THE EXTREMAL TRUNCATED MOMENT PROBLEM

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Abstract. For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \leq 2n}$ to have a *representing measure* μ, it is necessary for the associated moment matrix $\mathcal{M}(n)(\beta)$ to be positive semidefinite and for the algebraic variety associated to β, $\mathcal{V} \equiv \mathcal{V}_\beta$, to satisfy *rank* $\mathcal{M}(n) \leq$ *card* $\mathcal{V}$ as well as the following *consistency* condition: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on $\mathcal{V}$, then $\sum_{|i| \leq 2n} a_i \beta_i = 0$. We prove that for the *extremal* case (*rank* $\mathcal{M}(n) =$ *card* $\mathcal{V}$), positivity of $\mathcal{M}(n)$ and consistency are sufficient for the existence of a (unique, *rank* $\mathcal{M}(n)$-atomic) representing measure. We also show that in the preceding result, consistency cannot always be replaced by recursiveness of $\mathcal{M}(n)$.

1. Introduction

Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in \mathbb{Z}_+^d, |i| \leq 2n}$ denote a real d-dimensional multisequence of degree $2n$. The *truncated moment problem* for β concerns the existence of a positive Borel measure μ, supported in $\mathbb{R}^d$, such that

$$\beta_i = \int_{\mathbb{R}^d} x^i \, d\mu, \ |i| \leq 2n; \tag{1.1}$$

(here, for $x \equiv (x_1, ..., x_d) \in \mathbb{R}^d$ and $i \equiv (i_1, ..., i_d) \in \mathbb{Z}_+^d$, we let $x^i := x_1^{i_1} \cdots x_d^{i_d}$). A measure μ as in (1.1) is a *representing measure* for β.

Let $\mathcal{P} \equiv \mathbb{R}^d[x] = \mathbb{R}[x_1, ..., x_d]$ denote the space of real valued d-variable polynomials, and for $k \geq 1$, let $\mathcal{P}_k \equiv \mathbb{R}_k^d[x]$ denote the subspace of $\mathcal{P}$ consisting of polynomials p with $\deg p \leq k$. Corresponding to β we have the *Riesz functional* $\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R}$, which associates to an element p of $\mathcal{P}_{2n}$, $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$, the value $\Lambda(p) := \sum_{|i| \leq 2n} a_i \beta_i$; of course, in the presence of a representing measure μ, we have $\Lambda(p) = \int p \, d\mu$. In the sequel, $\hat{p}$ denotes the coefficient vector (a_i) of p.

1991 *Mathematics Subject Classification.* Primary 47A57, 44A60, 42A70, 30A05; Secondary 15A57, 15-04, 47N40, 47A20.

Key words and phrases. extremal truncated moment problems, moment matrix extension, Riesz functional, real ideals, affine Hilbert function, Hilbert polynomial of a real ideal.

The first-named author's research was partially supported by NSF Research Grants DMS-0099357 and DMS-0400741. The second-named author's research was partially supported by NSF Research Grant DMS-0201430 and DMS-0457138.

Following [CuFi2], we associate to β the *moment matrix* $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$, with rows and columns X^i indexed by the monomials of $\mathcal{P}_n$ in degree-lexicographic order; for example, with $d = n = 2$, the columns of $\mathcal{M}(2)$ are denoted as $1, X_1, X_2, X_1^2, X_2X_1$, X_2^2. The entry in row X^i, column X^j of $\mathcal{M}(n)$ is β_{i+j}, so $\mathcal{M}(n)$ is a real symmetric matrix characterized by

$$\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle = \Lambda(pq) \ (p, q \in \mathcal{P}_n). \tag{1.2}$$

If μ is a representing measure for β, then $\langle \mathcal{M}(n)\hat{p}, \hat{p}\rangle = \Lambda(p^2) = \int p^2 d\mu \geq 0$; since $\mathcal{M}(n)$ is real symmetric, it follows that $\mathcal{M}(n)$ is positive semidefinite (in symbols, $\mathcal{M}(n) \geq 0$).

The *algebraic variety* of β (or of $\mathcal{M}(n)(\beta)$) is defined by

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p\in\mathcal{P}_n, \hat{p}\in\ker\mathcal{M}(n)} \mathcal{Z}(p),$$

where $\mathcal{Z}(p) := \{x \in \mathbb{R}^d : p(x) = 0\}$. (We sometimes denote $\mathcal{V}_\beta$ as $\mathcal{V}(\mathcal{M}(n)(\beta))$.) If β admits a representing measure μ, then $p \in \mathcal{P}_n$ satisfies $\hat{p} \in \ker \mathcal{M}(n)$ if and only if *supp* $\mu \subseteq \mathcal{Z}(p)$ [CuFi2, Proposition 3.1]. Thus *supp* $\mu \subseteq \mathcal{V}$, and it follows from [CuFi4, (1.7)] that $r :=$ *rank* $\mathcal{M}(n)$ and $v :=$ *card* $\mathcal{V}$ satisfy $r \leq$ *card supp* $\mu \leq v$. Further, in this case, if $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then clearly $\Lambda(p) = \int p \, d\mu = 0$. To summarize the preceding discussion, we have the following basic necessary conditions for the existence of a representing measure for $\beta^{(2n)}$:

$$\text{(Positivity) } \mathcal{M}(n) \geq 0 \tag{1.3}$$

$$\text{(Consistency) } p \in \mathcal{P}_{2n}, \ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0 \tag{1.4}$$

$$\text{(Variety Condition) } r \leq v, \text{ i.e., } \mathit{rank}\ \mathcal{M}(n) \leq \mathit{card}\ \mathcal{V}. \tag{1.5}$$

As we show below (Section 3), consistency implies the following condition:

$$\text{(Recursiveness) } p, q, pq \in \mathcal{P}_n, \hat{p} \in \ker \mathcal{M}(n) \Longrightarrow \widehat{pq} \in \ker \mathcal{M}(n). \tag{1.6}$$

Consistency is a new condition; previously, in [CuFi2, p. 5], we considered only recursiveness (when (1.6) holds, we say that β (or $\mathcal{M}(n)(\beta)$) is *recursively generated*). In [CuFi2, Theorem 3.19] we showed that for $d = 1$ (the truncated *Hamburger moment problem* for $\mathbb{R}$), positivity and recursiveness are sufficient to imply the existence of representing measures. For $d = 2$ (the plane), there exists $\mathcal{M}(3) > 0$ (positive definite) for which β has no representing measure [CuFi3, Section 4]. Since an invertible moment matrix satisfies (1.4) and (1.5) vacuously, it follows that in general (1.3)-(1.5) are not sufficient conditions for representing measures. By contrast, the results of [CuFi6], [CuFi8], and [CuFi10] together show that when $d = 2$ and ker $\mathcal{M}(n)$ contains an element $\hat{p}$ with $\deg p \leq 2$, then β has a representing measure if and only if $\mathcal{M}(n)$ is positive, recursively generated and satisfies the variety condition. This result motivated the following question of [Fia3, Conjecture 1.2].

Question 1.1. *Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, recursively generated, and $r \le v$, does β admit a representing measure?*

In the present note we focus on the following refinement of Question 1.1.

Question 1.2. *Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, β is consistent, and $r \le v$, does β admit a representing measure?*

Our main result provides an affirmative answer to Question 1.2 in the *extremal* case, when $r = v$.

Theorem 1.3. *For $\beta \equiv \beta^{(2n)}$ extremal, i.e., $r = v$, the following are equivalent:*
(i) β has a representing measure;
(ii) β has a unique representing measure, which is rank $\mathcal{M}(n)$-atomic;
(iii) $\mathcal{M}(n) \ge 0$ and β is consistent.

In many cases, the conditions of Theorem 1.3 provide a concrete solution to the extremal case of the truncated moment problem. Indeed, only elementary linear algebra is required to verify that $\mathcal{M}(n)$ is positive semidefinite, to compute its rank, and to identify the dependence relations which enter into the definition of the variety $\mathcal{V}$. Further, as we show in Section 3, if the points of the variety can be computed exactly (which may be feasible in specific examples by using computer algebra), then only elementary linear algebra is required to verify that β is consistent. The proof of Theorem 1.3 is included in Theorem 4.2 (Section 4), which also provides a simple procedure for computing the unique representing measure for β.

If the points of $\mathcal{V}(\mathcal{M}(n))$ are not known exactly, then it may be difficult to verify consistency directly; for this reason, it is of interest to identify cases in which recursiveness, which is easy to check, actually implies consistency. In Sections 3, 5 and 6 we study the extent to which "consistency" in Theorem 1.3 can be replaced by "recursiveness," or by a simplified consistency condition. Consider a planar moment matrix $\mathcal{M}(3) \ge 0$ with $\mathcal{M}(2) > 0$ and a column dependence relation $Y = X^3$. In Section 5 we show that if $\mathcal{M}(3)$ (as above) is extremal with $r = v = 7$, then recursiveness is indeed sufficient for a representing measure. By contrast, in Section 6 we show that for an extremal $\mathcal{M}(3)$ as above, but with $r = v = 8$, it may happen that there is no representing measure (Theorem 6.2). This result provides a perhaps surprising negative answer to Question 1.1, and also shows that in general consistency is a strictly stronger property than recursiveness. In Theorem 6.3 we show that for the preceding $r = v = 8$ extremal problem, consistency reduces to checking that $\Lambda(h) = 0$ for a particular polynomial $h \in \mathbb{R}[x, y]$ of degree 4.

We next observe that the extremal case is inherent in the truncated moment problem. A recent result of C. Bayer and J. Teichmann [BaTe] (extending a classical theorem of V. Tchakaloff [Tch] and its successive generalizations in [Mys], [Put] and [CuFi9]) implies that if $\beta^{(2n)}$ has a representing measure, then it has a finitely atomic representing measure. In [CuFi4] it was shown that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $\mathcal{M}(n)$ admits an extension to a positive moment matrix $\mathcal{M}(n+k)$ (for some $k \geq 0$), which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $\mathcal{M}(n+k+1)$. Further, [CuFi11, Theorem 1.2] shows that any flat extension $\mathcal{M}(n+k+1)$ is an extremal moment matrix for which there is a computable *rank* $\mathcal{M}(n+k)$-atomic representing measure μ. Clearly, μ is also a finitely atomic representing measure for $\beta^{(2n)}$, and every finitely atomic representing measure for $\beta^{(2n)}$ arises in this way. In this sense, the existence of a representing measure for $\beta^{(2n)}$ is intimately related to the solution of an extremal truncated moment problem.

We conclude this section with two examples related to the extremal truncated moment problem. In the first example we illustrate extremal truncated moment problems of arbitrarily large degree. To ease the exposition of this example, we will present it in terms of the *truncated complex moment problem*. Let $\gamma \equiv \gamma^{(2n)} = \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+^d,|i|+|j|\leq 2n}$ denote a d-dimensional complex multisequence of degree $2n$. The truncated complex moment problem for γ concerns the existence of a positive Borel measure ν on $\mathbb{C}^d$ such that

$$\gamma_{ij} = \int_{\mathbf{C}^d} \bar{z}^i z^j d\nu \ (i,j \in Z_+^d, |i|+|j| \leq 2n), \tag{1.7}$$

(where $z \equiv (z_1, ..., z_d)$, $\bar{z} \equiv (\bar{z}_1, ..., \bar{z}_d) \in \mathbb{C}^d$, $i \equiv (i_1, ..., i_d)$, $j \equiv (j_1, ..., j_d) \in \mathbb{Z}_+^d$, and $\bar{z}^i z^j := \bar{z}_1^{i_1} \cdots \bar{z}_d^{i_d} z_1^{j_1} \cdots z_d^{j_d}$). The Riesz functional for γ is defined by $\Lambda_\gamma(\bar{z}^i z^j) := \gamma_{i,j}$. The mapping $\mathbb{C}^d \times \mathbb{C}^d \mapsto \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ defined by $(z, \bar{z}) \mapsto (x, y)$ (where $x := (z + \bar{z})/2$ and $y := (z - \bar{z})/2i$) induces a correspondence between truncated moment problems on $\mathbb{C}^d$ and truncated moment problems on $\mathbb{R}^{2d}$. Under this correspondence, γ is associated to a $2d$-dimensional real multisequence β (also of degree $2n$) via the formula $\Lambda_\beta((x,y)^{(k,j)}) := \Lambda_\gamma(((z+\bar{z})/2)^k((z-\bar{z})/2i)^j)$ $(k, j \in \mathbb{Z}_+^d, |k| + |j| \leq 2n)$; we write $\beta \equiv \mathcal{S}(\gamma)$. Let $\mathbb{C}^d[z,\bar{z}] = \mathbb{C}[z_1, ..., z_d, \bar{z}_1, ..., \bar{z}_d]$ and let $\mathbb{C}_k^d[z,\bar{z}]$ denote the subspace of polynomials $p(z,\bar{z})$ with *deg* $p \leq k$. The *complex moment matrix* $M(n) \equiv M(n)(\gamma)$ has rows and columns indexed by monomials in z and $\bar{z}$ up to degree n in degree-lexicographic order, such that $\langle M(n)\hat{p}, \hat{q}\rangle = \Lambda(p\bar{q})$ $(p, q \in \mathbb{C}_n^d[z,\bar{z}])$. The *variety* of γ is defined as $V(\gamma) := \bigcap_{p\in\mathbb{C}_n^d[z,\bar{z}],\hat{p}\in ker\ M(n)} Z(p)$, where $Z(p) := \{z \in \mathbb{C}^d : p(z,\bar{z}) = 0\}$. The close connection between $M(n)(\gamma)$ and $\mathcal{M}(n)(\mathcal{S}(\gamma))$ is described in detail in [CuFi11, Section 2]; in particular, both moment matrices share the same positivity, rank, recursiveness, and consistency, and, up to the identification $\mathbb{C}^d \approx \mathbb{R}^{2d}$, the same variety and representing measures. For this reason, results such as Theorem 1.3 admit direct analogues for the truncated complex moment problem. (For related instances of this, the reader is referred to [CuFi11, Theorems 2.19 and 2.21]).

