all functions on X which vanish at infinity, equipped with the norm $||f||_\infty := \sup_{x \in X} |f(x)|$. The space $C_c(X)$ of continuous functions with compact support is norm dense in $C_0(X)$ [Co, III.1, Exercise 13]; when X is compact, $C_0(X) = C_c(X) = C(X)$, where $C(X)$ denotes the space of continuous real-valued functions on X equipped with the $|| \cdot ||_\infty$ norm. The Riesz Representation Theorem [Co, C.18] states that $C_0(X)^*$, the dual space of $C_0(X)$, is isomorphically isomorphic to $M(X)$, the space of finite regular Borel measures on X (equipped with the norm $||\mu|| := |\mu|(X)$); under this duality, corresponding to $\mu \in M(X)$ is the functional $\hat{\mu}$ on $C_0(X)$ defined by $\hat{\mu}(f) := \int f \, d\mu$. For the case when $X \subseteq \mathbb{R}^d$, $B_1(C_0(X)^*)$, the closed unit ball of $C_0(X)^*$, is weak-* compact and metrizable [Co, Theorems V.3.1 and V.5.1]. The following result is due to J. Stochel [St2], where it is stated in terms of complex variables and the complex moment problem.

Proposition 2.3. *(Real version of* [St2, Proposition 1]*) Let K be a nonempty closed subset of $\mathbb{R}^d$ and let ρ be a nonnegative continuous function on K. Assume that $\{\mu_\omega\}_{\omega\in\Omega}$ is a net of finite positive Borel measures on K and μ is a finite positive Borel measure on K such that*
(i) $\lim_{\omega\in\Omega}\int_K f\, d\mu_\omega = \int_K f\, d\mu$ $(f \in C_c(K))$, *and*
(ii) $\sup_{\omega\in\Omega}\int_K \rho\, d\mu_\omega < +\infty$.
Then $\int_K \rho\, d\mu \le \sup_{\omega\in\Omega}\int_K \rho\, d\mu_\omega$ *and* $\int_K f\rho\, d\mu = \lim_{\omega\in\Omega}\int_K f\rho\, d\mu_\omega$ $(f \in C_0(K))$. *Moreover, if the set* $\{x \in K : \rho(x) \le r\}$ *is compact for some* $r > 0$, *then* $\int_K f\, d\mu = \lim_{\omega\in\Omega}\int_K f\, d\mu_\omega$ *for every* $f : K \longrightarrow \mathbb{R}$ *such that* $f/(1+\rho) \in C_0(K)$.

The next result is our main tool for proving Theorem 2.2.

Theorem 2.4. *Let $\beta \equiv \beta^{(2n)}$ and let K be a nonempty closed subset of $\mathbb{R}^d$. If the Riesz functional L_β is K-positive, then $\beta^{(2n-1)}$ has a K-representing measure.*

Proof. Let $f_1 \equiv 1,\ f_2, \dots, f_N$ denote a listing of the monomials in $\mathcal{P}_{2n}$ in degree-lexicographic order; thus $\beta = \{\beta(f_i)\}_{i=1}^N$, where $\beta(f_j) = L_\beta(f_j)$, and we set $\zeta_\beta = (\beta(f_1), \dots, \beta(f_N))$. For $x \in K$, let $\zeta(x) := (f_1(x), \dots, f_N(x)) \in \mathbb{R}^N$, and let $\zeta(K) := \{\zeta(x) : x \in K\}$. Let $\operatorname{cone}\zeta(K)$ denote the convex cone in $\mathbb{R}^N$ generated by $\zeta(K)$, i.e.,

$$\text{cone } \zeta(K) := \{\sum_{i=1}^k a_i\zeta(x_i) :\ k \ge 1,\ a_i \ge 0,\ x_i \in K\}.$$

If $\zeta_\beta \in$ cone $\zeta(K)$, then β has a finitely atomic K-representing measure. Indeed, if $\zeta_\beta = \sum_{i=1}^k a_i\zeta(x_i)$ (for some $k \ge 1,\ a_i > 0,\ x_i \in K\ (1 \le i \le k)$), let $\mu := \sum_{i=1}^k a_i\delta_{x_i}$ (where δ_x is the unit-mass measure supported at x). Clearly,

$$\int f_j\, d\mu = \sum_{i=1}^k a_i f_j(x_i) = [\sum_{i=1}^k a_i\zeta(x_i)]_j = [\zeta_\beta]_j = \beta(f_j)(1 \le j \le N),$$

so μ is a K-representing measure for β.

Next, consider the closed convex cone $\mathcal{C} \equiv \overline{\text{cone } \zeta(K)}$. We will show below that $\zeta_\beta \in \mathcal{C}$, but we first show that if $\zeta_\beta \in \mathcal{C}$, then $\beta^{(2n-1)}$ has a K-representing measure. To see this, suppose $\zeta_\beta = \lim_{p\to+\infty}\sum_{i=1}^{k_p} a_{ip}\zeta(x_{ip})$ $(k_p \ge 1,\ a_{ip} \ge 0,\ x_{ip} \in K)$. Let $\mu_p = \sum_{i=1}^{k_p} a_{ip}\delta_{x_{ip}}$, so that $\int f_j\, d\mu_p = \sum_{i=1}^{k_p} a_{ip}f_j(x_{ip}) = [\sum_{i=1}^{k_p} a_{ip}\zeta(x_{ip})]_j$. Now,

$$\lim_{p\to+\infty}\int f_j\, d\mu_p = [\zeta_\beta]_j = \beta(f_j)\ (1 \le j \le N). \tag{2.1}$$

In particular, $\lim_{p\to+\infty}||\mu_p|| = \lim_{p\to+\infty}\int f_1\, d\mu_p = \beta(f_1)$, so $\{\mu_p\}$ is bounded in $C_0(K)^*$. Since the unit ball of $C_0(K)^*$ is compact and metrizable, it follows that some subsequence (which we also denote by $\{\mu_p\}$) is weak-* convergent, i.e., there exists $\Lambda \in C_0(K)^*$ such that $\lim_{p\to+\infty}\int f\, d\mu_p = \Lambda(f)$ $(f \in C_0(K))$. Since each $\mu_p \ge 0$, the Riesz Representation Theorem implies that $\Lambda(f) = \int f\, d\mu$ $(f \in C_0(K))$ for some positive finite regular Borel measure μ with supp $\mu \subseteq K$. Let $\sigma_p = \int ||x||^{2n}\, d\mu_p$ and set $\sigma = \sup_p \sigma_p$. Since $\rho(x) \equiv ||x||^{2n} \in \mathcal{P}_{2n}$, (2.1) implies that $\{\sigma_p\}$ is convergent, whence $\sigma < +\infty$. Proposition 2.3 now implies that $\lim_{p\to+\infty}\int f\, d\mu_p = \int f\, d\mu$ for every f such that $f/(1+||x||^{2n}) \in C_0(K)$. It follows exactly as in the proof of [CF9, Proposition 3.2] that for $f \in \mathcal{P}_{2n-1}$, $f/(1+||x||^{2n}) \in C_0(K)$. Thus, for each f_j satisfying deg $f_j \le 2n-1$, we have $\lim_{p\to+\infty}\int f_j\, d\mu_p = \int f_j\, d\mu$, while (2.1) implies $\lim_{p\to+\infty}\int f_j\, d\mu_p = \beta(f_j)$. Thus $\beta(f_j) = \int f_j\, d\mu$ whenever deg $f_j \le 2n-1$, whence μ is a K-representing measure for $\beta^{(2n-1)}$.

To complete the proof we will show that $\zeta_\beta \in \mathcal{C}$. For otherwise, since $\mathcal{C}$ is a closed cone in $\mathbb{R}^N$, the Minkowski separation theorem (cf. [T, p. 124], [B, (34.2)]) implies that there exists $a \equiv (a_j)_{1\le j\le N} \in \mathbb{R}^N$ such that $\langle a, v\rangle \ge 0$ $(v \in \mathcal{C})$ and $\langle a, \zeta_\beta\rangle < 0$. Let $p(x) = \sum_{j=1}^N a_j f_j(x) \in \mathcal{P}_{2n}$.