Example 1.4. For $n > 0$, we exhibit an extremal $\gamma \equiv \gamma^{(2n)}$ in one complex variable with *rank* $M(n)(\gamma) = $ *card* $V(\gamma) = 2n$. The rows and columns of $M(n)$ are indexed by $1, Z, \bar{Z}, ..., Z^n, \bar{Z}Z^{n-1}, ..., \bar{Z}^{n-1}Z, \bar{Z}^n$. We set $\gamma_{ii} = 1$ $(0 \le i \le n)$, and for $0 < a < 1$, we set $\gamma_{0,2n-1} = \gamma_{2n-1,0} := a$ and $\gamma_{0,2n} = \gamma_{2n,0} := 1 - a^2$; the remaining γ_{ij} equal 0. For example, with $n = 3$ we have

$$M(3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 1 & 0 & 0 & 1-a^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 1-a^2 & 0 & 0 & 1 \end{pmatrix}.$$

Observe that in the column space of $M(n)$ we have $\bar{Z}Z = 1$, $\bar{Z}^n - Z^n = a(Z^{n-1} - \bar{Z}^{n-1})$, and a basis for the column space is given by $\mathcal{B} \equiv \{1, Z, \bar{Z}, Z^2, \bar{Z}^2, ..., Z^i, \bar{Z}^i, ..., Z^{n-1}, \bar{Z}^{n-1}, Z^n\}$. It follows readily that $M(n)$ is recursively generated. Note that $M_{\mathcal{B}}$, the compression of $M(n)$ to the rows and columns indexed by $\mathcal{B}$, is of the form $J \oplus \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$, where J is an identity matrix of size $2n - 2$. Thus $M_{\mathcal{B}}$ is a positive definite matrix, with *rank* $M_{\mathcal{B}} = 2n$. Since *rank* $M(n) =$ *rank* $M_{\mathcal{B}}$, it follows from [CuFi1, Proposition 2.3] that $M(n)$ is positive semidefinite. (In the language of [CuFi2], $M(n)$ is a *flat extension* of $M_{\mathcal{B}}$.) Now $\bar{Z}Z = 1$, so we may apply the analysis of the *truncated trigonometric moment problem* from [CuFi6]. Since $M(n)$ is positive and recursively generated, $\bar{Z}Z = 1$, and *rank* $M(n) = 2n$, [CuFi6, Theorem 3.5] implies that γ has a unique representing measure, which is $2n$-atomic; in particular, *card* $V(\gamma) \ge$ *rank* $M(n)(\gamma) = 2n$. Now $V(\gamma)$ consists of common solutions of the equations $\bar{z}z = 1$ and $\bar{z}^n - z^n = a(z^{n-1} - \bar{z}^{n-1})$, so $V(\gamma) \subseteq Z(p)$, where $p(z, \bar{z}) = z^{2n} + az^{2n-1} - az - 1$. Thus, *card* $V(\gamma) \le$ *card* $Z(p) \le 2n$, and it follows that *card* $V(\gamma) = 2n =$ *rank* $M(n)(\gamma)$, whence γ is extremal. □

The preceding example does not illustrate Theorem 1.3, because we did not conclude that *card* $V(\gamma) =$ *rank* $M(n)(\gamma)$ until after we had established the existence of a representing measure using [CuFi6]. Moment theory can sometimes be used to estimate the number and location of the zeros of a prescribed polynomial; indeed, as a by-product of Example 1.4, we see that the polynomial $p(z) \equiv z^{2n} + az^{2n-1} - az - 1$ $(0 < a < 1)$ has $2n$ distinct zeros, all in the unit circle. (In response to our question, Professor Srdjan Petrovic has provided a direct proof of this fact.)

The next example does illustrate how Theorem 1.3 can be used to solve an extremal problem; in particular, it shows how to verify consistency and how to compute the unique representing measure.

Example 1.5. Consider the 2-dimensional real moment matrix

$$\mathcal{M}(2) = \begin{pmatrix} 1 & 0 & 0 & 1/2 & 0 & 3/2 \\ 0 & 1/2 & 0 & -5/4 & 0 & -3/4 \\ 0 & 0 & 3/2 & 0 & -3/4 & 0 \\ 1/2 & -5/4 & 0 & 45/8 & 0 & 3/8 \\ 0 & 0 & -3/4 & 0 & 3/8 & 0 \\ 3/2 & -3/4 & 0 & 3/8 & 0 & 45/8 \end{pmatrix}.$$

We denote the rows and columns of $\mathcal{M}(2)$ as $1,\ X,\ Y,\ X^2,\ YX,\ Y^2$ and we denote the moment corresponding to $x^i y^j$ by β_{ij}. Since the upper left 4×4 corner of $\mathcal{M}(2)$ is positive definite and we have column relations $YX = -(1/2)Y$ and $Y^2 = 2 - 4X - X^2$, it follows that $\mathcal{M}(2)$ is positive semidefinite with $rank\ \mathcal{M}(2) = 4$. The variety $\mathcal{V} \equiv \mathcal{V}_\beta$ consists of the common zeros of $f(x) := yx + \frac{1}{2}y$ and $g(x) := y^2 + x^2 + 4x - 2$; these are the points $w_k \equiv (x_k, y_k)$ $(1 \le k \le 4)$, given by $x_1 = x_2 = -\frac{1}{2}$, $y_1 = \frac{\sqrt{15}}{2}$, $y_2 = -y_1$, $x_3 = -2 - \sqrt{6}$, $x_4 = -2 + \sqrt{6}$, $y_3 = y_4 = 0$, so $\beta^{(4)}$ is extremal. We next apply the method of Section 3 to verify that β is consistent, and to this end we will compute a basis for $\mathcal{I}_4 := \{p \in \mathcal{P}_4 : p|_{\mathcal{V}} \equiv 0\}$. Let $W_4 \equiv W_4(\mathcal{V})$ denote the matrix with 4 rows and 15 columns defined as follows. The columns are indexed by the monomials in $\mathcal{P}_4$ in degree-lexicographic order, and the entry in row k, column $Y^i X^j$ is $y_k^i x_k^j$ $(1 \le k \le 4, i, j \ge 0, i + j \le 4)$. Clearly, a polynomial $p \equiv \sum_{0 \le i+j \le 4} a_{ij} x^i y^j \in \mathcal{P}_4$ vanishes on $\mathcal{V}$ if and only if $\hat{p} \equiv (a_{ij}) \in ker\ W_4$. Row-reducing W_4, we obtain

$$W_{red} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & -1 & 0 & -\frac{9}{2} & 0 & \frac{1}{2} & 0 & \frac{15}{2} \\ 0 & 1 & 0 & 0 & 0 & -4 & 0 & 0 & 2 & 0 & 1 & 0 & -1 & 0 & -15 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & \frac{15}{4} & 0 & -\frac{1}{8} & 0 & -\frac{15}{8} & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -\frac{9}{2} & 0 & \frac{1}{2} & 0 & \frac{81}{4} & 0 & -\frac{1}{4} & 0 & -\frac{15}{4} \end{pmatrix},$$

from which it follows that $\dim \ker W_{red} = 11$. The form of W_{red} implies that there is a basis for $\ker W_{red}$ $(= \ker W_4)$ of the form $\{\hat{f}_i\}_{i=1}^{11}$, where $\hat{f}_i \equiv (a_{i,1}, ..., a_{i,15})$ satisfies $a_{i,4+j} = \delta_{ij}$ $(1 \le j \le 11)$. By explicitly computing this basis, we derive the following basis for $\mathcal{I}_4$: $f_1 := \frac{1}{2}y + yx$, $f_2 := -2 + 4x + x^2 + y^2$, $f_3 := -1 + \frac{9}{2}x^2 + x^3$, $f_4 := -\frac{1}{4}y + yx^2$, $f_5 := 1 - 2x - \frac{1}{2}x^2 + y^2 x$, $f_6 := -\frac{15}{4}y + y^3$, $f_7 := \frac{9}{2} - x - \frac{81}{4}x^2 + x^4$, $f_8 := \frac{1}{8}y + yx^3$, $f_9 := -\frac{1}{2} + x + \frac{1}{4}x^2 + y^2 x^2$, $f_{10} := \frac{15}{8}y + y^3 x$, $f_{11} := -\frac{15}{2} + 15x + \frac{15}{4}x^2 + y^4$. Using the moment data, it is now straightforward to verify that $\Lambda_\beta(f_i) = 0$ $(1 \le i \le 11)$, so β is consistent.

Theorem 1.3 now implies that β has a unique representing measure. To compute this measure we follow the procedure described in the proof of Theorem 4.2.

Consider the following basis for the column space of $\mathcal{M}(2)$, $\mathcal{B} = \{1,\ X,\ Y,\ X^2\}$. Let

$$V_{\mathcal{B}} \equiv V_{\mathcal{B}}[\mathcal{V}] := \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{pmatrix}$$

We show in Lemma 4.1 that $V_{\mathcal{B}}$ is necessarily invertible, and in the proof of Theorem 4.2 we show that the unique representing measure for β is of the form $\mu = \sum_{k=1}^{4} \rho_k \delta_{w_k}$, whence the densities ρ_k are determined by

$$(\rho_1, \rho_2, \rho_3, \rho_4)^T = V_{\mathcal{B}}^{-1}(\beta_{00}, \beta_{01}, \beta_{10}, \beta_{02})^T$$

(where $(\cdot)^T$ denotes transpose). Using the given moment values, we find $\rho_1 = \rho_2 = \frac{1}{5},\ \rho_3 = \frac{9\sqrt{2}-7\sqrt{3}}{30\sqrt{2}} \cong 0.0142262,\ \rho_4 = \frac{9\sqrt{2}+7\sqrt{3}}{30\sqrt{2}} \cong 0.585774.$ □

2. Real Ideals and Necessary Conditions

If $\beta^{(2n)}$ has a representing measure μ, then the Riesz functional

$$\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R}, \quad \Lambda(x^i) := \beta_i \ (= \int_{\mathbb{R}^d} x^i \, d\mu \ \ (|i| \leq 2n)),$$

is *square positive*, that is,

$$p \in \mathcal{P}_n \Rightarrow \Lambda(p^2) \geq 0$$

(equivalently, $\mathcal{M}(n)(\beta)$ is positive semidefinite, cf. (1.2)). If we assume, in addition, that for a representing measure μ all moments

$$\int_{\mathbb{R}^d} x^i d\mu \ \ (i \in \mathbb{Z}_+^d)$$

are convergent, then we can extend Λ to $\mathcal{P}$ by letting

$$\Lambda(x^i) := \int_{\mathbb{R}^d} x^i d\mu, \quad i \in \mathbb{Z}_+^d,$$

thus obtaining a square positive functional over $\mathcal{P}$ (e.g., if μ is an m-atomic measure with support $\{w_1, ..., w_m\} \subseteq \mathbb{R}^d$, then $\Lambda(p) = \sum_{i=1}^{m} p(w_i)\mu(\{w_i\})$ for all polynomials p). If Λ_β does extend to a square positive linear functional Λ on $\mathcal{P}$, then, as shown in [Moe1], the set

$$\mathcal{I} := \{p \in \mathcal{P} : \Lambda(p^2) = 0\}$$

is a *real ideal*, i.e., it is an ideal ($p_1, p_2 \in \mathcal{I} \Rightarrow p_1 + p_2 \in \mathcal{I}$ and $p \in \mathcal{I},\ q \in \mathcal{P} \Rightarrow pq \in \mathcal{I}$) and satisfies one of the following two equivalent conditions:

(i) For $s \in \mathbb{Z}_+, p_1, \ldots, p_s \in \mathcal{P} : \sum_{i=1}^{s} p_i^2 \in \mathcal{I} \Rightarrow \{p_1, \ldots, p_s\} \subseteq \mathcal{I}$;
(ii) There exists $G \subseteq \mathbb{R}^d$ such that for all $p \in \mathcal{P} : p|_G \equiv 0 \Rightarrow p \in \mathcal{I}$.

If $\mathcal{I}$ is a real ideal, then one may take for G the *real variety*

$$V_{\mathbb{R}}(\mathcal{I}) := \{w \in \mathbb{R}^d : f(w) = 0 \quad (\text{all } f \in \mathcal{I})\}.$$

But one may also take for G any subset of $V_{\mathbb{R}}(\mathcal{I})$ containing sufficiently many points, such that

$$p \in \mathcal{P},\ p|_G \equiv 0 \Rightarrow p|_{V_{\mathbb{R}}(\mathcal{I})} \equiv 0.$$

For instance, if the real variety is a (real) line, one may take for G a subset of infinitely many points on that line. On the other hand, if $V_{\mathbb{R}}(\mathcal{I})$ is a finite set of points, then necessarily $G = V_{\mathbb{R}}(\mathcal{I})$. (We note that in the full moment problem for $\beta \equiv \beta^{(\infty)}$, M. Laurent [Lau2] independently showed that $\mathcal{J} := \{p \in \mathcal{P} : M(\infty)\hat{p} = 0\}$ is a *radical* ideal; equivalently, $p \in \mathcal{J} \Leftrightarrow p^2 \in \mathcal{J}$.)

If $\mathcal{I}$ is an ideal, its subset $\mathcal{I}_k := \mathcal{I} \cap \mathcal{P}_k$ is an $\mathbb{R}$-vector subspace of $\mathcal{P}_k$. One can then introduce the *Hilbert function* of $\mathcal{I}$ by

$$H_{\mathcal{I}}(k) := \dim \mathcal{P}_k - \dim \mathcal{I}_k, \quad k \in \mathbb{Z}_+;$$

in [CLO] this is called the *affine Hilbert function*. As shown for instance in [CLO], both $k \mapsto \dim \mathcal{I}_k$ and $k \mapsto H_{\mathcal{I}}(k)$ are nondecreasing functions, and for sufficiently large k, say $k \geq k_0$, $H_{\mathcal{I}}(k)$ becomes a polynomial in k, the so-called *Hilbert polynomial of* $\mathcal{I}$, whose degree equals the *dimension* of $\mathcal{I}$.

Example 2.1. Let $G \equiv \{w_1, \ldots, w_m\} \subseteq \mathbb{R}^d$. Then $\mathcal{I} := \{f \in \mathcal{P} : f|_G \equiv 0\}$ is a real ideal with $V_{\mathbb{R}}(\mathcal{I}) = G$. Let $t_1, t_2, t_3, \ldots$ denote the monomials x^i in degree-lexicographic order, so that for each $k \in \mathbb{Z}_+$ $t_1, \ldots, t_K$ (with $K := \dim \mathcal{P}_k$) form a basis of the $\mathbb{R}$-vector space $\mathcal{P}_k$. For $p \in \mathcal{P}_k$, $p \equiv \sum_{i=1}^{K} a_i t_i$, let $\hat{p} := (a_1, \ldots, a_K)$ (the coefficient vector of p). Then $p(x)$ can be written as

$$p(x) = \langle \hat{p}, t(x) \rangle,$$

where $t(x) := (t_1(x), ..., t_K(x))$, so

$$p \in \mathcal{I} \cap \mathcal{P}_k \Leftrightarrow \hat{p} \perp t(w_i),\ i = 1, \ldots, m.$$

Arranging the rows $t(w_i)$ $(= (t_1(w_i), \ldots, t_K(w_i)))$ in a matrix

$$W_k \equiv W_k[G] := (t_j(w_i))_{i=1,\ldots,m,\ j=1,\ldots,K},$$

one gets $p \in \mathcal{I} \cap \mathcal{P}_k \Leftrightarrow \hat{p} \in \ker\ W_k$, whence $\dim \mathcal{I}_k + \mathit{rank}\ W_k = \dim \mathcal{P}_k$, or using the Hilbert function,

$$H_{\mathcal{I}}(k) = \mathit{rank}\ W_k, \quad k \in \mathbb{Z}_+ .$$

By construction, W_k is a submatrix of W_{k+1}. Hence $\mathit{rank}\ W_k \leq \mathit{rank}\ W_{k+1}$, reflecting the fact that the Hilbert function increases. If, for a given k, the rank of W_k is less than m, then one row of W_k, say the last one, depends on the others. This means that every polynomial which vanishes in $w_1, \ldots, w_{m-1}$ also vanishes in w_m. Using Lagrange interpolation polynomials, we see that for all sufficiently large k this cannot happen. Hence $\mathit{rank}\ W_k = m$ for all sufficiently large k. This m is the constant (degree-0) polynomial in k which coincides with $H_{\mathcal{I}}(k)$ for all $k \geq k_0$; hence, $\mathcal{I}$ is a zero dimensional ideal. □

Now we will study the consistency condition (1.4). We consider an arbitrary real d-dimensional multisequence $\beta \equiv \beta^{(2n)}$ of degree $2n$. Associated with β one has the Riesz functional Λ, the moment matrix $\mathcal{M}(n)$, and the algebraic variety $\mathcal{V} \equiv \mathcal{V}_\beta$ (or $\mathcal{V}(\mathcal{M}(n))$). One can then define the ideal

$$\mathcal{I}(\mathcal{V}) := \{p \in \mathcal{P} : p|_{\mathcal{V}} \equiv 0\}. \tag{2.1}$$

Since $\mathcal{V}$ is a set of real points, $\mathcal{I}(\mathcal{V})$ is a real ideal, which we will call the *real ideal* of β.