For $x \in K$, since $\zeta(x) \in \mathcal{C}$, then $p(x) = \sum_{j=1}^{N} a_j f_j(x) = \langle a, \zeta(x) \rangle \geq 0$, so $p|_K \geq 0$. Since L_β is K-positive, it follows that $L_\beta(p) \geq 0$. On the other hand, since $L_\beta(p) = \langle a, \zeta_\beta \rangle < 0$, we have a contradiction. Thus $\zeta_\beta \in \mathcal{C}$ and $\beta^{(2n-1)}$ has a K-representing measure. $\square$

Let μ denote a positive Borel measure on $\mathbb{R}^d$ having convergent moments up to at least degree m, and let $\beta^{(m)}[\mu]$ denote the sequence of moments of μ, defined by $\beta_i := \int x^i \, d\mu$ $(|i| \leq m)$. A *cubature rule* for μ of degree m is a finitely atomic representing measure for $\beta^{(m)}[\mu]$. A classical result of Tchakaloff [T, Théorème II; p. 129] shows that if supp μ is compact, then for each m, μ has a cubature rule ν of degree m with supp $\nu \subseteq$ supp μ and card supp $\nu \leq \dim \mathcal{P}_m$. Extensions of this result to the case where supp μ is unbounded appear in [Pu2] and [CF9], but the cubature rules in these papers only extend to degree $m-1$. Recently, Bayer and Teichmann [BT, Theorem 2] obtained the full extension of Tchakaloff's Theorem: μ always has a cubature rule ν of degree m such that supp $\nu \subseteq$ supp μ (and satisfying card supp $\nu \leq 1 + \dim \mathcal{P}_m$). This result has an important consequence for the truncated moment problem, for it implies that if $\beta^{(2n)}$ has a K-representing measure, then $\beta^{(2n)}$ has a finitely atomic K-representing measure. By combining this result with Theorem 2.4, we obtain our proof of Theorem 2.2, as follows.

Proof of Theorem 2.2. Let $\beta \equiv \beta^{(2n)}$ and suppose that L_β admits a K-positive extension $L : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$. Theorem 2.4 implies that there is a positive Borel measure μ supported in K such that $L(p) = \int p \, d\mu$ $(p \in \mathcal{P}_{2n+1})$. In particular, for $|i| \leq 2n$, $\beta_i = L_\beta(x^i) = L(x^i) = \int x^i \, d\mu$, so μ is a K-representing measure for β. Conversely, suppose $\beta^{(2n)}$ admits a K-representing measure. The Bayer-Teichmann theorem [BT] implies that $\beta^{(2n)}$ admits a finitely atomic K-representing measure ν. Since ν has convergent moments of all orders, we may define $\tilde{\beta} \equiv \tilde{\beta}^{(2n+2)}$ by $\tilde{\beta}_i := \int x^i \, d\nu$ $(|i| \leq 2n+2)$. Now supp $\nu \subseteq K$, so it follows immediately that $L_{\tilde{\beta}} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$ is K-positive, and since $\tilde{\beta}_i = \beta_i$ $(|i| \leq 2n)$, $L_{\tilde{\beta}}$ is an extension of L_β. $\square$

Since each nonnegative polynomial on $\mathbb{R}$ is a sum of two squares, for $d = 1$ positivity of the Riesz functional $L_{\beta^{(2m)}}$ is equivalent to positivity of the corresponding moment matrix $\mathcal{M}(m)$, which can be checked by concrete tests. For $d > 1$, positive polynomials are not necessarily sums of squares, so it is difficult to directly verify that $L_{\beta^{(2m)}}$ is positive. In the remainder of this section and in Section 3 we study several concrete criteria which guarantee the existence of K-positive extensions (and K-representing measures). We begin by recalling some concrete (i.e., computable) necessary conditions for representing measures in the truncated moment problem. As in the full moment problem, the basic necessary condition for a representing measure for $\beta^{(2n)}$ is positivity of the moment matrix. Indeed, if μ is a representing measure, then for $p \in \mathcal{P}_n$, $\langle \mathcal{M}(n)\hat{p}, \hat{p} \rangle = L_\beta(p^2) = \int p^2 \, d\mu \geq 0$. Note that $\mathcal{M}(n) \geq 0$ is equivalent to the condition that L_β is *square positive*, i.e., for $p \in \mathcal{P}_n$, $L_\beta(p^2) \geq 0$. Let $\mathcal{C}_{\mathcal{M}(n)}$ denote the column space of $\mathcal{M}(n)$. For $p(x) \equiv \sum_{|i| \leq n} a_i x^i \in \mathcal{P}_n$, we denote by $p(X)$ the element of $\mathcal{C}_{\mathcal{M}(n)}$ defined by $p(X) := \sum_{|i| \leq n} a_i X^i$; note that $p(X) = \mathcal{M}(n)\hat{p}$. Each dependence relation in the columns of $\mathcal{M}(n)$ may be expressed as $p(X) = 0$ for some $p \in \mathcal{P}_n$. Let $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n))$ denote the *variety* of $\mathcal{M}(n)$, defined by $\mathcal{V} := \bigcap_{p \in \mathcal{P}_n, p(X)=0} \mathcal{Z}(p)$, where $\mathcal{Z}_p := \{x \in \mathbb{R}^d : p(x) = 0\}$ (cf. [CF4, p. 6]).

Proposition 2.5. *(*[CF2, Proposition 3.1 and Corollary 3.7]*) Suppose* $\beta \equiv \beta^{(2n)}$ *has a representing measure* μ.
(i) $p \in \mathcal{P}_n$ *satisfies* $p(X) = 0$ *in* $\mathcal{C}_{\mathcal{M}(n)}$ *if and only if* supp $\mu \subseteq \mathcal{Z}(p)$*;*
(ii) supp $\mu \subseteq \mathcal{V}(\mathcal{M}(n))$ *;*
(iii) rank $\mathcal{M}(n) \leq$ card supp $\mu \leq$ card $\mathcal{V}(\mathcal{M}(n))$.

Remark 2.6. Note that if $\beta \equiv \beta^{(2n)}$ admits a representing measure μ, then L_β is $\mathcal{V}$-positive; for, if $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \geq 0$, then $p|_{\text{supp } \mu} \geq 0$ from Proposition 2.5(ii), so $L(p) = \int p \, d\mu \geq 0$. We may also introduce the variety in the full moment problem for $\beta^{(\infty)}$, $\mathcal{V}(\mathcal{M}(\infty)) = \bigcap_{p \in \mathcal{P}, p(X)=0} \mathcal{Z}(p)$. If $\beta^{(\infty)}$ admits a representing measure μ, then supp $\mu \subseteq \mathcal{V}(\mathcal{M}(\infty))$. Indeed, if $p \in \mathcal{P}$ and $p(X) = 0$

in $\mathcal{C}_{\mathcal{M}(\infty)}$, then with $n = \deg\ p$, μ is a representing measure for $\beta^{(2n)}$ and $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, so Proposition 2.5 implies supp $\mu \subseteq \mathcal{Z}(p)$.

Theorem 1.1 now admits the following reformulation.

Proposition 2.7. *Let $K \subseteq \mathbb{R}^d$ be a nonempty closed set. The following are equivalent for $\beta \equiv \beta^{(\infty)}$:*
(i) β admits a K-representing measure;
(ii) L_β is K-positive;
(iii) L_β is $K \bigcap \mathcal{V}(\mathcal{M}(\infty))$-positive.

Proof. Clearly, (iii) implies (ii), so in view of Theorem 1.1, it suffices to show that (i) implies (iii). Suppose μ is a K-representing measure for β and suppose $f \in \mathcal{P}$ satisfies $f|_{K \bigcap \mathcal{V}(\mathcal{M}(\infty))} \geq 0$. Since supp $\mu \subseteq K \bigcap \mathcal{V}(\mathcal{M}(\infty))$ by Remark 2.6, then $f|_{\text{supp}\ \mu} \geq 0$, so we have $L(f) = \int f\ d\mu \geq 0$. □

Returning to the truncated moment problem for $\beta \equiv \beta^{(2n)}$, the same argument as in the preceding proof shows that if β admits a K-representing measure, then L_β is $K \bigcap \mathcal{V}(\mathcal{M}(n))$-positive. Motivated by Example 2.1 and Proposition 2.7, in the sequel we study whether the analogue of Proposition 2.7(iii) $\Longrightarrow$ (i) holds for the truncated moment problem. In particular, for $K = \mathbb{R}^d$, we consider the following question.