Lemma 2.2. *Assume that $\beta \equiv \beta^{(2n)}$ satisfies (1.4). Then*

$$\mathcal{N}_n := \{p \in \mathcal{P}_n : \mathcal{M}(n)\hat{p} = 0\} = \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_n. \tag{2.2}$$

If $t_1, ..., t_N$ denote the monomials $x^i \in \mathcal{P}_n$ in degree-lexicographic order, then the row vectors of $\mathcal{M}(n)$ and the row vectors $\{t(w) := (t_1(w), ..., t_N(w)) : w \in \mathcal{V}\}$, span the same subspace of $\mathbb{R}^N$; in particular, rank $\mathcal{M}(n) = H_{\mathcal{I}(\mathcal{V})}(n)$.

Proof. If $p \in \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_n$ and $q \in \mathcal{P}_n$, then $pq \in \mathcal{P}_{2n}$ and $(pq)|_{\mathcal{V}} \equiv 0$, whence by the consistency property (1.4) we must have $\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle = \Lambda(pq) = 0$; thus, $\mathcal{M}(n)\hat{p} = 0$. Conversely, if $p \in \mathcal{P}_n$ and $\mathcal{M}(n)\hat{p} = 0$, then $p|_{\mathcal{V}} \equiv 0$ by the definition of $\mathcal{V}$. Hence $p \in \mathcal{I}(\mathcal{V})$. Now, using (2.2) and proceeding as in Example 2.1, we see that

$$\begin{aligned} \hat{p} \ &\in \ \ker\ \mathcal{M}(n) \Leftrightarrow p \in \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_n \\ &\Leftrightarrow \ \hat{p} \perp t(w) \ \ (\text{all } w \in \mathcal{V}). \end{aligned}$$

This means that the rows of $\mathcal{M}(n)$ span the same space (namely, $\mathbb{R}^N \ominus \ker\ \mathcal{M}(n)$) as the rows $(t_1(w), \ldots, t_N(w))$, $w \in \mathcal{V}$. It also follows that

$$\textit{rank}\ \mathcal{M}(n) = \dim\ \mathcal{P}_n - \dim\ \ker\ \mathcal{M}(n) = \dim\ \mathcal{P}_n - \dim\ \mathcal{I}(\mathcal{V}) \cap \mathcal{P}_n = H_{\mathcal{I}(\mathcal{V})}(n).$$

□

As the following lemma will show, consistency is a very strong condition, already yielding an atomic measure (though one which may have some negative densities).

Lemma 2.3. *Let $\Lambda : \mathcal{P}_{2n} \to \mathbb{R}$ be a linear functional and let $\mathcal{V} \subseteq \mathbb{R}^d$. The following statements are equivalent.*
(a) There exist $\alpha_1, ..., \alpha_m \in \mathbb{R}$ and there exist $w_1, ..., w_m \in \mathcal{V}$ such that $\Lambda(p) = \sum_{i=1}^m \alpha_i p(w_i)$ (all $p \in \mathcal{P}_{2n}$).
(b) If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = 0$.

Proof. The implication (a) $\Rightarrow$ (b) is obvious. Therefore assume that (b) holds, and fix the basis of monomials x^i of $\mathcal{P}_{2n}$. For notational convenience, denote this basis by $t_1, \ldots, t_K$. Then b) is equivalent to

$$\text{(c)}\;\; \text{For all } c_1, ..., c_K \in \mathbb{R}^K : \sum_{j=1}^{K} c_j t_j(w) = 0 \;\; (\text{all } w \in \mathcal{V}) \Rightarrow \sum_{j=1}^{K} c_j \Lambda(t_j) = 0.$$

Using $\hat{c} := (c_1, \ldots, c_K)$, $t(w) := (t_1(w), \ldots, t_K(w))$, and $\hat{\Lambda} := (\Lambda(t_1), \ldots, \Lambda(t_K))$, (b) is thus equivalent to

$$\text{(d)}\;\; \text{For all } \hat{c} \in \mathbb{R}^K : \hat{c} \perp t(w) \;\; (\text{all } w \in \mathcal{V}) \Rightarrow \hat{c} \perp \hat{\Lambda}.$$

Recall that for subspaces $\mathcal{R}$ and $\mathcal{S}$ of $\mathbb{R}^K$, $\mathcal{R}^\perp \subseteq \mathcal{S}^\perp \Leftrightarrow \mathcal{S} \subseteq \mathcal{R}$. Hence $\hat{\Lambda}$ is in the $\mathbb{R}$-linear subspace of $\mathbb{R}^K$ spanned by $\{t(w) : w \in \mathcal{V}\}$. As such, this subspace has a basis of m $(\leq K)$ vectors $t(w_1), ..., t(w_m)$, where $w_1, ..., w_m \in \mathcal{V}$. Hence there exist $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that $\hat{\Lambda} = \sum_{i=1}^m \alpha_i t(w_i)$, or equivalently,

$$\Lambda(t_j) = \sum_{i=1}^{m} \alpha_i t_j(w_i) \;\; (1 \leq j \leq K).$$

This is a linear relation holding for a basis of $\mathcal{P}_{2n}$, hence it holds true for all $p \in \mathcal{P}_{2n}$, that is,

$$p \in \mathcal{P}_{2n} \Rightarrow \Lambda(p) = \sum_{i=1}^{m} \alpha_i p(w_i).$$

□

Remark 2.4. If Λ is the Riesz functional Λ_β corresponding to $\beta \equiv \beta^{(2n)}$, then Lemma 2.3(b) is the consistency condition (1.4). We remark that in the proof of Lemma 2.3(b) we did not assume the square positivity of Λ (which corresponds to the positivity condition (1.3) when $\Lambda = \Lambda_\beta$).

When $\Lambda = \Lambda_\beta$, $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n)) = \{w_1, ..., w_m\}$, and *rank* $\mathcal{M}(n) = m$ (the extremal case), we next show that in the representation of Lemma 2.3(a), the square positivity of Λ is equivalent to the positivity of the α_i's. We have noted above that Λ is square positive if and only if $\mathcal{M}(n)$ is positive semidefinite; in this case, we also have $\{p \in \mathcal{P}_n : \mathcal{M}(n)\hat{p} = 0\} = \{p \in \mathcal{P}_n : \Lambda(p^2) = 0\}$.

Lemma 2.5. *Let* $\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R}$ *be given by*

$$\Lambda(p) := \sum_{i=1}^{m} \alpha_i p(w_i) \;\; (p \in \mathcal{P}_{2n}),$$

with $\mathcal{V}_\beta \equiv \{w_1, ..., w_m\} \subseteq \mathbb{R}^d$ *and* $\alpha_1, ..., \alpha_m \in \mathbb{R}$. *If rank* $\mathcal{M}(n) = m$, *the following statements are equivalent:*

(i) $\alpha_i > 0$ *(all* $i = 1, ..., m$*);*
(ii) Λ *is square positive.*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. Conversely, assume that Λ is square positive, i.e., $\mathcal{M}(n)$ is positive semi-definite. Let $t_1, ..., t_N$ be the basis of monomials in $\mathcal{P}_n$ in degree-lexicographic order, so that the (j,k)-entry of $\mathcal{M}(n)$ is $\Lambda(t_j t_k)$. It follows that $\mathcal{M}(n)$ can be decomposed as

$$\mathcal{M}(n) = W_n^T \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_m \end{pmatrix} W_n, \tag{2.3}$$

where W_n is the $m \times N$ matrix with rows $t(w_i) \equiv (t_1(w_i), ..., t_N(w_i))$ $(1 \le i \le m)$. Since *rank* $\mathcal{M}(n) = m$, (2.3) implies that *rank* $W_n = m$. Hence the columns of W_n span $\mathbb{R}^m$; in particular, every unit vector in $\mathbb{R}^m$ is a linear combination of columns of W_n. This means that there exist polynomials $\ell_i \in \mathcal{P}_n$ satisfying $\ell_i(w_j) = \delta_{ij}$ $(1 \le i, j \le m)$, where δ_{ij} denotes the Kronecker symbol. Now, $\alpha_i = \Lambda(\ell_i^2) = \left\langle \mathcal{M}(n)\hat{\ell}_i, \hat{\ell}_i \right\rangle \ge 0$ (since $\mathcal{M}(n) \ge 0$). Finally, no α_i can be zero, because otherwise *rank* $\mathcal{M}(n) < m$, a contradiction. □

Remark 2.6. (i) The preceding results yield a first proof of Theorem 1.3(iii) ⇒ (i). Indeed, Lemma 2.3 shows that if β is consistent, then β admits an atomic representing measure μ, while Lemma 2.5 shows that if $\mathcal{M}(n)$ is also positive semi-definite and extremal, then μ is *rank* $\mathcal{M}(n)$-atomic and $\mu \ge 0$.
(ii) A decomposition similar to (2.3) was used by Laurent [Lau2] in her study of the full moment problem for $\beta^{(\infty)}$ in the case when *card* $\mathcal{V}(\mathcal{M}(\infty)) < +\infty$.

We conclude this section with some additional observations about ideals and consistency. Given a real d-dimensional multisequence β of degree $2n$, let $\{p_1, ..., p_s\}$ denote a basis for $\mathcal{N}_n := \{p \in \mathcal{P}_n : \mathcal{M}(n)\hat{p} = 0\}$. Denote by $\mathcal{J} \equiv \mathcal{J}_\beta$ the smallest ideal containing the polynomials $p_1, ..., p_s$. Since $\mathcal{V} \equiv \mathcal{V}_\beta$ is the set of all real common zeros of $p_1, ..., p_s$, we have $\mathcal{J} \subseteq \mathcal{I}(\mathcal{V})$. If β is consistent, then Lemma 2.2 gives $\mathcal{J} \bigcap \mathcal{P}_n = \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_n$, whence

$$\dim(\mathcal{J} \bigcap \mathcal{P}_k) \le \dim(\mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_k) \ \ (k \ge 0),$$

with equality when $k = 0, ..., n$. For general β, the consistency condition (1.4) can be rephrased in terms of $\mathcal{I}(\mathcal{V})$ as

$$p \in \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n} \Rightarrow \Lambda_\beta(p) = 0.$$

Now, since $\mathcal{J} \bigcap \mathcal{P}_{2n}$ is a subset of $\mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n}$, we can find

$$M := \dim(\mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n}) - \dim(\mathcal{J} \bigcap \mathcal{P}_{2n}) \ (= H_{\mathcal{I}(\mathcal{V})}(2n) - H_{\mathcal{J}}(2n))$$

polynomials $h_1, ..., h_M \in \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n}$ enlarging a basis for $\mathcal{J} \bigcap \mathcal{P}_{2n}$ to a basis for $\mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n}$. Then (1.4) can be rephrased again as

$$p \in \mathcal{J} \bigcap \mathcal{P}_{2n} \Rightarrow \Lambda(p) = 0, \text{ and } \Lambda(h_i) = 0 \ \ (1 \le i \le M). \tag{2.4}$$

Note that if $f \in \mathcal{N}_n$ and $g \in \mathcal{P}_n$, then $p := fg \in \mathcal{J} \bigcap \mathcal{P}_{2n}$ and $\Lambda(p) = \left\langle \mathcal{M}(n)\hat{f}, \hat{g} \right\rangle = 0$. In Sections 3 and 6 we will identify situations in which $p \in \mathcal{J} \bigcap \mathcal{P}_{2n}$ always implies $\Lambda(p) = 0$, so that consistency reduces to the test $\Lambda(h_i) = 0 \ \ (1 \le i \le M)$.

3. Moment Matrices and Consistency

A basic result of [CuFi2] shows that $\beta \equiv \beta^{(2n)}$ has a *minimal representing measure*, i.e., a representing measure whose support consists of exactly *rank* $\mathcal{M}(n)$ atoms, if and only if $\mathcal{M}(n) \ge 0$ and $\mathcal{M}(n)$ admits an extension to a moment matrix $\mathcal{M}(n+1)$ with *rank* $\mathcal{M}(n+1) =$ *rank* $\mathcal{M}(n)$. Following [CuFi2], we refer to such an extension as a *flat extension*. There is at present no concrete set of necessary and sufficient conditions for the existence of flat extensions $\mathcal{M}(n+1)$; one useful sufficient condition is that $\mathcal{M}(n) \ge 0$ satisfy *rank* $\mathcal{M}(n) =$ *rank* $\mathcal{M}(n-1)$ [CuFi2, Theorem 5.4]. More generally, β has a *finitely atomic* representing measure (a representing measure with finite support) if and only if $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+k)$ (for some $k \ge 0$), which in turn admits a flat extension $\mathcal{M}(n+k+1)$ (cf. [CuFi4, Theorem 1.5]). Since $\mathcal{M}(n+k+1)$ then admits unique successive flat extensions $\mathcal{M}(n+k+2)$, $\mathcal{M}(n+k+3)$, ... [CuFi2], this condition is equivalent to the existence of a finite rank positive extension $\mathcal{M}(\infty)$. Further, a recent result of C. Bayer and J. Teichmann [BaTe] (cf. Section 1) implies that if β has a representing measure, then β admits a finitely atomic representing measure as just described.