Question 2.8. *Let $\beta = \beta^{(2n)}$ and $\mathcal{V} = \mathcal{V}(\mathcal{M}(n))$. If L_β is $\mathcal{V}$-positive, does β have a representing measure (necessarily supported in $\mathcal{V}$)?*

Suppose $\mathcal{M}(n) > 0$ ($\mathcal{M}(n)$ is positive definite), so that $\mathcal{V}(\mathcal{M}(n)) = \mathbb{R}^d$. In view of Theorem 1.2, in this case Question 2.8 is equivalent to the following question.

Question 2.9. *If $\mathcal{M}(n) > 0$ and L_β is positive, does L_β admit a positive extension $L : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$.*

In Question 2.9, the hypothesis that L_β is positive is essential. Indeed, [CF3] illustrates a case with $d = 2$ where $\mathcal{M}(3) > 0$, but there is no representing measure. In this example, the Riesz functional is not positive.

We devote the remainder of this section to illustrating cases where Question 2.8 has a positive answer, and to this end we require some preliminary results. Let $\beta \equiv \beta^{(2n)}$ and recall from [F1] and [CF2] that $\mathcal{M}(n)$ is *recursively generated* if whenever $p, q, pq \in \mathcal{P}_n$ and $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, then $(pq)(X) = 0$. Motivated by the proof of Example 2.1, and by a suggestion from Professor Michael Möller, we say that $L \equiv L_\beta$ is *strongly recursively generated* if the following property holds: if $p \in \mathcal{P}_n$, $L(p^2) = 0$, and $q \in \mathcal{P}$ satisfies $pq \in P_{2n}$, then $L(pq) = 0$. We next show that strong recursiveness is a necessary condition for representing measures. In Example 2.1, L is not strongly recursively generated, since $L((1-x)^2) = 0$ but $L((x-1)x^3) \neq 0$.

Proposition 2.10. *If $\beta \equiv \beta^{(2n)}$ has a representing measure, then L_β is strongly recursively generated.*

Proof. Suppose $p \in \mathcal{P}_n$, $L_\beta(p^2) = 0$, and $q \in \mathcal{P}$ satisfies $pq \in P_{2n}$. Since β has a representing measure, $\mathcal{M}(n) \geq 0$, so $0 = L_\beta(p^2) = \langle \mathcal{M}(n)\hat{p}, \hat{p} \rangle = ||\mathcal{M}(n)^{1/2}\hat{p}||^2$, whence $p(X) = \mathcal{M}(n)^{1/2}(\mathcal{M}(n)^{1/2}\hat{p}) = 0$. Proposition 2.5(i) implies that supp $\mu \subseteq \mathcal{Z}(p)$, i.e., $p|_{\text{supp}\ \mu} \equiv 0$. Now $pq|_{\text{supp}\ \mu} \equiv 0$, so $L_\beta(pq) = \int pq\ d\mu = 0$. □

Proposition 2.11. *For $\beta \equiv \beta^{(2n)}$, if $L \equiv L_\beta$ is strongly recursively generated, then $\mathcal{M}(n)$ is recursively generated.*

Proof. Suppose $p, q, pq \in \mathcal{P}_n$ and $p(X) = 0$. Since $\mathcal{M}(n)\hat{p} = p(X) = 0$, then $L(p^2) = \langle \mathcal{M}(n)\hat{p}, \hat{p} \rangle = 0$. Let $r \in \mathcal{P}_n$, so that $pqr \in \mathcal{P}_{2n}$. Since L is strongly recursively generated, we have $\langle \mathcal{M}(n)\hat{pq}, \hat{r} \rangle = L(pqr) = 0$. Thus, $\langle \mathcal{M}(n)\hat{pq}, \hat{r} \rangle = 0$ (all $r \in \mathcal{P}_n$), so $(pq)(X) = \mathcal{M}(n)\hat{pq} = 0$, as desired. □

Recall from [CFM] that β is *consistent* if whenever $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $L_\beta(p) = 0$.

Proposition 2.12. *Let $\beta \equiv \beta^{(2n)}$ and $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n))$. If $L \equiv L_\beta$ is $\mathcal{V}$-positive, then β is consistent, $\mathcal{M}(n) \geq 0$, L is strongly recursively generated, and $r :=$ rank $\mathcal{M}(n) \leq v :=$ card supp $\mathcal{V}$.*

Proof. Suppose L is $\mathcal{V}$-positive. If $p \in \mathcal{P}_{2n}$ satisfies $p|_{\mathcal{V}} \equiv 0$, then $\mathcal{V}$-positivity implies $L(p) \geq 0$ and $L(-p) \geq 0$, whence $L(p) = 0$; thus β is consistent. For $p \in \mathcal{P}_n$, $\mathcal{V}$-positivity implies that $L(p^2) \geq 0$, so $\langle \mathcal{M}(n)\hat{p}, \hat{p}\rangle \geq 0$, whence $\mathcal{M}(n) \geq 0$. To show that L is strongly recursively generated, let $p \in \mathcal{P}_n$ with $L(p^2) = 0$. Since $\mathcal{M}(n) \geq 0$, we have $||\mathcal{M}(n)^{1/2}\hat{p}||^2 = \langle \mathcal{M}(n)^{1/2}\hat{p}, \mathcal{M}(n)^{1/2}\hat{p}\rangle = \langle \mathcal{M}(n)\hat{p}, \hat{p}\rangle = L(p^2) = 0$, whence $p(X) = \mathcal{M}(n)\hat{p} = \mathcal{M}(n)^{1/2}(\mathcal{M}(n)^{1/2}\hat{p}) = 0$. Thus, $\mathcal{V} \subseteq \mathcal{Z}(p)$, i.e., $p|_{\mathcal{V}} \equiv 0$. Now for $q \in \mathcal{P}$ such that $pq \in \mathcal{P}_{2n}$, since $pq|_{\mathcal{V}} \equiv 0$, $\mathcal{V}$-positivity for L implies that $L(pq) = 0$; thus L is strongly recursively generated.

To complete the proof, we will show that $r \leq v$. The linear map $\psi : \mathcal{C}_{\mathcal{M}(n)} \longrightarrow \mathcal{P}_n|_{\mathcal{V}}$, given by $\psi(p(X)) := p|_{\mathcal{V}}$ ($p \in \mathcal{P}_n$), is well-defined (by the definition of $\mathcal{V}$). Since L is $\mathcal{V}$-positive, β is consistent, hence *weakly consistent* (cf. [CFM]), i.e., if $p \in \mathcal{P}_n$ and $p|_{\mathcal{V}} \equiv 0$, then $p(X) = 0 \in \mathcal{C}_{\mathcal{M}(n)}$. Thus, ψ is one-to-one, whence $r \equiv \dim\ \mathcal{C}_{\mathcal{M}(n)} \leq \dim\ \mathcal{P}_n|_{\mathcal{V}}$. Since we may assume that $v < +\infty$, then $\dim\ \mathcal{P}_n|_{\mathcal{V}} \leq \dim\ \mathbb{R}^v = v$ (where we view a polynomial restricted to $\mathcal{V}$ as the sequence of its values at the points of $\mathcal{V}$); thus, $r \leq v$. ☐

In the proof of Proposition 2.16 we will need the following result.

Corollary 2.13. *Let $\mathcal{V} := \mathcal{V}(\mathcal{M}(n))$. If L_β is $\mathcal{V}$-positive and $\mathcal{V} \subseteq \mathcal{Z}(p)$ for some $p \in \mathcal{P}_n$, then $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$.*

Proof. Proposition 2.12 implies that β is consistent, so [CFM, Formula (2.2) in Lemma 2.2] implies that $\mathcal{M}(n)$ is *weakly consistent*, i.e., $q \in \mathcal{P}_n, q|_{\mathcal{V}} \equiv 0 \Longrightarrow q(X) = 0$ (in $\mathcal{C}_{\mathcal{M}(n)}$). Since by hypothesis $\mathcal{V} \subseteq \mathcal{Z}(p)$, we must have $p|_{\mathcal{V}} \equiv 0$, so the result follows. ☐

The preceding results show that $\mathcal{V}(\mathcal{M}(n))$-positivity for $L_{\beta^{(2n)}}$ implies many of the known necessary conditions for representing measures. Using these results we next present some cases where Question 2.8 has an affirmative answer.

Proposition 2.14. *(Truncated Hamburger Moment problem) Let $d = 1$ (one real variable). The following are equivalent for $\beta \equiv \beta^{(2n)}$:*
(i) β has a representing measure;
(ii) $\mathcal{M}(n)(\beta)$ is positive semidefinite and recursively generated;
(iii) β has a rank *$\mathcal{M}(n)$-atomic representing measure;*
(iv) L_β is $\mathcal{V}(\mathcal{M}(n))$-positive.