Recall that the columns of $\mathcal{M}(n)$ are denoted as X^i, $|i| \le n$, following the degree-lexicographic ordering of the monomials x^i in $\mathcal{P}_n$. Let $p \in \mathcal{P}_n$, $p(x) \equiv \sum_{|i| \le n} a_i x^i$; the general element of $\mathcal{C}_{\mathcal{M}(n)}$, the column space of $\mathcal{M}(n)$, may thus be denoted as $p(X) := \sum_{|i| \le n} a_i X^i$. Let $\hat{p} \equiv (a_i)$ denote the coefficient vector of p relative to the basis of monomials of $\mathcal{P}_n$ in degree-lexicographic order, and note that $p(X) = \mathcal{M}(n)\hat{p}$. Now recall the variety of β,

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, p(X)=0} \mathcal{Z}(p),$$

where $\mathcal{Z}(p) := \{x \in \mathbb{R}^d : p(x) = 0\}$. Let $\mathcal{P}_n|_{\mathcal{V}}$ denote the restriction to $\mathcal{V}$ of the polynomials in $\mathcal{P}_n$, and consider the mapping $\phi_\beta : \mathcal{C}_{\mathcal{M}(n)} \to \mathcal{P}_n|_{\mathcal{V}}$ given by $p(X) \mapsto p|_{\mathcal{V}}$. The map ϕ_β is well-defined, for if $p, q \in \mathcal{P}_n$ with $p(X) = q(X)$, then $\mathcal{V} \subseteq \mathcal{Z}(p-q)$, whence $p|_{\mathcal{V}} = q|_{\mathcal{V}}$. Note that if β has a representing measure μ, then ϕ_β is 1-1; for, if $p \in \mathcal{P}_n$ and $p|_{\mathcal{V}} \equiv 0$, then since *supp* $\mu \subseteq \mathcal{V}$ (cf. Section 1), we have $p|_{supp\ \mu} \equiv 0$, whence [CuFi2, Proposition 3.1] implies $p(X) = 0$. Consider also the following property of β:

$$(3.1) \qquad p \in \mathcal{P}_n, q \in \mathcal{P}, pq \in \mathcal{P}_{2n}, p(X) = 0 \Rightarrow \Lambda_\beta(pq) = 0$$

(where Λ_β is the Riesz functional associated to β; cf. Section 1).

The following result will be used in the proof of Theorem 1.3.

Proposition 3.1. *Let β, ϕ_β and $\mathcal{M}(n)(\beta)$ be as above. Then*
(i) β consistent $\Longrightarrow \phi_\beta$ 1-1 $\Longrightarrow \mathcal{M}(n)(\beta)$ recursively generated.
(ii) β consistent $\Longrightarrow \beta$ satisfies (3.1) $\Longrightarrow \mathcal{M}(n)(\beta)$ recursively generated.

Proof. (i) Suppose β is consistent. Formula (2.2) in Lemma 2.2 implies that ϕ_β is 1-1. We next assume that ϕ_β is 1-1 and we show that $\mathcal{M}(n)$ is recursively generated. Let $p,\ q,\ pq \in \mathcal{P}_n$ and suppose $p(X) = 0$. Since $\mathcal{V} \subseteq \mathcal{Z}(p)$, then $p|_\mathcal{V} \equiv 0$, whence $pq|_\mathcal{V} \equiv 0$. Since $pq \in \mathcal{P}_n$ and ϕ_β is 1-1, it follows that $(pq)(X) = 0$.

(ii) Suppose β is consistent. Let $p \in \mathcal{P}_n$ and let $q \in \mathcal{P}$, with $pq \in \mathcal{P}_{2n}$. If $p(X) = 0$, then clearly $\mathcal{V}_\beta \subseteq \mathcal{Z}(p)$, whence $(pq)|_{\mathcal{V}_\beta} \equiv 0$. Now, consistency implies that $\Lambda_\beta(pq) = 0$, so (3.1) holds.

Assume now that (3.1) holds and suppose $p, q, pq \in \mathcal{P}_n$ with $p(X) = 0$. Now, for each $s \in \mathcal{P}_n$, $p(qs) \in \mathcal{P}_{2n}$, so (3.1) implies

$$\langle \mathcal{M}(n)\widehat{pq}, \hat{s}\rangle = \Lambda_\beta((pq)s) = \Lambda_\beta(p(qs)) = 0 \ \text{(by (3.1))}.$$

Thus $(pq)(X) = \mathcal{M}(n)\widehat{pq} = 0$, so $\mathcal{M}(n)$ is recursively generated. □

It is not difficult to see that Lemma 2.2 remains true if the hypothesis that β is consistent is replaced by the condition that ϕ_β is 1-1. Indeed, we see that ϕ_β is 1-1 $\Leftrightarrow$ ker $\mathcal{M}(n) =$ ker $W_n \Leftrightarrow$ *rank* $\mathcal{M}(n) =$ *rank* W_n.

For the case when $\mathcal{V} \equiv \mathcal{V}_\beta$ is finite and the elements of $\mathcal{V}$ can be computed exactly, we next describe an elementary procedure for determining whether or not β is consistent. Denote the distinct points of $\mathcal{V}$ as $\{w_j\}_{j=1}^m$. Recall the matrix $W \equiv W_{2n}[\mathcal{V}_\beta]$, with m rows and with columns indexed by the monomials in $\mathcal{P}_{2n}$ (indexed, as usual, in degree-lexicographic order). The entry of W in row k, column x^i is w_k^i. Clearly, a polynomial $p(x) \equiv \sum_{|i|\le 2n} a_i x^i \in \mathcal{P}_{2n}$ satisfies $p|_\mathcal{V} \equiv 0$ if and only if $W\hat{p} = 0$. Using Gaussian elimination, we may row-reduce W so as to find a basis for $\ker W$, say $\{\hat{p}_1, ..., \hat{p}_s\}$. It follows that $\{p_1, ..., p_s\}$ is a basis for $\{p \in \mathcal{P}_{2n} : p|_\mathcal{V} \equiv 0\}$. Let $\hat{p}_j := (a_{ji})_{|i|\le 2n}$ $(1 \le j \le s)$. We now see that β is consistent if, and only if, for each j, $\Lambda_\beta(p_j) = \sum_{|i|\le 2n} a_{ji}\beta_i = 0$.

In Example 1.5 (above) we were able to compute the points of $\mathcal{V}$ exactly and to then check the consistency of β using the preceding method. In other examples we may be able to determine that $\mathcal{V}$ is finite (from the form of the polynomial relations which determine $\mathcal{V}$) without being able to exactly compute the points of the variety. In such cases we cannot employ the above procedure for checking consistency.

The concluding remarks of Section 2, particularly (2.4), suggest alternate, more algebraic, approaches to verifying consistency that we will pursue below and in Sections 5 and 6. Let $\mathcal{N}_n := \{p \in \mathcal{P}_n : \mathcal{M}(n)\hat{p} = 0\}$ and let $\mathcal{J}_\beta := (\mathcal{N}_n)$ denote the ideal of $\mathcal{P}$ generated by $\mathcal{N}_n$. For $\mathcal{S} \subseteq \mathcal{P}$, let $\mathcal{V}(\mathcal{S}) := \{x \in \mathbb{R}^d : p(x) = 0 \text{ for every } p \in \mathcal{S}\}$; we have

$$\mathcal{V}_\beta = \mathcal{V}(\mathcal{N}_n) = \mathcal{V}(\mathcal{J}_\beta).$$

Let $\mathcal{I}(\mathcal{V}) := \{p \in \mathbb{R}^d[x] : p|_{\mathcal{V}_\beta} \equiv 0\}$ ($\supseteq \mathcal{J}_\beta$), and set $\mathcal{K}_n := \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_n$. Clearly $\mathcal{N}_n \subseteq \mathcal{K}_n$, and ϕ_β is one-to-one if and only if $\mathcal{N}_n = \mathcal{K}_n$.

Consider a polynomial ideal $\mathcal{I} \subseteq \mathcal{P}$. Recall from [MoSa] that $\{p_1, ..., p_k\} \subseteq \mathcal{I}$ forms an *H-basis* for $\mathcal{I}$ if for every $p \in \mathcal{I}$ there exist polynomials $q_1, ..., q_k$ such that $p = \sum_{i=1}^k p_i q_i$ and $\deg p_i q_i \le \deg p$ $(1 \le i \le k)$. Every Gröbner basis is an H-basis; in particular, every polynomial ideal has an H-basis [MoSa]. We will utilize the following weak H-basis condition for elements of $\mathcal{P}_{2n}$:

(3.2) For each $p \in \mathcal{I}(\mathcal{V})$ with $\deg p \le 2n$, there exist $m > 0$, polynomials $h_1, ..., h_m \in \mathcal{N}_n$, and polynomials $f_1, ..., f_m$ with $\deg f_i h_i \le 2n$ $(1 \le i \le m)$ (where m, h_i and f_i may depend on p) such that $p = \sum_{i=1}^m f_i h_i$.

Note that if $\mathcal{N}_n$ contains an H-basis for $\mathcal{I}(\mathcal{V})$, then (3.2) is satisfied.

The following result is proved in [Fia4].

Theorem 3.2. *(*[Fia4]*) If $\mathcal{M}(n)$ is recursively generated and satisfies (3.2), then $\beta^{(2n)}$ is consistent.*

Corollary 3.3. *If $\mathcal{M}(n)$ is recursively generated and $\mathcal{N}_n$ contains an H-basis for $\mathcal{I}(\mathcal{V})$, then $\beta^{(2n)}$ is consistent.*

We next present some examples which illustrate Corollary 3.3.

Proposition 3.4. *For $d = 2$ (the plane), if $\mathcal{M}(n)(\beta)$ is recursively generated and $\mathcal{V}_\beta$ is a proper, infinite irreducible curve, then β is consistent.*

Proof. There is an irreducible polynomial $f \in \mathcal{P}_n$ such that $f(X, Y) = 0$ and $\mathcal{V}_\beta = \mathcal{Z}(f)$. [Ful, Corollary 1, p. 18] implies that $\mathcal{I}(\mathcal{V}) = (f)$ (the ideal generated by f), and clearly $\{f\}$ $(\subseteq \mathcal{N}_n)$ is an H-basis for $\mathcal{I}(\mathcal{V})$. The result now follows from Corollary 3.3. $\square$

Example 3.5. We illustrate Proposition 3.4 with an example from [CuFi10, Example 5.2]. Consider the moment matrix

$$\mathcal{M}(3)=\begin{pmatrix} 1&0&0&1&1&2&0&0&0&0\\ 0&1&1&0&0&0&3&1&1&2\\ 0&1&2&0&0&0&1&1&2&5\\ 1&0&0&3&1&1&0&0&0&0\\ 1&0&0&1&1&2&0&0&0&0\\ 2&0&0&1&2&5&0&0&0&3\\ 0&3&1&0&0&0&14&3&1&1\\ 0&1&1&0&0&0&3&1&1&2\\ 0&1&2&0&0&0&1&1&2&5\\ 0&2&5&0&0&3&1&2&5&33 \end{pmatrix}.$$

It is straightforward to check that $\mathcal{M}(3)$ is positive and recursively generated, with column relations $YX=1$, $YX^2=X$, $Y^2X=Y$ in $\mathcal{C}_{\mathcal{M}(3)}$, and $rank\ \mathcal{M}(3)=7$. Then $\mathcal{V}_\beta$ is the hyperbola $yx=1$, and Proposition 3.4 implies that β is consistent. (The existence of a representing measure for β follows from [CuFi10, Theorem 2.1].) $\square$

Let $\mathcal{F}\equiv\{r_1,...,r_d\}\subseteq\mathcal{P}\equiv\mathbb{R}[x_1,...,x_d]$ and assume that $r_1,...,r_d$ have no common zeros at infinity. This means that the leading homogeneous forms $Lf(r_1),...,Lf(r_d)$ have no common zeros except $(0,...,0)\in\mathbb{R}^d$ [MoSa]. In this case, [MoSa, Theorem 5.3] implies that $\mathcal{F}$ is an H-basis for $\mathcal{I}:=(r_1,...,r_d)$, and $\mathcal{V}:=V_{\mathbb{R}}(\mathcal{I})$ is finite [MoSa, Section 7]. Further, $H_{\mathcal{I}}(k)=\delta:=\deg r_1\cdot...\cdot\deg r_d$ for $k\geq\deg r_1+...+\deg r_d-d+1$, and $H_{\mathcal{I}}(k)<\delta$ for $k<\deg r_1+...+\deg r_d-d+1$ [MoSa, Lemma 5.4]. Recall that a common zero w of $r_1,...,r_d$ is simple if the Jacobian $(\frac{\partial r_i}{\partial x_j}(w))_{1\leq i,j\leq d}$ has rank d. Specializing to $d=2$, a theorem of M. Noether implies that if r_1 and r_2 have no common zeros at infinity and the common zeros are all real and simple, then there are exactly $M:=\deg r_1\deg r_2$ common zeros, $\mathcal{V}=\{w_1,...,w_m\}$, and if $p\in\mathcal{P}$ satisfies $p|_{\mathcal{V}}\equiv 0$, then p has a representation $p=a_1r_1+a_2r_2$, where $a_i\in\mathcal{P}$ satisfies $\deg a_i\leq\deg p-\deg r_i$ $(i=1,2)$. These observations, together with Corollary 3.3, lead to the following criterion for consistency.

Proposition 3.6. *Suppose $d=2$. Let $\mathcal{M}(n)$ be recursively generated, and suppose a basis for* $\ker\mathcal{M}(n)$ *consists of $\hat{r}_1$ and $\hat{r}_2$, where r_1 and r_2 have no common zeros at infinity and whose common zeros are all real and simple. Then $\beta^{(2n)}$ is consistent.*

Proof. The above mentioned results of [MoSa] show that $\{r_1,r_2\}$ forms and H-basis for $\mathcal{I}:=(r_1,r_2)$, and since the common zeros of r_1 and r_2 are real and simple, Noether's Theorem implies that $\mathcal{I}$ coincides with $\mathcal{I}(V)$. The result now follows from Corollary 3.3. $\square$

Example 7.1 (below) illustrates Proposition 3.6.

We conclude this section by illustrating a broad class of extremal moment matrices having flat extensions (and representing measures). Suppose $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+1)$. If $f \in \mathcal{P}_n$ and $f(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, then $f(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n+1)}$, i.e., $\mathcal{N}_n \subseteq \mathcal{N}_{n+1}$ [Fia1]. If, further, $\mathcal{M}(n+1)$ is recursively generated, then it follows that $\mathcal{J}_\beta \bigcap \mathcal{P}_{n+1} \subseteq \mathcal{N}_{n+1}$. Motivated by [Moe2], we say that $\mathcal{M}(n+1)$ is a *tight* extension of $\mathcal{M}(n)$ if $\mathcal{N}_{n+1} = \mathcal{J}_\beta \bigcap \mathcal{P}_{n+1}$. ([Moe2] discusses "tight extensions" of linear functionals on $\mathcal{P}_n$.)