Proof. The equivalence of (i), (ii), and (iii) is established in [CF1, Theorem 3.9], and (i) $\Longrightarrow$ (iv) was derived in Remark 2.6. Finally, (iv) $\Longrightarrow$ (ii) follows from Propositions 2.11 and 2.12. ☐

Remark 2.15. (i) Theorem 1.2 can be used to give an alternate proof of (ii) $\Longrightarrow$ (i) in Proposition 2.14. Indeed, if $\mathcal{M}(n)$ is positive and recursively generated, then [CF1, Theorem 2.6] shows that $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+1)$. Since $\mathcal{M}(n+1) \geq 0$, the sum-of-squares representation for nonnegative polynomials on $\mathbb{R}$ [PS] implies that the Riesz functional corresponding to $\mathcal{M}(n+1)$ is positive, whence Theorem 1.2 implies that $\beta^{(2n)}$ admits a representing measure.
(ii) Proposition 2.14 can be used to illustrate that the truncated moment problem is more general than the full moment problem. Hamburger's Theorem for $d = 1$ states that $\beta^{(\infty)}$ has a representing measure supported in $\mathbb{R}$ if and only if $\mathcal{M}(\infty) \geq 0$ (cf. [H, p. 166], [ShT, p. 5]). To prove this via the truncated moment problem, note that if $\mathcal{M}(\infty) \geq 0$, then for each n, $\mathcal{M}(n) \geq 0$ and $\mathcal{M}(n)$ is recursively generated (cf. [F3, Proposition 4.2]). Proposition 2.14 thus implies that $\beta^{(2n)}$ has a representing measure for each n, so the existence of a representing measure for $\beta^{(\infty)}$ follows from Stochel's theorem [St2].

We next resolve Question 2.8 for the case when $d = 2$ and $\mathcal{V}(\mathcal{M}(n)) \subseteq \mathcal{Z}(p)$, where deg $p \leq 2 \leq n$.

Proposition 2.16. *(The truncated moment problem on planar curves of degree at most 2) Let $d = 2$. Suppose* deg $p(x,y) \le 2 \le n$ *and* $\mathcal{V}(\mathcal{M}(n)) \subseteq \mathcal{Z}(p)$). *The following are equivalent for* $\beta \equiv \beta^{(2n)}$:
(i) β has a representing measure;
(ii) β has a finitely atomic representing measure;
(iii) $\mathcal{M}(n)$ is positive and recursively generated, and $r \le v$;
(iv) L_β is $\mathcal{V}(\mathcal{M}(n))$-positive.

Proof. Corollary 2.13 shows that (iv) implies $p(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. Under this condition, [CF11, Theorem 1.2] implies that (i), (ii) and (iii) are equivalent, and (i) always implies (iv). Since Propositions 2.11-2.12 show that (iv) implies (iii), the proof is complete. □

We next consider the case when $d \ge 1$ and $\mathcal{V}(\mathcal{M}(n))$ is compact.

Proposition 2.17. *Let $d \ge 1$ and suppose $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n))$ is compact. If L_β is $\mathcal{V}$-positive, then β admits a $\mathcal{V}$-representing measure μ with* card supp $\mu \le$ dim $\mathcal{P}_{2n}$.

Proof. The proof is implicit in the proof of Tchakaloff's Theorem, concerning the existence of cubature rules over compact sets in $\mathbb{R}^d$ [T, Théorème II, p. 129]. The present formulation in terms of $\mathcal{V}$-positivity is a special case of [CF9, Proposition 3.6] (adapted to real moment problems). □

We conclude this section by resolving Question 2.8 for the case when $\mathcal{M}(n)$ is *extremal*, i.e., $r :=$ rank $\mathcal{M}(n)$ and $v :=$ card $\mathcal{V}$ satisfy $r = v$ (where $\mathcal{V} := \mathcal{V}(\mathcal{M}(n))$).

Proposition 2.18. *Let $d \ge 1$ and suppose $r = v$. If L_β is $\mathcal{V}$-positive, then β admits a $\mathcal{V}$-representing measure.*

Proof. Proposition 2.12 implies that $\mathcal{M}(n) \ge 0$ and β is consistent, so since $r = v$, the existence of a unique representing measure μ, with supp $\mu = \mathcal{V}$, follows from [CFM, Theorem 4.2]. □

3. An equivalence for truncated moment problems on semialgebraic sets

In this section we establish analogues of Theorem 1.4 for the truncated moment problem. The following result (which restates Theorem 1.5) is an analogue of the "only if" part of Theorem 1.4, but without the requirement of compactness.

Theorem 3.1. *(i) Assume that $K_\mathcal{Q}$ satisfies ($\mathbf{S}_{n,k}$) for some n and k. Then every polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_\mathcal{Q}$ belongs to $\Sigma_{\mathcal{Q},n+k}$.*
(ii) Assume that $K_\mathcal{Q}$ satisfies ($\mathbf{R}_{n,k}$) for some n and k. Then each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_\mathcal{Q}$ belongs to $\Sigma_{\mathcal{Q},n+k+1}$.

Theorem 3.1 provides a sufficient condition for boundedly finite convergence in the polynomial optimization method of J. Lasserre [Las]. Let $f \in \mathcal{P}_{2n}$ and let $f^* :=$ inf $\{f(x) : x \in K_\mathcal{Q}\}$. For $k \ge 0$, define the $(n+k)$-th Lasserre relaxation by

$$f^*_{n+k} := \inf \{L_\beta(f) : \beta \equiv \beta^{(2n+2k)} \text{ satisfies } L_\beta(1) = 1 \text{ and } \mathcal{M}_{q_i}(n+k)(\beta) \ge 0 \ (1 \le i \le m)\}.$$

Then $f^*_{n+k} \le f^*_{n+k+1} \le \cdots \le f^*$, and if $K_\mathcal{Q}$ satisfies ($\mathbf{P}$), then $\lim_{k\to\infty} f^*_{n+k} = f^*$ [Las] (cf. [Lau]). Now suppose, as in Theorem 3.1, that $K_\mathcal{Q}$ satisfies ($\mathbf{S}_{n,k}$) for some fixed n and k. Let $f \in \mathcal{P}_{2n}$, so for $\varepsilon > 0$, $g := f - f^* + \varepsilon$ satisfies $g|_{K_\mathcal{Q}} > 0$. From Theorem 3.1, g admits a decomposition $g = \sum_i q_i \sum_j g_{ij}^2$, with $\deg q_i g_{ij}^2 \le 2n + 2k$. If $\beta \equiv \beta^{(2n+2k)}$ satisfies $L_\beta(1) = 1$ and $\mathcal{M}_{q_i}(n+k)(\beta) \ge 0$ $(1 \le i \le m)$, then $L_\beta(g) = \sum_i \sum_j \langle \mathcal{M}_{q_i}(n+k)(\beta)\hat{g}_{ij}, \hat{g}_{ij}\rangle \ge 0$, whence $L_\beta(f) \ge f^* - \varepsilon$. It now follows that $f^*_{n+k} = f^*$, so we have convergence after k steps.

The next result, adapted from [T, p. 126], shows that in Theorem 3.1, an element in any of the cones $\Sigma_{\mathcal{Q},n}$ can be expressed using at most dim $\mathcal{P}_{2n}$ generators (cf. [Ro, Theorem 17.1]).

Lemma 3.2. *Let $\mathcal{W}$ be a finite dimensional real vector space and let $m :=$ dim $\mathcal{W}$. Let $\mathcal{C}$ be a convex cone in $\mathcal{W}$ generated by vectors $\{f_\alpha\}_{\alpha\in I}$, i.e., $\mathcal{C} \equiv \{\sum_{i=1}^n a_i f_{\alpha_i} : n \ge 1,\ a_i \ge 0\}$. Then each element f of $\mathcal{C}$ has a representation as above with $n \le m$.*

To prove Theorem 3.1 we will need the following two results, which are part of Cassier's technique [Ca] and are also implicit in [PuV] and [Sm1]. Since it is difficult to find explicit statements and proofs in the literature, we include those here for the reader's convenience.