Theorem 3.7. *(*[Fia4]*) If $\mathcal{M}(n) \geq 0$ admits a tight flat extension, then $\mathcal{M}(n)$ is extremal.*

Recall that $\mathcal{M}(n)$ is *flat* if *rank* $\mathcal{M}(n) =$ *rank* $\mathcal{M}(n-1)$; the proof of [CuFi2, Theorem 5.4] shows that if $\mathcal{M}(n)$ (≥ 0) is flat, then $\mathcal{M}(n)$ admits a tight flat extension, so $\mathcal{M}(n)$ is also extremal. Remarkably, examination of the proofs of [CuFi7], [CuFi8], [CuFi10] and [Fia2] reveals that in each extremal case studied therein, $\mathcal{M}(n)$ admits a tight flat extension $\mathcal{M}(n+1)$. We can further illustrate this phenomenon as follows.

Example 3.8. The extremal matrices $\mathcal{M}(n)$ of Example 1.4 admit tight flat extensions. For simplicity of notation, we consider only $\mathcal{M}(3)$ and $\beta \equiv \beta^{(6)}$. We have *rank* $\mathcal{M}(3) = 6$, with column relations $\bar{Z}Z = 1$, $\bar{Z}Z^2 = Z$, $\bar{Z}^2 Z = \bar{Z}$, and $\bar{Z}^3 - Z^3 = a(Z^2 - \bar{Z}^2)$. Thus $\mathcal{N}_3$ has a basis $\mathcal{B}_3 = \{\bar{z}z - 1, \bar{z}z^2 - z, \bar{z}^2 z - \bar{z}, \bar{z}^3 - z^3 - a(z^2 - \bar{z}^2)\}$. In Example 1.4 we showed that $\gamma^{(2n)}$ has a (unique) *rank* $\mathcal{M}(n)$-atomic representing measure, so [CuFi2], [CuFi3] imply that $\mathcal{M}(n)$ has a (unique, recursively generated) flat extension $\mathcal{M}(n+1)$. For the unique flat extension $\mathcal{M}(4)$ we have $\mathcal{N}_4 \supseteq \mathcal{J}_\beta \bigcap \mathcal{P}_4 \supseteq \mathcal{B}_4 := \mathcal{B}_3 \bigcup \{\bar{z}^2 z^2 - \bar{z}z, \bar{z}^3 z - \bar{z}^2, \bar{z}z^3 - z^2, \bar{z}^3 z - z^4 - a(z^3 - \bar{z}^2 z), \bar{z}^4 - \bar{z}z^3 - a(\bar{z}z^2 - \bar{z}^3)\}$. Since $\mathcal{B}_4$ is independent in $\mathcal{P}_4$, and $\dim\ \mathcal{N}_4 = \dim\ \mathcal{P}_4 -$ *rank* $\mathcal{M}(4) = \dim\ \mathcal{P}_4 -$ *rank* $\mathcal{M}(3) = 15 - 6 = 9$, we have $9 = \dim\ \mathcal{N}_4 \geq \dim\ \mathcal{J}_\beta \bigcap \mathcal{P}_4 \geq$ *card* $\mathcal{B}_4 = 9$, whence $\mathcal{M}(4)$ is tight. □

Despite Theorem 3.7 and the preceding examples, we will show in Section 6 (Proposition 6.1) that there exists a positive, recursively generated, extremal $\mathcal{M}(3)$, admitting a flat extension, but having no tight flat extension.

4. The Extremal Moment Problem

Assume that $\beta \equiv \beta^{(2n)}$ is extremal, i.e., $r :=$ *rank* $\mathcal{M}(n)$ and $v :=$ *card* $\mathcal{V}_\beta$ satisfy $r = v$. Let $\mathcal{V} \equiv \{w_1, ..., w_r\}$ denote the distinct points of $\mathcal{V}_\beta$. If μ is a representing measure for β, then *supp* $\mu \subseteq \mathcal{V}$ and $r \leq$ *card supp* $\mu \leq v$, so the extremal hypothesis $r = v$ implies that *supp* $\mu = \mathcal{V}$. Thus μ is necessarily is of the form

$$\mu = \sum_{i=1}^{r} \rho_i \delta_{w_i}. \tag{4.1}$$

We begin by establishing a criterion which allows us to compute the densities ρ_i.

Let $p_1, ..., p_r$ be polynomials in $\mathcal{P}_n$ such that $\mathcal{B} \equiv \{p_1(X), ..., p_r(X)\}$ is a basis for the column space of $\mathcal{M}(n)$, and set

$$V \equiv V_{\mathcal{B}}[\mathcal{V}] := \begin{pmatrix} p_1(w_1) & . & . & . & p_1(w_r) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ p_r(w_1) & . & . & . & p_r(w_r) \end{pmatrix}.$$

Now V is singular if and only if there exist scalars $\alpha_1, ..., \alpha_r$, not all 0, such that $\alpha_1 p_1(w_i) + \cdots \alpha_r p_r(w_i) = 0$ $(1 \le i \le r)$. Equivalently, the polynomial $p \in \mathcal{P}_n$ defined by $p := \alpha_1 p_1 + \cdots + \alpha_r p_r$ satisfies $p|_{\mathcal{V}} \equiv 0$. Since $\mathcal{B}$ is a basis, it follows that $p(X) \equiv \alpha_1 p_1(X) + \cdots + \alpha_r p_r(X) \neq 0$, so ϕ_β is not 1-1. Conversely, suppose ϕ_β is not 1-1, i.e., there exists $q \in \mathcal{P}_n$ with $q|_{\mathcal{V}} \equiv 0$ and $q(X) \neq 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. Since $\mathcal{B}$ is a basis, there exist scalars $a_1, ..., a_r$, not all 0, such that $q(X) = \sum_{i=1}^r a_i p_i(X)$, and since ϕ_β is well-defined, we may assume that $q = \sum_{i=1}^r a_i p_i$. Now $q|_{\mathcal{V}} \equiv 0$ implies that $\sum_{i=1}^r a_i p_i(w_j) = 0$ $(1 \le j \le r)$, whence V is singular. Thus we have

Lemma 4.1. *The following are equivalent for β extremal:*
i) ϕ_β is 1-1, i.e., $p \in \mathcal{P}_n$, $p|_{\mathcal{V}} \equiv 0 \Longrightarrow p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$;
ii) For any basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$, V is invertible;
iii) There exists a basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$ such that V is invertible.

Suppose now that β is extremal and let $\mathcal{B}$ be any basis for $\mathcal{C}_{\mathcal{M}(n)}$; thus there exist polynomials $p_1, ..., p_r \in \mathcal{P}_n$ such that $\mathcal{B} = \{p_1(X), ..., p_r(X)\}$. If β has a representing measure μ, then ϕ_β is 1-1 (cf. Section 1), so Lemma 4.1 shows that V is invertible, whence μ is uniquely determined from (4.1) by

$$(\rho_1, ..., \rho_r)^T = V^{-1}(\Lambda_\beta(p_1), ..., \Lambda_\beta(p_r))^T. \tag{4.2}$$

Assuming only that β is extremal and that ϕ_β is 1-1, let $\mu_{\mathcal{B}}$ denote the measure defined by (4.1) and (4.2). Our main result, which follows, includes a proof of Theorem 1.3.

Theorem 4.2. *For $\beta \equiv \beta^{(2n)}$ extremal, the following are equivalent:*
(i) β has a representing measure;
(ii) β has a unique representing measure, which is rank $\mathcal{M}(n)$-atomic;
(iii) For some (respectively, for every) basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$, V is invertible and $\mu_{\mathcal{B}}$ is a representing measure for β;
(iv) β is consistent and $\mathcal{M}(n) \ge 0$;
(v) $\mathcal{M}(n) \ge 0$ has a flat extension $\mathcal{M}(n+1)$;
(vi) $\mathcal{M}(n) \ge 0$ has a unique flat extension $\mathcal{M}(n+1)$.

(Note that a proof of (iv) $\Rightarrow$ (i) is contained in Remark 2.6; we present a different proof below.)

Proof. The implications (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iv) are clear, so it suffices to prove (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii), and to then prove (ii) $\Longrightarrow$ (vi) $\Longrightarrow$ (v) $\Longrightarrow$ (i) ($\Leftrightarrow$ (ii)) . We begin with the proof of (iv) $\Longrightarrow$ (iii). Let $\mathcal{B}$ be a basis for $\mathcal{C}_{\mathcal{M}(n)}$, and, as above, denote $\mathcal{B} \equiv \{p_1(X), ..., p_r(X)\}$, where $p_1, ..., p_r$ are polynomials in $\mathcal{P}_n$. Let $\mathcal{V} \equiv \mathcal{V}_\beta = \{w_1, ..., w_r\}$ and consider V as defined above. Since β is consistent, Proposition 3.1 implies that ϕ_β is 1-1, so Lemma 4.1 shows that V is invertible, and we may thus consider $\mu_{\mathcal{B}}$ as defined by (4.1) and (4.2). To show that $\mu_{\mathcal{B}}$ is a representing measure for β, we first show that for $f \in \mathcal{P}_{2n}$, $\int f(x) d\mu_{\mathcal{B}}(x) = \Lambda_\beta(f)$. Let $v_f := (f(w_1), ..., f(w_r))$. Since V is invertible, there exists $a_f \equiv (a_1, ..., a_r) \in \mathbb{R}^r$ such that $V^T a_f^T = v_f^T$. Thus $p \equiv \sum_{i=1}^r a_i p_i \in \mathcal{P}_n$ satisfies $p(w_i) = f(w_i)$ $(1 \le i \le r)$. Now

$$\begin{aligned}
\int f(x) d\mu_{\mathcal{B}}(x) &= \sum_{k=1}^r \rho_k f(w_k) = \sum_{k=1}^r \rho_k p(w_k) \\
&= \sum_{k=1}^r \rho_k \sum_{i=1}^r a_i p_i(w_k) = \sum_{i=1}^r a_i \sum_{k=1}^r \rho_k p_i(w_k) \\
&= \sum_{i=1}^r a_i \Lambda_\beta(p_i) \ \text{ (from (4.2))} \\
&= \Lambda_\beta(\sum_{i=1}^r a_i p_i) = \Lambda_\beta(f)
\end{aligned}$$

(since β is consistent and $f - p \in \mathcal{P}_{2n}$ satisfies $(f-p)|_{\mathcal{V}} \equiv 0$).

To complete the proof that $\mu_{\mathcal{B}}$ is a representing measure, it remains to show that $\mu_{\mathcal{B}} \ge 0$. For $1 \le k \le r$, let $V_k \equiv V_k(x)$ denote the matrix obtained from V by replacing w_k (in column k) by the variable x, and let $f_k \in \mathcal{P}_n$ be defined by $f_k(x) := \det\ V_k(x)$. Clearly, $f_k(w_j) = \delta_{kj} \det\ V$ $(1 \le k, j \le r)$. Now

$$\begin{aligned}
0 &\le \left\langle \mathcal{M}(n) \hat{f}_k, \hat{f}_k \right\rangle = \Lambda_\beta(f_k^2) = \int f_k^2 d\mu_{\mathcal{B}} \ \text{ (from the preceding paragraph)} \\
&= \sum_{j=1}^r \rho_j f_k^2(w_j) = \rho_k (\det\ V)^2,
\end{aligned}$$

and since $\det\ V \neq 0$, it follows that $\rho_k \ge 0$. (Since *card supp* $\mu_{\mathcal{B}} = r$, it then follows that $\rho_k > 0$ $(1 \le k \le r)$.)

To prove (iii) $\Longrightarrow$ (ii), assume that ν is a representing measure for β. Since β is extremal, ν is of the form $\nu = \sum_{i=1}^r \sigma_i \delta_{w_i}$ for $\sigma_i > 0$ $(1 \le i \le r)$. Suppose $\mathcal{B} \equiv \{p_1(X), ..., p_r(X)\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$ (as above) such that V is invertible and $\mu_{\mathcal{B}}$ is a representing measure for β. Since ν and $\mu_{\mathcal{B}}$ are representing measures, we have

$$\begin{aligned} V(\rho_1, ..., \rho_r)^T &= (\int p_1 d\mu_{\mathcal{B}}, ..., \int p_r d\mu_{\mathcal{B}})^T \\ &= (\Lambda_\beta(p_1), ..., \Lambda_\beta(p_r))^T = (\int p_1 d\nu, ..., \int p_r d\nu)^T \\ &= V(\sigma_1, ..., \sigma_r)^T, \end{aligned}$$

and since V is invertible, it follows that $\nu = \mu_{\mathcal{B}}$. This completes the equivalence of (i), (ii), (iii) and (iv).

Now recall that β has a *rank* $\mathcal{M}(n)$-atomic representing measure if and only if $\mathcal{M}(n) \geq 0$ admits a flat extension $\mathcal{M}(n+1)$ [CuFi2, Theorem 5.13], and clearly distinct flat extensions correspond to distinct *rank* $\mathcal{M}(n)$-atomic representing measures. Thus we have (ii) $\Rightarrow$ (vi) $\Rightarrow$ (v) $\Rightarrow$ (i), and since (i) $\Leftrightarrow$ (ii), the proof is complete. $\square$

Remark 4.3. For a positive, extremal $\mathcal{M}(n)$ for which the points of the variety are known, Theorem 4.2 provides two ways to determine whether or not β has a representing measure. Following Theorem 4.2(iv) one can use the method of Section 3 to determine whether or not β is consistent. Alternatively, one can select any basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$ and check whether V is invertible. If V is not invertible, there is no representing measure. If V is invertible, then $\mu_{\mathcal{B}}$ automatically interpolates all moments up to degree n, so the proof of Theorem 4.2(iv) $\Rightarrow$ (iii) shows that β has a representing measure if and only if $\mu_{\mathcal{B}}$ interpolates all moments of degrees $n+1, n+2, ..., 2n$, in which case $\mu_\beta \geq 0$. In a given numerical problem, one approach or the other may be easier to implement, depending on the size of n and the value of *rank* $\mathcal{M}(n)$.

5. Solution of the $\mathcal{M}(3)$ Extremal Problem with $Y = X^3 : r = v = 7$

In this section (and the next) we return to the question as to whether a positive, extremal, recursively generated moment matrix has a representing measure (cf., Question 1.1). We also consider the extent to which recursiveness implies consistency in an extremal moment problem. Our motivation is the observation that it is generally much easier to verify recursiveness than consistency. We examine these issues in detail for an extremal planar moment matrix $\mathcal{M}(3)$ with $\mathcal{M}(3) \geq 0$, $\mathcal{M}(2) > 0$, and $Y = X^3$ in $\mathcal{C}_{\mathcal{M}(3)}$. Our first result illustrates an extremal problem in which recursiveness does imply consistency.