Lemma 3.3. *Let $m \geq 1$ and let C be a convex cone in $\mathbb{R}^m$ such that $C - C := \{f - g : f, g \in C\} = \mathbb{R}^m$. Then the interior of C is nonempty (relative to the Euclidean topology on $\mathbb{R}^m$).*

Proof. Let $\mathcal{B} \equiv \{f_1, \cdots, f_r\}$ be a maximal linearly independent subset of C. We first claim that $\mathcal{B}$ is a basis for $\mathbb{R}^m$. Consider the subspace $\mathcal{W}$ of $\mathbb{R}^m$ generated by $\mathcal{B}$, and let $h \in \mathcal{W}$. Then $h = \Sigma_{i=1}^r a_i f_i$, where $a_i \in \mathbb{R}$ $(i = 1, \cdots, r)$. Write $\{1, \cdots, r\} \equiv I \bigcup J := \{i : a_i \geq 0\} \bigcup \{i : a_i < 0\}$. Since C is a convex cone, it follows that $h = f - g$, where $f := \Sigma_{i \in I} a_i f_i \in C$ and $g := -\Sigma_{i \in J} a_i f_i \in C$. It follows that $h \in C - C$, and thus $\mathcal{W} = C - C = \mathbb{R}^m$. Therefore, $r = m$ and $\mathcal{B}$ is a basis for $\mathbb{R}^m$.

To complete the proof it suffices to show that C has nonempty interior. Since $\mathcal{B}$ is a basis for $\mathbb{R}^m$, and all norms on $\mathbb{R}^m$ are equivalent [Co, Theorem III.3.1], we may endow $\mathbb{R}^m$ with the norm $\|\sum_{i=1}^m a_i f_i\| := \max_{1 \leq i \leq m} |a_i|$. Now let $f := \sum_{i=1}^m a_i f_i$, where $a_i > 0$ $(1 \leq i \leq m)$. If $g \equiv \sum_{i=1}^m b_i f_i \in \mathbb{R}^m$ satisfies $\|f - g\| < \min_{1 \leq i \leq m} a_i$, then for each i, $|b_i - a_i| \leq \|f - g\| < a_i$, so $b_i > 0$ $(1 \leq i \leq m)$, whence $g \in C$. Thus, $f \in \text{int } C$. □

Lemma 3.4. *Let $m \geq 1$, let A be an open convex cone in $\mathbb{R}^m$ (endowed with the usual Euclidean topology), and let $q \notin A$. Then there exists a continuous linear functional $L : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $L|_A > 0$ and $L(q) \leq 0$.*

Proof. Since A is a convex set and $q \notin A$, Hahn-Banach separation (cf. [B, 30.6] and its proof) implies that there is a continuous linear functional L on $\mathbb{R}^m$ such that $\alpha := \inf\ \{L(a) : a \in A\}$ satisfies $L|_A \geq \alpha \geq L(q)$. Further, since A is open, [B, 30.5(3)] implies that $L|_A > \alpha$. To complete the proof, we show that $\alpha = 0$. For suppose first that $\alpha < 0$. By the definition of α, we may choose $a_0 \in A$ with $L(a_0) < 0$. Since A is a convex cone, for each positive $t \in \mathbb{R}$ we have $ta_0 \in A$, so $tL(a_0) = L(ta_0) > L(q)$, a contradiction (since $L(a_0) < 0$). Thus $\alpha \geq 0$. Further, for $a \in A$, the cone property gives $\frac{1}{n}a \in A$, and $L(\frac{1}{n}a) = \frac{1}{n}L(a) \to 0$, whence $\alpha \leq 0$. It follows that $\alpha = 0$, as desired. □

Remark 3.5. Observe that Lemma 3.4 remains true if the singleton $\{q\}$ is replaced by any convex subset C disjoint from A, as a straightforward modification of the above proof shows.

Proof of Theorem 3.1. (i) We adapt part of the proof of [Sm1, Corollary 3] (cf. Theorem 1.4), which, in turn, is based on the proof of [Ca, Théorème 4]. In the sequel, we view $\mathcal{P}_{2n}$ as the Euclidean space $\mathbb{R}^m$ $(m := \dim\ \mathcal{P}_{2n})$, endowed with the usual Euclidean topology. Assume that $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$. We first show that $\mathcal{P}_{2(n+k)} = \Sigma_{\mathcal{Q},n+k} - \Sigma_{\mathcal{Q},n+k}$. Indeed, we may express a monomial $p \in \mathcal{P}_{2(n+k)}$ as $p = x_{i_1} \cdots x_{i_\ell} x_{i_\ell+1} \cdots x_{i_{\ell+m}}$, where $\ell, m \leq n + k$ and each $i_j \in \{1, \ldots, d\}$. Since $\Sigma_{\mathcal{Q},n+k}$ contains all squares of polynomials in $\mathcal{P}_{n+k}$ (here is where we use that $q_0 \equiv 1$), we have

$$\begin{aligned} 2x_{i_1} \cdots x_{i_\ell} x_{i_\ell+1} \cdots x_{i_{\ell+m}} \quad &= \quad (x_{i_1} \cdots x_{i_\ell} + x_{i_\ell+1} \cdots x_{i_{\ell+m}})^2 \\ &\quad -((x_{i_1} \cdots x_{i_\ell})^2 + (x_{i_\ell+1} \cdots x_{i_{\ell+m}})^2) \in \Sigma_{\mathcal{Q},n+k} - \Sigma_{\mathcal{Q},n+k}. \end{aligned}$$

For a general element of $\mathcal{P}_{2(n+k)}$, apply this argument separately to the terms with positive coefficients and to the terms with negative coefficients. If we view $\mathcal{P}_{2(n+k)}$ as a finite dimensional Euclidean space, then since $\mathcal{P}_{2(n+k)} = \Sigma_{\mathcal{Q},n+k} - \Sigma_{\mathcal{Q},n+k}$, it follows from Lemma 3.3 that $\Sigma_{\mathcal{Q},n+k}$ has nonempty interior (relative to $\mathcal{P}_{2(n+k)}$).

Now let q be a polynomial in $\mathcal{P}_{2(n+k)}$ that is strictly positive on $K_{\mathcal{Q}}$, and suppose $q \notin \Sigma_{\mathcal{Q},n+k}$. Let $A := \text{int}\ \Sigma_{\mathcal{Q},n+k}$. Since A is a nonempty open cone and $q \notin A$, Lemma 3.4 implies that there exists a continuous linear functional $L : \mathcal{P}_{2(n+k)} \longrightarrow \mathbb{R}$ such that $L|_A > 0$ and $L(q) \leq 0$. Continuity implies that $L|_{\Sigma_{\mathcal{Q},n+k}} \geq 0$. Consider the moment problem for the sequence $\beta \equiv \beta^{(2n+2k)}$

defined by $\beta_i := L(x^i)$ ($|i| \le 2n+2k$), so that $L_\beta = L$. For an arbitrary element p of $\Sigma_{\mathcal{Q},n+k}$, consider the decomposition $p = \sum_{i=0}^m q_i \sum_j g_{ij}^2$ ($q_i g_{ij}^2 \in \mathcal{P}_{2(n+k)}$). We have $L(p) \ge 0$ and, as in (1.2), $L(p) = \sum_i \sum_j \langle \mathcal{M}_{q_i}(n)\hat{g_{ij}}, \hat{g_{ij}}\rangle$. Since the g_{ij} are arbitrary polynomials in $\mathcal{P}_{n-k_i}$ (so that $2\deg\, g_{ij} + \deg\, q_i \le 2(n+k)$), it follows that $\mathcal{M}(n+k) \ge 0$ and that $\mathcal{M}_{q_i}(n+k) \ge 0$ for $i = 1, \cdots, m$. Property ($\mathbf{S}_{n,k}$) now implies that β has a $K_{\mathcal{Q}}$-representing measure μ. Since q is strictly positive on $K_{\mathcal{Q}}$, this leads to the contradiction $L(q) = \int q\, d\mu > 0$. Thus, $q \in \Sigma_{\mathcal{Q},n+k}$.

(ii) The result follows immediately from (i) and the fact that ($\mathbf{R}_{n,k}$) implies ($\mathbf{S}_{n,k+1}$). □

We next present several examples of semialgebraic sets $K_{\mathcal{Q}}$ and corresponding $\tilde{\mathcal{Q}}$ which satisfy ($\mathbf{S}_{n,k}$) or ($\mathbf{R}_{n,k}$) for certain n and k. For these sets, Theorem 3.1 immediately yields corresponding representations for polynomials that are strictly positive on $K_{\mathcal{Q}}$. Combining each example with the following proposition yields a corresponding result for the full moment problem.

Proposition 3.6. *Suppose that for each $n \ge 1$ there exists $k_n \ge 0$ such that $K_{\mathcal{Q}}$ satisfies ($\mathbf{S}_{n,k_n}$). Then $K_{\mathcal{Q}}$ satisfies ($\mathbf{S}$) with $\tilde{\mathcal{Q}} = \mathcal{Q}$.*

Proof. Suppose $\mathcal{M}(\infty) \ge 0$ and $\mathcal{M}_{q_i}(\infty) \ge 0$ for $i = 1, \cdots, m$. Then $\mathcal{M}(n+k_n)$ is a positive and recursively generated extension of $\mathcal{M}(n)$, and $\mathcal{M}_{q_i}(n+k_n) \ge 0$. Property ($\mathbf{S}_{n,k_n}$) implies that $\mathcal{M}(n)$ has a $K_{\mathcal{Q}}$-representing measure. The result now follows from Stochel's Theorem [St2]. □

We begin the examples by illustrating Theorem 3.1 in the setting of a closed interval on the real line.

Proposition 3.7. *(Truncated Hausdorff Moment Problem* [KN, Theorem II.2.3]*) Let $d = 1$. For $a < b$, let $q(x) := (x-a)(b-x)$ and $\mathcal{Q} := \{1, q\}$, so that $K_{\mathcal{Q}} = [a, b]$. For $n \ge 1$, $K_{\mathcal{Q}}$ satisfies ($\mathbf{S}_{n,0}$). Given $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i=0}^{2n}$, $\mathcal{M}(n)$ is the Hankel matrix $(\beta_{i+j})_{i,j=0}^n$ and $\mathcal{M}_q(n) = \Delta := (a+b)H(n-1) - K(n-1) - ab\mathcal{M}(n-1)$, where $H(n-1) = (\beta_{i+j+1})_{i,j=0}^{n-1}$ and $K(n-1) = (\beta_{i+j})_{i,j=1}^n$. Thus, β has a representing measure supported in $[a, b]$ if and only if $\mathcal{M}(n) \ge 0$ and $(a+b)H(n-1) \ge K(n-1) + ab\mathcal{M}(n-1)$.*

This formulation of the Truncated Hausdorff Moment Problem is given in [KN, Theorem II.2.3], although the fact that $M_q(n)$ coincides with Δ depends on a calculation based on [CF10, Theorem 3.6]. The special case $a = 0$, $b = 1$ is given in [A, p. 74], and the case when $a = -1$, $b = 1$ is treated in [ShT, Theorem 3.1; p. 77]. In the setting of an interval $[a, b]$, the equivalent statement from Theorem 3.1, that $\Sigma_{\mathcal{Q},n}$ contains each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $[a, b]$, admits a stronger formulation. Indeed, the Markov-Lukács Theorem shows that a polynomial $p(x)$ of degree $2n$ that is nonnegative on $[a, b]$ admits a representation $p(x) = r(x)^2 + q(x)s(x)^2$, where $r \in \mathcal{P}_n$ and $s \in \mathcal{P}_{n-1}$ (cf. [ShT, p. 77]). A proof of this result for $[-1, 1]$ appears in [PS, Problem 47; pp. 78, 259], and the general case follows by a simple change-of-variables argument.

The interval $[a, b]$ admits a different presentation as $K_{\mathcal{Q}}$ if we take $\mathcal{Q} := \{1, q_1, q_2\}$, where $q_1(x) := x - a$ and $q_2(x) : b - x$. In this case, we can establish that for every $n \ge 1$, $K_{\mathcal{Q}}$ satisfies ($\mathbf{S}_{n,1}$). Indeed, if $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+1)$ satisfying $M_{q_1}(n+1) \ge 0$ and $M_{q_2}(n+1) \ge 0$, then the conditions of [CF1, Theorem 4.1(iv)] are satisfied, whence β admits a representing measure supported in $[a, b]$.

For the case of the unit circle $\mathbb{T}$, we have the following result.

Proposition 3.8. *Let $p(x, y) := 1 - x^2 - y^2$ and let $\mathcal{Q} := \{1, p, -p\}$, so that $K_{\mathcal{Q}} = \mathbb{T}$. For every $n \ge 2$, $\mathbb{T}$ satisfies ($\mathbf{S}_{n,0}$).*

Proof. Suppose $\mathcal{M}(n) \ge 0$, $\mathcal{M}_p(n) \ge 0$ and $\mathcal{M}_{-p}(n) \ge 0$. Then $\mathcal{M}_p(n) = 0$. We now appeal to the equivalence between the real and complex truncated moment problems [CF10, Propositions 2.17 - 2.19] (cf. also the proof of [CF10, Theorem 5.2]). In the corresponding complex moment matrix $M(n)[\gamma]$ we have $M(n)[\gamma] \ge 0$ and $M_{1-Z\bar{Z}}(n) = 0$. [CF5, Proposition 3.9(i)] now implies that $Z\bar{Z} = 1$ in the column space. It follows from [CF7, Propositions 2.2 and 2.3] that these are the conditions for a representing measure for γ supported in $\mathbb{T}$, and by equivalence, this measure corresponds to a representing measure for β supported on $\mathbb{T}$. ☐

We next consider lines in the plane.

Proposition 3.9. *Let $q \in \mathcal{P}_1$ be given by $q(x,y) \equiv ax+by+c$ and let $\mathcal{Q} := \{1, q, -q\}$. Then $K_{\mathcal{Q}} \equiv \{(x,y) : ax+by+c = 0\}$ satisfies $(\mathbf{R}_{n,0})$ for all $n \geq 1$.*

Proof. Assume that $\mathcal{M}(n) \geq 0$ is positive and recursively generated, $\mathcal{M}_q(n) \geq 0$ and $\mathcal{M}_{-q}(n) \geq 0$. It follows that $\mathcal{M}_q(n) = 0$, so we conclude that $q(X,Y) \equiv aX+bY+c1 = 0$, using the real version of [CF5, Proposition 3.9(i)-(ii)]. Since $\mathcal{M}(n)$ is positive, recursively generated, and $q(X,Y) = 0$, we can appeal to the real version of [CF3, Theorem 2.1] to conclude that $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$. Thus, there exists a representing measure for $\mathcal{M}(n)$, which is necessarily supported in $K_{\mathcal{Q}}$. ☐

Remark 3.10. If we set $a = 0$, $b = 1$, $c = 0$ in Proposition 3.9 we see that the x-axis satisfies $(\mathbf{R}_{n,0})$ for all $n \geq 1$. If we let $q(x,y) := y$ and $\mathcal{Q} := \{1, q, -q\}$, Theorem 3.1(ii) implies that every polynomial $p \in \mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}} \equiv \{(x,0) : x \in \mathbb{R}\}$ belongs to the cone $\Sigma_{\mathcal{Q},n+1}$, that is, p admits a representation of the form $p(x,y) = \sum_i [f_i(x,y)]^2 + y \sum_i \{[g_i(x,y)]^2 - [h_i(x,y)]^2\}$. Note that this generalizes the well known result for one-variable polynomials, strictly positive on $\mathbb{R}$. For, given such a polynomial r, we may define $R(x,y) := r(x)$. The above representation, when evaluated at $y = 0$, yields $r(x) = \sum_i [f_i(x,0)]^2$, as desired.

Proposition 3.11. *Let $p \in \mathbb{R}[x,y]$ be a quadratic polynomial such that $\mathcal{Z}(p)$ is an ellipse in the plane, that is, for $\mathcal{Q} := \{1, p, -p\}$, the set $K_{\mathcal{Q}}$ is an ellipse. Then $K_{\mathcal{Q}}$ satisfies $(\mathbf{R}_{n,0})$ for $n \geq 2$.*

Proof. Assume $n \geq 2$ and let $\beta \equiv \beta^{(2n)}$ be given, for which the associated moment matrix $\mathcal{M}(n)$ is positive, $\mathcal{M}_p(n) \geq 0$, and $\mathcal{M}_{-p}(n) \geq 0$. Now $\mathcal{M}_p(n) = 0$, and it follows as in the proof of Proposition 3.9 that $p(X,Y) = 0$. We consider two cases.