Theorem 5.1. *Let $d = 2$. Suppose $Y = X^3$ in $\mathcal{C}_{\mathcal{M}(3)}$. If $\mathcal{M}(3)$ is positive, recursively generated, and $v = r = 7$, then $\beta^{(6)}$ has a unique, 7-atomic, representing measure; equivalently, $\beta^{(6)}$ is consistent.*

Example 5.2. We illustrate Theorem 5.1 with the following moment matrix:

$$\mathcal{M}(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 200 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 200 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 200 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 200 & 5868 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 200 & 5868 & 386568 \\ 0 & 42 & 200 & 0 & 0 & 0 & 200 & 5868 & 386568 & 26992856 \end{pmatrix}.$$

$\mathcal{M}(3)$ is positive and recursively generated, with column basis $\mathcal{B} := \{1,\ X,\ Y,\ X^2,\ YX,\ Y^2,\ YX^2\}$, and column relations $Y = X^3$, $Y^2X = 208X - 282Y + 74YX^2$, and $Y^3 = 15392X - 20660Y + 5194YX^2$. A calculation shows that $\mathcal{V}_\beta$ consists of exactly 7 points in $\mathbb{R}^2$, $\{(x_i, x_i^3)\}_{i=1}^7$, with $x_1 = 0$, $x_2 \cong 8.36748$, $x_3 \cong 0.996357$, $x_4 \cong 1.7299$, and $x_{4+j} = -x_{j+1}$ $(1 \le j \le 3)$. Thus β is extremal, so Theorem 5.1 implies that β has a representing measure. Indeed, following the method of Section 4, a calculation shows that $V_\mathcal{B}$ is invertible and that $\mu_\mathcal{B}$ has densities $\rho_1 \cong 0.331731$, $\rho_2 \cong 3.3378229 \times 10^{-10}$, $\rho_3 \cong 0.249980$, $\rho_4 \cong 0.08415439$, and $\rho_{4+j} = \rho_{j+1}$ $(1 \le j \le 3)$. $\square$

We begin the proof of Theorem 5.1 with some preliminary results. Recall from Section 3 the map $\phi_\beta : \mathcal{C}_{\mathcal{M}(n)} \to \mathcal{P}_n|_{\mathcal{V}_\beta}$, given by $p(X) \mapsto p|_{\mathcal{V}_\beta}$ $(p \in \mathcal{P}_n)$. As noted in Section 3, ϕ_β is 1-1 if and only if $\mathcal{N}_n = \mathcal{K}_n$ (where $\mathcal{N}_n := \{p \in \mathcal{P}_n : p(X) = 0\}$ and $\mathcal{K}_n := \mathcal{I}(\mathcal{V}_\beta) \bigcap \mathcal{P}_n = \{p \in \mathcal{P}_n : p|_{\mathcal{V}_\beta} \equiv 0\}$); we always have $\mathcal{N}_n \subseteq \mathcal{K}_n$.

Lemma 5.3. *If $\mathcal{M}(n)(\beta)$ satisfies $r \le v$ and $\dim \mathcal{K}_n \le \dim \mathcal{P}_n - v$, then $\mathcal{M}(n)(\beta)$ is extremal and ϕ_β is 1-1.*

Proof. We have $v \le \dim \mathcal{P}_n - \dim \mathcal{K}_n \le \dim \mathcal{P}_n - \dim \mathcal{N}_n = r \le v$. It follows that $r = v$ and $\mathcal{N}_n = \mathcal{K}_n$, so $\mathcal{M}(n)(\beta)$ is extremal and ϕ_β is 1-1. $\square$

Lemma 5.4. *If $\mathcal{M}(3)(\beta)$ satisfies $Y = X^3$ and $r \le v = 7$, then ϕ_β is 1-1.*

Proof. Suppose $p(x, y) \equiv c_1 + c_2x + c_3y + c_4x^2 + c_5yx + c_6y^2 + c_7x^3 + c_8yx^2 + c_9y^2x + c_{10}y^3$ is an element of $\mathcal{K}_3$, i.e., $p|_{\mathcal{V}_\beta} \equiv 0$. Denote the distinct points of $\mathcal{V}_\beta$ by $\{(x_i, y_i)\}_{i=1}^7$; since $y_i = x_i^3$ $(1 \le i \le 7)$, the x_i's are distinct. Consider the linear map $\Psi : \mathcal{K}_3 \to \mathbb{R}^3$ defined by $\Psi(p) = (c_7, c_9, c_{10})$. We claim that Ψ is 1-1; for, suppose $c_7 = c_9 = c_{10} = 0$ and let $f(x) := p(x, x^3) \equiv c_1 + c_2x + c_3x^3 + c_4x^2 + c_5x^4 + c_6x^6 + c_8x^5$. Since f has the seven distinct roots $\{x_i\}_{i=1}^7$, it follows that $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_8 = 0$, whence $p \equiv 0$ and Ψ is 1-1. Thus $\dim \mathcal{K}_3 \le \dim \mathbb{R}^3 = 3 = 10 - 7 = \dim \mathcal{P}_3 - v$, so Lemma 5.3 implies that ϕ_β is 1-1. $\square$

Proposition 5.5. *Let $\mathcal{M}(3)(\beta) \geq 0$, with $Y = X^3$ in $\mathcal{C}_{\mathcal{M}(3)}$. If $\mathcal{B} := \{1,\ X,\ Y,\ X^2,\ YX,\ Y^2,\ YX^2\}$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$ and $v = r$, then $\beta^{(6)}$ has a representing measure.*

Proof. Let $\mathcal{V} \equiv \mathcal{V}_\beta$; Lemmas 4.1 and 5.4 imply that $V_{\mathcal{B}}[\mathcal{V}]$ is invertible, so, as in the proof of Theorem 4.2, to prove that $\mu_{\mathcal{B}}$ is a representing measure, it suffices to prove that $\mu_{\mathcal{B}}$ is *interpolating* for $\beta^{(6)}$, i.e., $\beta_{ij} = \int y^j x^i \, d\mu_{\mathcal{B}}$ $(i, j \geq 0,\ i + j \leq 6)$. Relation (4.2) shows that $\mu_{\mathcal{B}}$ interpolates the moments corresponding to elements of $\mathcal{B}$, namely $\beta_{00},\ \beta_{10},\ \beta_{01},\ \beta_{20},\ \beta_{11},\ \beta_{02}$, and β_{21}. From the hypothesis, we have

$$Y = X^3. \tag{5.1}$$

Also, there exist $\alpha,\ \gamma \in \mathbb{R}$ and $p, q \in \mathcal{P}_2$, such that we have column relations

$$Y^2X = \alpha YX^2 + p(X, Y), \tag{5.2}$$

and

$$Y^3 = \gamma YX^2 + q(X, Y). \tag{5.3}$$

In *supp* $\mu_{\mathcal{B}}$ we have $y = x^3$, so $\int x^3 d\mu_{\mathcal{B}} = \int y d\mu_{\mathcal{B}} = \beta_{01} = \langle Y, 1\rangle = \langle X^3, 1\rangle = \beta_{30}$ (by (5.1)); thus $\int x^3 d\mu_{\mathcal{B}} = \beta_{30}$. Similarly,

$$\begin{aligned} \int y^2 x \, d\mu_{\mathcal{B}} &= \int (\alpha y x^2 + p(x, y)) d\mu_{\mathcal{B}} = \alpha \beta_{21} + \Lambda_\beta(p) \\ &= \left\langle \alpha YX^2 + p(X, Y), 1 \right\rangle = \left\langle Y^2 X, 1 \right\rangle = \beta_{12} \end{aligned}$$

(by (4.2) and (5.2)), and

$$\begin{aligned} \int y^3 d\mu_{\mathcal{B}} &= \int (\gamma y x^2 + q(x, y)) d\mu_{\mathcal{B}} = \gamma \beta_{21} + \Lambda_\beta(q) \\ &= \left\langle \gamma YX^2 + q(X, Y), 1 \right\rangle = \left\langle Y^3, 1 \right\rangle = \beta_{03} \end{aligned}$$

(by (4.2) and (5.3)). Thus, $\mu_{\mathcal{B}}$ interpolates all moments up to degree 3.

The proof now continues inductively, using the results for all degrees $< k$ to obtain the result for degree k, and using (5.1)-(5.3) in successive rows of $\mathcal{M}(3)$. For example, to obtain results for degree 4, we start with the relations $y = x^3$, $y^2x = \alpha y x^2 + p(x, y)$, and $y^3 = \gamma y x^2 + q(x, y)$, valid in $\mathcal{V}_\beta$, to get new relations of degree 4 in $\mathcal{V}_\beta$: $x^4 = yx$, $yx^3 = y^2$, $y^2x^2 = \alpha y x^3 + x p(x, y)$, $y^3 x = \gamma y x^3 + x q(x, y)$, $y^4 = \gamma y^2 x^2 + y q(x, y)$. Using (5.1)-(5.3) and the results for degrees 1, 2 and 3, we may now successively integrate these new relations to obtain $\beta_{i+j} = \int y^j x^i \, d\mu_{\mathcal{B}}$ $(i, j \geq 0,\ i + j = 4)$; for example, $\int x^4 d\mu_{\mathcal{B}} = \int yx d\mu_{\mathcal{B}} = \beta_{11} = \langle Y, X\rangle = \langle X^3, X\rangle = \beta_{40}$. Degrees 5 and 6 are treated similarly. □

Proof of Theorem 5.1. In view of Theorem 4.2(i) ⇔ (ii), it suffices to show that $\beta^{(6)}$ has a representing measure. The results in [CuFi6], [CuFi8] and [CuFi10] show that if $\mathcal{M}(n)$ is positive, recursively generated, satisfies $r \leq v$ and has a column relation of degree one or two, then $\beta^{(2n)}$ admits a representing measure. We may thus assume that $\mathcal{M}(2)$ is positive and invertible; indeed, positivity in $\mathcal{M}(3)$ implies that any dependence relation in the columns of $\mathcal{M}(2)$ extends to the columns of $\mathcal{M}(3)$ [CuFi4]. In particular, we may assume in the sequel that a basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(3)}$ includes $\{1, X, Y, X^2, YX, Y^2\}$.

Lemma 5.4 implies that ϕ_β is 1-1. As in Section 4, we may thus form $\mu_{\mathcal{B}}$, and as in the proof of Theorem 4.2, it suffices to show that $\mu_{\mathcal{B}}$ is interpolating for β. The proof of Proposition 5.5 shows that this is the case if $\mathcal{B} = \{1,\ X,\ Y,\ X^2,\ YX,\ Y^2,\ YX^2\}$. This proof shows, more generally, that $\mu_{\mathcal{B}}$ is interpolating if $\mathcal{B}$ contains $\{1, X, Y, X^2, YX, Y^2\}$ and there exist column relations of the form (5.2) and (5.3).

We consider next the case when $\mathcal{B} = \{1,\ X,\ Y,\ X^2,\ YX,\ Y^2,\ Y^2X\}$, with column relations $YX^2 = u(X,Y) + \gamma Y^2X$ ($\gamma \in \mathbb{R}$, deg $u \le 2$) and $Y^3 = \delta Y^2X + t(X,Y)$ ($\delta \in \mathbb{R}$, deg $t \le 2$). Let $h(x,y) := x^2y - u(x,y) - \gamma xy^2$, so that $h(X,Y) = 0$ and $\mathcal{V}_\beta \subseteq \{(x,y) \in \mathbb{R}^2 : y = x^3$ and $h(x,y) = 0\}$. If $\gamma = 0$, then $h(x,y) = 0$ has at most 6 real roots of the form (x, x^3), contradicting $r = v = 7$. Thus $\gamma \ne 0$, and we may derive a system as in (5.2)-(5.3); indeed, $Y^3 = \frac{\delta}{\gamma} YX^2 + (t - \frac{\delta}{\gamma} u)(X,Y)$. Using this system, we may now proceed as in the proof of Proposition 5.5 to conclude that $\mu_{\mathcal{B}}$ is interpolating. Finally, we consider the case $\mathcal{B} = \{1,\ X,\ Y,\ X^2,\ YX,\ Y^2,\ Y^3\}$, with relations

$$YX^2 = s(X,Y) + \delta Y^3 \quad (\delta \in \mathbb{R},\ \deg s \le 2) \tag{5.4}$$

and

$$Y^2X = t(X,Y) + \epsilon Y^3 \quad (\epsilon \in \mathbb{R},\ \deg t \le 2). \tag{5.5}$$

Since $h(x,y) := yx^2 - s(x,y)$ has at most 6 roots of the form (x, x^3), then $v = 7$ implies $\delta \ne 0$. We may now successively transform (5.4) and (5.5) into (5.2) and (5.3) and then apply the method of the proof of Proposition 5.5. $\square$

6. The Extremal Problem for $\mathcal{M}(3)$ with $Y = X^3 : r = v = 8$

In this section we study the extremal moment problem for a moment matrix $\mathcal{M}(3)$ satisfying

$$\mathcal{M}(3) \ge 0, \mathcal{M}(2) > 0, Y = X^3 \text{ in } \mathcal{C}_{\mathcal{M}(3)}, \text{ and } r = v = 8. \tag{6.1}$$

In Proposition 6.1 we illustrate (6.1) with the first example of an extremal moment matrix $\mathcal{M}(n)$, which admits a representing measure, but for which (i) the ideal $\mathcal{J}_\beta$ corresponding to ker $\mathcal{M}(n)$ is *not* a real ideal, and (ii) the unique flat extension $\mathcal{M}(n+1)$ is *not* a tight flat extension. In Theorem 6.2 we resolve Question 1.1 in the negative, by constructing a moment matrix $\mathcal{M}(3)$ which satisfies (6.1), but is not consistent, and thus admits no representing measure. In Theorem 6.3 we provide a simplified consistency test for moment matrices satisfying (6.1), thereby completing the analysis of the extremal moment problem for $\mathcal{M}(3)$ with $Y = X^3$ (cf. Remark 6.5(iii)).

We begin by introducing the objects that we will use in our examples. Let $f(x,y) := y - x^3$. Recall from Bezout's Theorem ([CLO, Theorem 8.7.10] that if deg $g = 3$, then f and g have exactly 9 common zeros (counting multiplicity), including complex zeros and zeros at infinity. To construct a variety that will serve as $\mathcal{V}(\mathcal{M}(3))$ in Proposition 6.1 and Theorem 6.2, we first seek a polynomial $g \in \mathbb{R}[x,y]$ of degree 3 such that f and g have exactly 8 distinct common real affine zeros, one of which is a zero of multiplicity 2. For this, let $\ell_i(x,y) = 0$ ($i = 1,2,3$) be lines in the plane such that ℓ_1 intersects $y = x^3$ in 3 distinct points ($(x_i, y_i), 1 \le i \le 3$), ℓ_2 intersects $y = x^3$ in 3 additional distinct points ($(x_i, y_i), 4 \le i \le 6$), and ℓ_3 intersects $y = x^3$ in 2 additional distinct points ($(x_i, y_i), 7 \le i \le 8$), such that ℓ_3 is the tangent line to $y = x^3$ at (x_8, y_8). Setting $g(x,y) := \ell_1(x,y)\ell_2(x,y)\ell_3(x,y)$, we have $\mathcal{V}((f,g)) = \{(x_i, y_i)\}_{i=1}^8$, and (x_8, y_8) is a common zero of f and g with multiplicity 2. Indeed, (x_8, y_8) is a multiple zero since $\ell_3(x,y) = 0$ is a common tangent line for $f(x,y) = 0$ and $g(x,y) = 0$ at (x_8, y_8); equivalently, there exist $a, b \in \mathbb{R}$ such that the differential $D : \mathcal{P} \to \mathbb{R}$ defined by

$$D(p) := a\frac{\partial p}{\partial x}(x_8, y_8) + b\frac{\partial p}{\partial y}(x_8, y_8) \tag{6.2}$$

satisfies $D(f) = D(g) = 0$ (cf. [CLO, Proposition 3.4.2], [MMM]). We next introduce some ideals which will be referenced in the sequel. Let $\mathcal{V} \equiv \mathcal{V}((f,g))$ $(= \{(x_i, y_i)\}_{i=1}^8)$ and set $\mathcal{A} := \mathcal{I}(\mathcal{V}) \equiv \{p \in \mathcal{P} : p|_{\mathcal{V}} \equiv 0\}$ and $\mathcal{D} := \{p \in \mathcal{A} : D(p) = 0\}$; $\mathcal{A}$ is a real ideal (cf. Section 2), and $\mathcal{D}$ is an ideal (which contains f and g). For the last assertion, note that if $p \in \mathcal{D}$ and $q \in \mathcal{P}$, then $(pq)|_{\mathcal{V}} \equiv 0$ and $D(pq) = q(x_8, y_8)D(p) + p(x_8, y_8)D(q) = 0$ (since $D(p) = 0$ and $p|_{\mathcal{V}} \equiv 0$).

As we show below, the conditions of (6.1) imply that $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2, Y^2X\}$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$, so we will further require that the points of $\mathcal{V}$ are in "general position" relative to the monomials $1, x, y, x^2, yx, y^2, yx^2$ and y^2x, i.e., we will require that $V \equiv V_{\mathcal{B}}[\mathcal{V}]$ is invertible (cf. Lemma 4.1). Let $W \equiv W_{\mathcal{B}}[\mathcal{V}] := V^T$. Now, if $H(x,y)$ is any real-valued function defined on $\mathcal{V}$, then there exist scalars $\alpha_1, ..., \alpha_8 \in \mathbb{R}$ such that

$$H(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 yx + \alpha_6 y^2 + \alpha_7 yx^2 + \alpha_8 y^2 x \ \ ((x,y) \in \mathcal{V});$$

indeed, $\alpha \equiv (\alpha_1, ..., \alpha_8)$ is uniquely determined from

$$\alpha^T = W^{-1}(H(x_1, y_1), .., H(x_8, y_8))^T. \tag{6.3}$$

In particular, there exist unique real numbers $a_1, ..., a_8$ such that

$$h(x,y) := y^2x^2 - (a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 yx + a_6 y^2 + a_7 yx^2 + a_8 y^2 x) \tag{6.4}$$

vanishes on $\mathcal{V}$.

For the sake of definiteness, let

$$\ell_1(x,y) := y - 4x$$
$$((x_1,y_1) = (-2,-8), (x_2,y_2) = (0,0), (x_3,y_3) = (2,8)),$$

$$\ell_2(x,y) := y - 4x + 3$$
$$((x_4,y_4) = (1,1), (x_5,y_5) = (-\frac{1}{2} + \frac{\sqrt{13}}{2}, -5 + 2\sqrt{13}), \tag{6.5}$$
$$(x_6,y_6) = (-\frac{1}{2} - \frac{\sqrt{13}}{2}, -5 - 2\sqrt{13})),$$

$$\ell_3(x,y) := y - \frac{3}{4}x + \frac{1}{4}$$
$$((x_7,y_7) = (-1,-1), (x_8,y_8) = (\frac{1}{2}, \frac{1}{8})).$$

Then

$$\begin{aligned} 4g(x,y) &= 4(y-4x)(y-4x+3)(y - \frac{3}{4}x + \frac{1}{4}) \\ &= -48x^3 + 88yx^2 - 35y^2x + 4y^3 + 52x^2 - 65yx + 13y^2 - 12x + 3y. \end{aligned} \tag{6.6}$$

A calculation shows that ℓ_3 is tangent to both f and g at $(x_8 y_8)$; indeed, $D(f) = D(g) = 0$, where D is the functional given by (6.2) with $a = 1, b = \frac{3}{4}$. Further, $\det V = \frac{98415}{4}\sqrt{13}$ $(\neq 0)$, so *rank* $V = 8$. Applying (6.3) with $H(x,y) = y^2x^2$, we see that in (6.4) we have

$$h(x,y) = y^2x^2 + 6x - 14x^2 - \frac{11}{2}y + \frac{43}{2}yx - yx^2 - \frac{17}{2}y^2 + \frac{1}{2}y^2x, \tag{6.7}$$

and $h|_{\mathcal{V}} \equiv 0$. A calculation shows that $D(h) = -\frac{405}{128}$ $(\neq 0)$, so $h \in (\mathcal{A} \bigcap \mathcal{P}_4) \setminus \mathcal{D}$.

Proposition 6.1. *Let* $\mu := \sum_{i=1}^{8} \delta_{(x_i,y_i)}$ *(with* (x_i,y_i) *from (6.5)) and let* $\mathcal{M}(3) := \mathcal{M}(3)[\mu]$. *Then* $\mathcal{M}(3)$ *satisfies (6.1) and has the following additional properties:*
(i) The ideal $\mathcal{J}_{\beta^{(6)}}$ *generated by* $\mathcal{N}_3 \equiv \{p \in \mathcal{P}_3 : \mathcal{M}(3)\hat{p} = 0\}$ *is not a real ideal;*
(ii) $\mathcal{M}(3)$ *has a flat extension, but* $\mathcal{M}(3)$ *does not admit a tight flat extension.*

Proof. A direct calculation using the points in (6.5) shows that $V_{\mathcal{B}}[\mathcal{V}]$ is invertible, so it follows as in the proof of Lemma 4.1 that $\mathcal{B}$ is independent in $\mathcal{C}_{\mathcal{M}(3)}$. Also, since μ is a representing measure for $\mathcal{M}(3)[\mu]$, *supp* $\mu = \mathcal{V}$, $f|_{\mathcal{V}} \equiv 0$ and $g|_{\mathcal{V}} \equiv 0$, we have $Y = X^3$ and $g(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$, whence $\mathcal{B}$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$, *rank* $\mathcal{M}(3) = 8$, and $\mathcal{V}(\mathcal{M}(3)) = \mathcal{Z}(f) \bigcap \mathcal{Z}(g) = \mathcal{V}$. Thus, $\mathcal{M}(3)$ satisfies (6.1).

(i) Let $\mathcal{J} \equiv \mathcal{J}_{\beta^{(6)}}$ denote the ideal generated by $\{p \in \mathcal{P}_3 : \mathcal{M}(3)\hat{p} = 0\}$, so that $\mathcal{J} = (f,g)$. We claim that $\mathcal{J}$ is not a real ideal. For, otherwise, there would exist

$G \subseteq \mathbb{R}^2$ such that for $p \in \mathcal{P}$, $p|_G \equiv 0 \iff p \in \mathcal{J}$ (cf. Section 2). In this case, since $f^2 + g^2 \in \mathcal{J}$, then $(f^2+g^2)|_G \equiv 0$, whence $G \subseteq \mathcal{V}$. Recall that the function h given by (6.7) satisfies $h|_{\mathcal{V}} \equiv 0$ and $D(h) \neq 0$. Since $p \in \mathcal{J} \Rightarrow D(p) = 0$, we see that $h \notin \mathcal{J}$; but since $h|_{\mathcal{V}} \equiv 0$, then $h|_G \equiv 0$, contradicting the defining property of G. Thus, $\mathcal{J}$ is not a real ideal.

(ii) Since $\mathcal{M}(3)$ is extremal and has a representing measure (that is, μ), it has a unique flat extension $\mathcal{M}(4)$, namely $\mathcal{M}(4)[\mu]$. Since $h|_{\mathcal{V}} \equiv 0$, we have $h(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(4)}$ [CuFi2, Proposition 3.1], so $h \in \mathcal{N}_4$. Since we have shown in the proof of (i) that $h \notin \mathcal{J}$, we must have $\mathcal{J} \bigcap \mathcal{P}_4 \neq \mathcal{N}_4$, so $\mathcal{M}(4)$ is not a tight flat extension. $\square$

We next present an example of $\mathcal{M}(3)$ satisfying (6.1), but not consistent, so that $\beta^{(6)}$ has no representing measure; this provides a negative answer to Question 1.1. We define a linear functional $L : \mathcal{P}_6 \to \mathbb{R}$ by

$$L(p) := a_0 D(p) + \sum_{i=1}^{8} a_i p(x_i, y_i) \quad (p \in \mathcal{P}_6) \tag{6.8}$$

(with D and $\{(x_i,y_i)\}_{i=1}^8$ as defined just previous to Proposition 6.1, and $a_i \in \mathbb{R}$ $(0 \le i \le 8)$). Let $\beta^{(6)}$ be the sequence corresponding to L, i.e., $\beta_{ij} := L(x^i y^j)$ $(i, j \geq 0, i+j \leq 6)$. Let $M \equiv \mathcal{M}(3)$ be the corresponding moment matrix, which is real symmetric since

$$\left\langle M\widehat{x^i y^j}, \widehat{x^k y^\ell}\right\rangle = L(x^{i+k} y^{j+\ell}) = \left\langle M\widehat{x^k y^\ell}, \widehat{x^i y^j}\right\rangle.$$

Recall $f(x,y) := y - x^3$ and note that $f(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$. Indeed, for $p \in \mathcal{P}_3$,

$$\langle f(X,Y), \hat{p}\rangle = \left\langle \mathcal{M}(3)\hat{f}, \hat{p}\right\rangle = L(fp) = 0$$

(since $D(f) = 0$ and $f|_{\mathcal{V}} \equiv 0$). Similarly, for g as defined earlier, since $D(g) = 0$ and $g|_{\mathcal{V}} \equiv 0$, we have $g(X,Y) = 0$. For the sake of definiteness, let $a_i := 1$ $(0 \le i \le 7)$.

Theorem 6.2. *There exists* α ($\cong 6.97093$) *such that if* $a_8 > \alpha$, *then* $\mathcal{M}(3)$ *satisfies (6.1) (and is thus positive, recursively generated, and extremal), but* $\beta^{(6)}$ *has no representing measure. In particular,* ϕ_β *is* 1-1, *but* β *is not consistent.*

Proof. Consider $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2, Y^2X\}$. Since $Y = X^3$ and $g(X,Y) = 0$, $\mathcal{B}$ spans $\mathcal{C}_{\mathcal{M}(3)}$. It follows from Smul'jan's Theorem that $\mathcal{M}(3)$ is positive semi-definite if and only if $M_{\mathcal{B}}$, the compression of $\mathcal{M}(3)$ to rows and columns indexed by $\mathcal{B}$, is positive semi-definite. Calculating nested determinants, we see that $M_{\mathcal{B}}$ is positive definite if and only if $a_8 > \alpha$, where $\alpha := \frac{6012817451}{862617600}$. In this case, since $M_8 > 0$ and $f(X,Y) = 0 = g(X,Y)$, it follows that *rank* $\mathcal{M}(3) = 8$ and $\mathcal{V}(\mathcal{M}(3)) = \mathcal{Z}(f) \bigcap \mathcal{Z}(g) = \mathcal{V}$. In particular, $\mathcal{M}(3)$ satisfies (6.1) (and is thus also recursively generated). Further, ϕ_β is 1-1 (see the proof of Proposition 6.1, or use Lemma 6.4 below). We claim that $\beta^{(6)}$ is not consistent. Indeed, the Riesz functional for $\beta^{(6)}$ is L.

The function h from (6.6) satisfies $h|_{\mathcal{V}} \equiv 0$ and $D(h) \neq 0$, whence $L(h) = D(h) \neq 0$. Now β is not consistent and thus has no representing measure. $\square$

In view of Theorem 4.2, the existence of a representing measure in the extremal moment problem (6.1) is equivalent to establishing that the Riesz functional Λ_β vanishes on a basis for $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})$, and we will show below that dim $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) = 20$. The substance of the next result is that, following (2.4) and the remarks following (2.4), the test for consistency in (6.1) can be reduced to checking that $\Lambda_\beta(h) = 0$ for h given by (6.4).

Theorem 6.3. *Suppose $\mathcal{M}(3)$ satisfies (6.1), with $\mathcal{V}(\mathcal{M}(3)) = \{(x_i, y_i)\}_{i=1}^8$ and column basis $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2, Y^2X\}$. Let h be as in (6.4). Then $\beta^{(6)}$ has a representing measure if and only if $\Lambda_\beta(h) = 0$.*

We require the following preliminary result.