Case 1. Assume $\mathcal{M}(1)$ is invertible. Here we appeal to the strategy in [CF7, pp. 348-349] to convert the given truncated moment problem into an equivalent problem, for which the column relation becomes $X^2 + Y^2 = 1$, that is, $K_{\mathcal{Q}}$ is the unit circle $\mathbb{T}$. This is accomplished via an affine transformation $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, which transforms the original ellipse into the unit circle. Once this is done, we can appeal to Proposition 3.8 to first obtain a measure on $\mathbb{T}$, and a fortiori a $K_{\mathcal{Q}}$-representing measure for β.

Case 2. Assume $\mathcal{M}(1)$ is singular. Without loss of generality, we can assume that $\mathcal{M}(1)$ admits a column relation of the form $aX + bY + c1 = 0$, with $a, b, c \in \mathbb{R}$, $(a,b,c) \neq (0,0,0)$. By the Extension Principle [F1], we then have $aX + bY + c1 = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. We now apply Proposition 3.9 to obtain a representing measure, which will necessarily be supported in the intersection of the line and the ellipse $K_{\mathcal{Q}}$. ☐

Proposition 3.12. *Let $p \in \mathbb{R}[x,y]$, suppose $\mathcal{Z}(p)$ is a parabola or hyperbola, and set $\mathcal{Q} := \{1, p, -p\}$. Then $K := K_{\mathcal{Q}}$ satisfies $(\mathbf{R}_{n,1})$ for $n \geq 2$.*

Proof. Given $\beta \equiv \beta^{(2n)}$, suppose $\mathcal{M}(n)$ admits a positive, recursively generated extension $\mathcal{M}(n+1)$ with $\mathcal{M}_p(n+1) \geq 0$ and $\mathcal{M}_{-p}(n+1) \geq 0$. Then $\mathcal{M}_p(n+1) = 0$. Since $n \geq 2$ and deg $p \leq 2$, the real version of [CF5, Proposition 3.9(i)] implies that $p(X,Y) = 0$. By [CF8, Theorem 2.2] (if K is a parabola) or [CF11, Theorem 1.5] (if K is a hyperbola), we see that β admits a K-representing measure. ☐

Proposition 3.12 and Theorem 3.1(ii) together imply that polynomials that are positive on parabolas or hyperbolas admit degree-bounded representations, a result that seems to be new. Combining Propositions 3.8, 3.9, 3.11 and 3.12 with Proposition 3.6 we also recover a result of J. Stochel [St1] that if $d = 2$ and $\mathcal{M}(\infty) \geq 0$ with $p(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(\infty)}$ (where deg $p \leq 2$), then $\beta^{(\infty)}$ has a representing measure supported in $\mathcal{Z}(p)$.

For truncated *complex* moment problems, one can define the obvious analogues of properties $(\mathbf{S}_{n,k})$ and $(\mathbf{R}_{n,k})$; since it will be clear from the context whether we are dealing with real or complex moment problems, we use the same notation for both situations.

Proposition 3.13. *For $m \geq 1$, let $q(z,\bar{z}) := z^m - p(z,\bar{z})$, where $p \in \mathbb{C}[z,\bar{z}]$ and $\deg\ p \leq m-1$. Let $\mathcal{Q} := \{1, q, -q\}$. Then for every n such that $m \leq [\frac{n}{2}]+1$, $K_{\mathcal{Q}} \equiv \{z \in \mathbb{C} : q(z,\bar{z}) = 0\}$ satisfies $(\mathbf{R}_{n,0})$.*

Proof. Assume that $M(n)$ is positive and recursively generated, and that $M_q(n)$, $M_{-q}(n) \geq 0$. It follows that $M_q(n) = 0$. By [CF5, Proposition 3.9(i)], $p(Z,\bar{Z}) = 0$ in the column space $\mathcal{C}_{M(n)}$. By [CF3, Theorem 3.1], $M(n)$ admits a flat extension $M(n+1)$, and therefore there exists a representing measure for γ, necessarily supported in $\mathcal{Z}(q)(= K_{\mathcal{Q}})$. □

We now establish suitable converses of the statements in Theorem 3.1. We begin with Theorem 1.6, which we restate for ease of reference.

Theorem 3.14. *(i) If $k \geq 1$ and each polynomial in $\mathcal{P}_{2n+2}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k}$, then $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$.*
(ii) If $k = 0$, $K_{\mathcal{Q}}$ is compact, and each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n}$, then $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,0})$.

Proof of Theorem 3.14. (i) By hypothesis, each polynomial p in $\mathcal{P}_{2n+2}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k}$, and thus admits the structure $p = \sum_i q_i \sum_j g_{ij}^2 \quad (q_i g_{ij}^2 \in \mathcal{P}_{2(n+k)})$. We aim to establish that $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$. To this end, assume that $\mathcal{M}(n)(\beta)$ admits a positive extension $\mathcal{M}(n+k)$ such that $\mathcal{M}_{q_i}(n+k) \geq 0$ $(i = 1, \cdots, m)$. Corresponding to $\mathcal{M}(n+k)$, we set $L' := L_{\beta^{(2n+2k)}}$. Let $\tilde{\beta} := \beta^{(2n+2)}$, so that $\tilde{L} := L_{\tilde{\beta}} = L'|_{\mathcal{P}_{2n+2}}$. We claim that $\tilde{L}$ is $K_{\mathcal{Q}}$-positive. For, let $p \in \mathcal{P}_{2n+2}$, $p|_{K_{\mathcal{Q}}} > 0$. We have $p = \sum_i q_i \sum_j g_{ij}^2$ (as above), so $\tilde{L}(p) = L'(p) = L'(\sum_i q_i \sum_j g_{ij}^2) = \sum_i \sum_j \langle M_{q_j}(n+k)\hat{g_{ij}}, \hat{g_{ij}} \rangle \geq 0$. It follows by continuity that L is $K_{\mathcal{Q}}$-positive. We now apply Theorem 1.2 to conclude that β has a representing measure supported on $K_{\mathcal{Q}}$.

(ii) Here we assume that $K_{\mathcal{Q}}$ is compact and that every polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n}$. To prove that $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,0})$, assume that $\mathcal{M}(n)(\beta) \geq 0$. As in the proof of (i) above, it follows that the Riesz functional $L \equiv L_\beta$ is $K_{\mathcal{Q}}$-positive. Since $K_{\mathcal{Q}}$ is compact, Tchakaloff's Theorem [T, Théorème II, p. 129] (or as reformulated in [CF9, Proposition 3.6]) implies that β admits a finitely atomic $K_{\mathcal{Q}}$-representing measure. □

Note that each of the results in Propositions 3.8 - 3.9, 3.11 - 3.13 can be "turned around" to illustrate Theorem 3.14.

Remark 3.15. (i) Although we do not know of a non-compact $K_{\mathcal{Q}}$ that satisfies $(\mathbf{S}_{n,0})$, we can illustrate a non-compact $K_{\mathcal{Q}}$ such that $\Sigma_{\mathcal{Q},n}$ contains each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$. Consider first the real line, with $d = 1$, $\mathcal{Q} = \{1\}$, and $K_{\mathcal{Q}} = \mathbb{R}$. As noted earlier, if $p \in \mathcal{P}_{2n}$ and $p|_{\mathbb{R}} \geq 0$, then there exist $r, s \in \mathcal{P}_n$ such that $p = r^2 + s^2$ [PS, Problem VI.44], so that $p \in \Sigma_{\mathcal{Q},n}$. Note that $K_{\mathcal{Q}}$ does not satisfy $(\mathbf{S}_{n,0})$. Indeed, the single condition of $(\mathbf{S}_{n,0})$ is $\mathcal{M}(n) \geq 0$, but Proposition 2.14 shows that this condition is not always sufficient for a representing measure (cf. Example 2.1). Note that Proposition 2.14 shows that $\mathbb{R} \equiv K_{\mathcal{Q}}$ does satisfy $(\mathbf{R}_{n,0})$.
(ii) Next, consider the half-line, with $d = 1$, $\mathcal{Q} = \{1,\ x\}$, $K_{\mathcal{Q}} = [0, +\infty)$. It follows from [PS, Problem 45; pp. 78, 259] that if $p \in \mathcal{P}_{2n}$ satisfies $p|_{[0,+\infty)} \geq 0$, then there exist $r, s \in \mathcal{P}_n$ and $u, v \in \mathcal{P}_{n-1}$ such that $p(x) = r(x)^2 + s(x)^2 + x(u(x)^2 + v(x)^2)$, whence $p \in \Sigma_{\mathcal{Q},n}$. Now the matrix conditions of $(\mathbf{S}_{n,0})$ entail $M(n) \geq 0$ and $H(n-1)(= M_x(n)) \geq 0$, but [CF1, Theorem 5.3] shows that the existence of a representing measure requires, in addition to these properties, the condition $(\beta_{n+1}, \ldots, \beta_{2n})^T \in \text{Ran}\ H(n-1)$. In Example 2.1 we have $M(2) \geq 0$ and $H(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0$, but $\mathbf{w} := (\beta_3, \beta_4)^T \equiv (1,2)^T$ does not satisfy $\mathbf{w} \in \text{Ran}\ H(1)$.

When the cone $\Sigma_{\mathcal{Q},n+k}$ is closed we can sharpen Theorem 1.5.

Theorem 3.16. *(i) Assume that $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$ for some n and k, and that the cone $\Sigma_{\mathcal{Q},n+k}$ is closed in $\mathcal{P}_{2(n+k)}$. Then every polynomial in $\mathcal{P}_{2n}$ that is nonnegative on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k}$.*
(ii) Assume that $K_{\mathcal{Q}}$ satisfies $(\mathbf{R}_{n,k})$ for some n and k, and that the cone $\Sigma_{\mathcal{Q},n+k}$ is closed in $\mathcal{P}_{2(n+k)}$. Then each polynomial in $\mathcal{P}_{2n}$ that is nonnegative on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k+1}$.

As in Theorem 3.1, the total number of terms $q_i f_{ij}^2$ in the representations of elements of $\Sigma_{\mathcal{Q},n+k}$ in Theorem 3.14 can always be taken to be at most dim $\mathcal{P}_{2n+2k}$ (cf. Lemma 3.2).

Proof. We focus on the proof of (i) above; the proof of (ii) is entirely similar. Let $m :=$ dim $\mathcal{P}_{2n}$. In the sequel we view $\mathcal{P}_{2n}$ as the Euclidean space $\mathbb{R}^m$ equipped with the usual inner product topology; for this, we identify a polynomial $p(x) \equiv \sum_{|i|\le 2n} a_i x^i$ with its vector of coefficients $\hat{p} \equiv (a_i)$ (with respect to the basis for $\mathcal{P}_{2n}$ consisting of the monomials in degree-lexicographic order). We assume that $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$ and that the convex cone $\Sigma_{\mathcal{Q},n+k}$ is closed in $\mathcal{P}_{2(n+k)}$. Suppose $p \in \mathcal{P}_{2n}$ satisfies $p|_{K_{\mathcal{Q}}} \ge 0$, but $p \notin \Sigma_{\mathcal{Q},n+k}$. The Minkowski separation theorem (cf. [T, p. 124]) implies that there is a polynomial $q \in \mathcal{P}_{2(n+k)}$ such that $\langle \hat{q}, \hat{s}\rangle \ge 0$ for every $s \in \Sigma_{\mathcal{Q},n+k}$ and $\langle \hat{q}, \hat{p}\rangle < 0$. Consider the linear functional $L : \mathcal{P}_{2(n+k)} \longrightarrow \mathbb{R}$ defined by $L(f) := \langle \hat{q}, \hat{f}\rangle$. Define $\tilde{\beta} \equiv \beta^{(2n+2k)}$ by $\beta_i := L(x^i)$ $(|i| \le 2n+2k)$. Let $\beta := \beta^{(2n)}$, so that $L_\beta = L|_{\mathcal{P}_{2n}}$. Since $L|_{\Sigma_{\mathcal{Q},n+k}} \ge 0$, it follows exactly as in the proof of Theorem 3.1 that $\mathcal{M}(n)(\beta)$ admits the positive extension $\mathcal{M}(n+k)(\tilde{\beta}) \ge 0$, and that $\mathcal{M}_{q_i}(n+k) \ge 0$ for $i = 1, \cdots, m$. The assumption that $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$ now implies that β admits a $K_{\mathcal{Q}}$-representing measure μ, whence $\langle \hat{q}, \hat{p}\rangle = L(p) = L_\beta(p) = \int p \, d\mu \ge 0$, a contradiction. Thus, $p \in \Sigma_{\mathcal{Q},n+k}$. □

In the Introduction we noted that the closed unit disk fails to satisfy $(\mathbf{S}_{3,k})$ for every $k \ge 0$. By contrast, the disk does satisfy $(\mathbf{S}_{1,0})$, and we have the following result.

Proposition 3.17. *Each polynomial $p \in \mathcal{P}_2$ satisfying $p|_{\overline{\mathbb{D}}} \ge 0$ admits a representation $p = \Sigma_{i=1}^5 f_i^2 + \alpha(1 - x^2 - y^2)$, where $\deg f_i \le 1$ $(1 \le i \le 6)$ and $\alpha \ge 0$.*

Proof. For $d = 2$ and $\mathcal{Q} = \{1, 1 - x^2 - y^2\}$, we have $K_{\mathcal{Q}} = \overline{\mathbb{D}}$. Let $q(x, y) = 1 - x^2 - y^2$. A calculation using [CF10, Theorem 3.6] shows that $\mathcal{M}_q(1) = (\beta_{00} - \beta_{20} - \beta_{02})$, and [CF6, Theorem 1.8(iv)] implies that $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{1,0})$. (The result in [CF6] is given in terms of the truncated complex moment problem for measures on $\mathbb{C}$, but the complete equivalence of this problem to the real truncated moment problem for measures on $\mathbb{R}^2$ is established in [CF10].) Since dim $\mathcal{P}_2 = 6$, to complete the proof using Theorem 3.16(i), it suffices to show that $\Sigma_{\mathcal{Q},1}$ is closed in $\mathcal{P}_2$ (relative to the Euclidean topology on $\mathcal{P}_2$ resulting from its identification with $\mathbb{R}^6$). Lemma 3.2 implies that each element of $\Sigma_{\mathcal{Q},1}$ is of the form $f(x, y) \equiv \sum_{i=1}^6 [(a_i + b_i x + c_i y)^2 + d_i^2(1 - x^2 - y^2)]$. Consider a sequence of such elements, $f_n \equiv \sum_{i=1}^6 [(a_i^{(n)} + b_i^{(n)} x + c_i^{(n)} y)^2 + (d_i^{(n)})^2(1 - x^2 - y^2)]$ $(n \ge 1)$, that is convergent to $f \in \mathcal{P}_2$. Note that

$$\begin{aligned} f_n &= \sum_{i=1}^6 [(a_i^{(n)})^2 + (d_i^{(n)})^2] + \sum_{i=1}^6 2a_i^{(n)} b_i^{(n)} x + \sum_{i=1}^6 2a_i^{(n)} c_i^{(n)} y \\ &\quad + \sum_{i=1}^6 [(b_i^{(n)})^2 - (d_i^{(n)})^2] x^2 + \sum_{i=1}^6 2b_i^{(n)} c_i^{(n)} xy + \sum_{i=1}^6 [(c_i^{(n)})^2 - (d_i^{(n)})^2] y^2. \end{aligned}$$

Convergence in $\mathcal{P}_2$ implies that each coefficient sequence is convergent, so the constant coefficient sequence $\{\sum_{i=1}^6 [(a_i^{(n)})^2 + (d_i^{(n)})^2]\}$ is convergent. Passing, if necessary, to subsequences, me may thus assume that $\{a_i^{(n)}\}$ and $\{d_i^{(n)}\}$ are convergent, say $a_i^{(n)} \to a_i$ and $d_i^{(n)} \to d_i$ $(i = 1, \cdots, 6)$. Next, the coefficient sequence of the x^2 terms, $\{\sum_{i=1}^6 [(b_i^{(n)})^2 - (d_i^{(n)})^2]\}$, is convergent, and since $\{d_i^{(n)}\}$ is convergent, it follows that (after passing to a further subsequence) $\{b_i^{(n)}\}$ is convergent, say $b_i^{(n)} \to b_i$ $(i = 1, \cdots, 6)$. Similarly, the coefficient sequence for y^2 is convergent, so we may assume that $c_i^{(n)} \to c_i$ $(i = 1, \cdots, 6)$. It now follows that $f_n \to \sum_{i=1}^{6}[(a_i+b_ix+c_iy)^2+d_i^2(1-x^2-y^2)] \in \Sigma_{\mathcal{Q},1}$. Thus, $\Sigma_{\mathcal{Q},1}$ is closed. □

It appears to be open whether the disk satisfies $(\mathbf{S}_{2,k})$ for some k.

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