Lemma 6.4. *If $Y = X^3$ in $\mathcal{C}_{\mathcal{M}(3)}$ and $r \leq v = 8$, then ϕ_β is 1-1.*

Proof. Let $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(3))$ and for $f \in \mathcal{K}_3 := \mathcal{P}_3 \bigcap \mathcal{I}(\mathcal{V})$, write

$$
\begin{aligned}
f(x,y) \;\equiv\;& a_1 + a_2x + a_3y + a_4x^2 + a_5yx + a_6y^2 \\
&+ a_7x^3 + a_8yx^2 + a_9y^2x + a_{10}y^3.
\end{aligned} \tag{6.9}
$$

Define a linear map $\Psi : \mathcal{K}_3 \to \mathbb{R}^2$ by $\Psi(f) := (a_7, a_{10})$. We claim that Ψ is 1-1. Suppose $a_7 = a_{10} = 0$ and define

$$p(x) := f(x, x^3) \equiv a_1 + a_2x + a_4x^2 + a_3x^3 + a_5x^4 + a_8x^5 + a_6x^6 + a_9x^7.$$

Since $\mathcal{V} \subseteq \mathcal{Z}(y - x^3)$, the eight points of $\mathcal{V}$ have distinct x-coordinates, and $f|_{\mathcal{V}} \equiv 0$, it follows that p has at least 8 distinct real roots. Since deg $p \leq 7$, we must have $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_8 = a_9 = 0$, whence $f \equiv 0$, so Ψ is 1-1. Now $\dim \mathcal{K}_3 \leq \dim \mathbb{R}^2 = 10 - 8 = \dim \mathcal{P}_3 - v$, so Lemma 5.3 implies that ϕ_β is 1-1. $\square$

Proof of Theorem 6.3. If $\beta \equiv \beta^{(6)}$ has a representing measure, then β is consistent, and since $h \in \mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})$, it follows that $\Lambda_\beta(h) = 0$. For the converse, we suppose that $\Lambda_\beta(h) = 0$ and we will show that β is consistent, i.e., $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \subseteq \ker\ \Lambda_\beta$ (cf. Theorem 4.2). To this end, we first compute $\dim(\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}))$. Consider $W \equiv W_6[\mathcal{V}]$ (cf. Section 2); clearly, $p \in \mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \iff \hat{p} \in \ker\ W$, so $\dim(\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})) = \dim\ \ker\ W = \dim\ \mathcal{P}_6 - \mathit{rank}\ W$. Lemma 6.4 shows that ϕ_β is 1-1, so Lemma 4.1 implies that $W_{\mathcal{B}}[\mathcal{V}]$ ($\equiv V_{\mathcal{B}}[\mathcal{V}]^T$) is invertible. Now $W_{\mathcal{B}}[\mathcal{V}]$ is the compression of W to columns indexed by the monomials corresponding to elements of $\mathcal{B}$, so $8 \geq$ row rank $W = \mathit{rank}\ W \geq \mathit{rank}\ W_{\mathcal{B}}[\mathcal{V}] = 8$, whence $\mathit{rank}\ W = 8$. Thus,

$$\dim(\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})) = \dim\ \mathcal{P}_6 - \mathit{rank}\ W = 28 - 8 = 20.$$

Let $f(x,y) := y - x^3$, so that $\mathcal{M}(3)\hat{f} = f(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$. Also, there exist $a, b \in \mathbb{R}$ and $p \in \mathcal{P}_2$ such that $g(x,y) := y^3 + ayx^2 + by^2x + p(x,y)$ satisfies

$\mathcal{M}(3)\hat{g} = g(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$. Since $r = 8$, it follows that $\mathcal{V} = \mathcal{Z}(f) \bigcap \mathcal{Z}(g)$, and clearly $f, g \in \mathcal{I}(\mathcal{V})$. Now, if $s, t \in \mathcal{P}_3$, then $sf + tg \in \mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})$ and $\Lambda_\beta(sf + tg) = \left\langle \mathcal{M}(3)\hat{f}, \hat{s} \right\rangle + \left\langle \mathcal{M}(3)\hat{g}, \hat{t} \right\rangle = 0$, whence $sf + tg \in \ker\ \Lambda_\beta$ (see the remarks following (2.4)).

We next identify 19 linearly independent elements of $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})$ of the form $sf + tg$ $(s, t \in \mathcal{P}_3)$. Consider the following 20 polynomials:

$$
\begin{array}{cc}
f_1 := g \equiv y^3 + ayx^2 + by^2x + p(x,y) & f_2 := f \equiv y - x^3 \\
f_3 := xg \equiv y^3x + ayx^3 + by^2x^2 + xp(x,y) & f_4 := xf \equiv yx - x^4 \\
f_5 := yg \equiv y^4 + ay^2x^2 + by^3x + yp(x,y) & f_6 := yf \equiv y^2 - yx^3 \\
f_7 := x^2g \equiv y^3x^2 + ayx^4 + by^2x^3 + x^2p(x,y) & f_8 := x^2f \equiv yx^2 - x^5 \\
f_9 := yxg \equiv y^4x + ay^2x^3 + by^3x^2 + yxp(x,y) & f_{10} := yxf \equiv y^2x - yx^4 \\
f_{11} := y^2g \equiv y^5 + ay^3x^2 + by^4x + y^2p(x,y) & f_{12} := y^2f \equiv y^3 - y^2x^3 \\
f_{13} := x^3g \equiv y^3x^3 + ayx^5 + by^2x^4 + x^3p(x,y) & f_{14} := x^3f \equiv yx^3 - x^6 \\
f_{15} := yx^2g \equiv y^4x^2 + ay^2x^4 + by^3x^3 + yx^2p(x,y) & f_{16} := yx^2f \equiv y^2x^2 - yx^5 \\
f_{17} := y^2xg \equiv y^5x + ay^3x^3 + by^4x^2 + y^2xp(x,y) & f_{18} := y^2xf \equiv y^3x - y^2x^4 \\
f_{19} := y^3g \equiv y^6 + ay^4x^2 + by^5x + y^3p(x,y) & f_{20} := y^3f \equiv y^4 - y^3x^3.
\end{array}
$$

We assert that $\mathcal{F} := \{f_i\}_{i=1}^{19}$ is linearly independent in $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \bigcap \ker\ \Lambda_\beta$. For $1 \le k \le 19$, set $\mathcal{F}_k := \{f_i\}_{i=1}^{k}$. Proceeding inductively, let $2 \le k \le 19$ and assume that $\mathcal{F}_{k-1}$ is linearly independent. Observe that, except when $k = 6, 10, 12, 16, 18$, f_k contains a monomial of highest degree that does not appear in any polynomial in $\mathcal{F}_{k-1}$, whence $\mathcal{F}_k$ is linearly independent. In the remaining cases, note that
(i) f_k contains a monomial of highest degree that also appears, among the elements of $\mathcal{F}_k$, only in f_{k-2};
(ii) f_{k-2} has a different monomial that also appears, among the elements of $\mathcal{F}_k$, only in f_{k-1};
(iii) f_{k-1} has a monomial of highest degree that appears in no other element of $\mathcal{F}_k$.
We thus see that $\mathcal{F}_k$ is independent in these cases. (Observe also that $\mathcal{F}_{20}$ is dependent, since

$$f_{20} = -f_{13} - af_{16} - bf_{18} + f_5 + p(y - x^3),$$

and $p(y - x^3) = pf_2 \in \langle f_2, f_4, f_6, f_8, f_{10}, f_{12} \rangle$.)

Now, $\dim[\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \bigcap \ker\ \Lambda_\beta] \ge 19$ and $\dim(\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})) = 20$. Since $h \in \mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \bigcap \ker\ \Lambda_\beta$, to complete the proof that $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \subseteq \ker\ \Lambda_\beta$, it suffices to verify that $h \notin \langle \{f_i\}_{i=1}^{19} \rangle$. Let $2 \le k \le 19$ and assume by induction that $h \notin \left\langle \{f_i\}_{i=1}^{k-1} \right\rangle$. Consider a linear combination $q := \alpha_1 f_1 + \cdots + \alpha_k f_k$, with $\alpha_k \ne 0$. Except when $k = 6, 10, 12, 16, 18$, f_k contains a monomial term of highest degree that does not appear in h or in any element of $\mathcal{F}_{k-1}$, so $q \ne h$. In the remaining cases, if $q = h$, then proceeding as in the proof that $\mathcal{F}$ is independent, we see that $\alpha_{k-2} \ne 0$, and then that $\alpha_{k-1} \ne 0$. Now f_{k-1} contains a monomial of highest degree that does not appear in

h or in any other element of $\mathcal{F}_k$, so we arrive at a contradiction. Thus $q \neq h$ in these cases also. Now, following (2.4), $\{f_i\}_{i=1}^{19} \bigcup \{h\}$ forms a basis for $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \bigcap \ker\ \Lambda_\beta$, whence $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \subseteq \ker\ \Lambda_\beta$, so β is consistent. The proof is now complete. □

Remark 6.5. (i) If the points of the variety $\{(x_i, y_i)\}_{i=1}^{8}$ are known explicitly, then h can be computed as in (6.4). In this case, Theorem 6.3 provides an effective test for the existence of a representing measure in the extremal problem (6.1). If the points of the variety are not known explicitly, there is still available a concrete test for the non-existence of a representing measure, as follows:
If $\mathcal{M}(3)$ (as in (6.1)) has a representing measure, then there is a unique flat extension $\mathcal{M}(4)$, and $\mathcal{V}(\mathcal{M}(4)) = \mathcal{V}(\mathcal{M}(3)) = \mathcal{V}$. In this case, there is a column relation in $\mathcal{C}_{\mathcal{M}(4)}$ of the form

$$Y^2X^2 = \alpha_1 1 + \alpha_2 X + \alpha_3 Y + \alpha_4 X^2 + \alpha_5 YX + \alpha_6 Y^2 + \alpha_7 YX^2 + \alpha_8 Y^2X.$$

To compute $\alpha_1, \cdots, \alpha_8$, let $\mathbf{v}$ denote the compression of column Y^2X^2 in $\mathcal{M}(4)$ to rows indexed by the basis $\mathcal{B}$, i.e., $\mathbf{v} : = (\beta_{22}, \beta_{32}, \beta_{23}, \beta_{42}, \beta_{33}, \beta_{24}, \beta_{43}, \beta_{34})^T$. Since $\mathcal{M}(4)$ is recursively generated, we have $X^4 = YX$ and $YX^3 = Y^2$ in $\mathcal{C}_{\mathcal{M}(4)}$, whence $\beta_{43} = \beta_{14}$ and $\beta_{34} = \beta_{05}$. Thus $\mathbf{v}$ is expressed in terms of the original data from $\beta^{(6)}$. Let J denote the compression of $\mathcal{M}(3)$ to rows and columns indexed by elements of $\mathcal{B}$; then J is invertible and $\alpha := (\alpha_1, \cdots, \alpha_8)$ is uniquely determined by

$$\alpha^T = J^{-1}\mathbf{v}. \tag{6.10}$$

Now let

$$k(x, y) := y^2x^2 - (\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 yx + \alpha_6 y^2 + \alpha_7 yx^2 + \alpha_8 y^2 x.$$

Since $k(X, Y) = 0$ in $\mathcal{C}_{\mathcal{M}(4)}$, then $k|_{\mathcal{V}} \equiv 0$, so it follows from (6.3) and (6.4) that $k \equiv h$, whence $\Lambda_\beta(k) = 0$. Thus, if k is computed as above (using (6.10)) and $\Lambda_\beta(k) \neq 0$, then β has no representing measure.
(ii) Let $k(x, y)$ be computed as above. Even without knowing the points of $\mathcal{V}$ explicitly, if we know that $k|_{\mathcal{V}} \equiv 0$, then from Lemma 6.4, Lemma 4.1, and (6.4) it follows that $k = h$, so β has representing measure if and only if $\Lambda_\beta(k) = 0$.
(iii) Finally, we note that for the extremal problem for $\mathcal{M}(3)$ with $Y = X^3$, $\mathcal{M}(3) \geq 0$, $\mathcal{M}(2) > 0$ and $r = v = 8$, we can always assume that $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2, Y^2X\}$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$. Indeed, suppose that a maximal linearly independent set of columns is $\{1, X, Y, X^2, YX, Y^2, YX^2, Y^3\}$. Then there is a column relation of the form $Y^2X = \alpha_1 YX^2 + \alpha_2 Y^3 + p(X, Y)$ (deg $p \leq 2$). If $\alpha_2 = 0$, then (since $Y = X^3$), $\mathcal{V}(\mathcal{M}(3))$ is a subset of the zeros of $x^7 = \alpha_1 x^5 + p(x, x^3)$, whence $v \leq 7$, a contradiction. Thus, $\alpha_2 \neq 0$, and since $r = 8$, it follows that $\mathcal{B}$ is a basis. A similar argument can be used in the case when $\{1, X, Y, X^2, YX, Y^2, Y^2X, Y^3\}$ is a basis. This completes the analysis of the extremal problem (6.1).

7. An Example with $r < v < +\infty$

In this section we present an example in which we solve a truncated moment problem with $r < v < +\infty$. Based on a number of examples and results in [CuFi4], [CuFi8] and [CuFi10], we conjecture that in such cases, if $\mathcal{M}(n)(\beta)$ has a representing measure, then a minimal representing measure is v-atomic, and corresponds to a rank-v positive extension $\mathcal{M}(n+k)$ (for some $k \leq v - r$), followed by a flat extension $\mathcal{M}(n+k+1)$. In [Fia4] we present an algorithm for determining the existence of representing measures in a broad class of truncated moment problems with $r < v < +\infty$; the following example may be viewed as an instance of this algorithm, and also illustrates Proposition 3.6.

Example 7.1. Consider

$$\mathcal{M}(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & 2000 \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & 2000 & 338881 \end{pmatrix}.$$

We have $\mathcal{M}(3) \geq 0$, $\mathcal{M}(2) > 0$ (positive and invertible), and $r := rank\ \mathcal{M}(3) = 8$, with column relations

$$Y = X^3, \tag{7.1}$$

and

$$Y^3 = q(X, Y), \tag{7.2}$$

where $q(x, y) := -2285x + 5720y - 34441yx^2 + 578y^2x$. Let $r_1(x, y) := y - x^3$ and $r_2(x, y) := y^3 + 2285x - 5720y + 3441x^2y - 578xy^2$. Then $\ker \mathcal{M}(3) = \langle \hat{r}_1,\ \hat{r}_2 \rangle$ and $\mathcal{V}_\beta \equiv \{(x, y) \in \mathbb{R}^2 : r_1(x, y) = r_2(x, y) = 0\}$. A calculation shows that $v := card\ \mathcal{V}_\beta = 9$. Now $\mathcal{M}(3)$ is positive, recursively generated (trivially, because $\mathcal{M}(2)$ is invertible), and $r < v$; further, Proposition 3.6 implies that $\beta^{(6)}$ is consistent. We will show that the minimal representing measure for $\beta^{(6)}$ is v-atomic (cf. Question 1.2).

If μ is a finitely atomic representing measure for β, then $\mathcal{M}(4)[\mu]$ is recursively generated [CuFi4]. Conversely, any recursively generated extension $\mathcal{M}(4)$ of $\mathcal{M}(3)$ must satisfy

$$X^4 = YX \tag{7.3}$$

and

$$YX^3 = Y^2. \tag{7.4}$$

Further, since $Y^3 = q(X,Y)$, in the column space of $\mathcal{M}(4)$ we must have

$$Y^3X = (xq)(X,Y) = q(X,Y)X \tag{7.5}$$

and

$$Y^4 = (yq)(X,Y) = Yq(X,Y). \tag{7.6}$$

Using these column relations, we see that $\mathcal{M}(4)$ is completely defined (i.e., all moments of degrees 7 and 8 are determined). On the other hand, a calculation shows that in $\mathcal{C}_{\mathcal{M}(4)}$, Y^2X^2 is independent of $\mathcal{B} := \{1,\ X,\ Y,\ X^2,\ XY,\ Y^2,\ X^2Y,\ XY^2\}$. Thus, *rank* $\mathcal{M}(4) = 9$. Since a flat extension of a positive, recursively generated moment matrix is necessarily recursively generated [CuFi4], it follows that there is no flat extension $\mathcal{M}(4)$ of $\mathcal{M}(3)$, and thus there is no 8-atomic representing measure for β.

Note that $\mathcal{M}(4)$ is extremal; indeed, the variety of $\mathcal{M}(4)$ consists of the common zeros of r_1, r_2, xr_1, yr_1, xr_2, and yr_2, and thus coincides with $\mathcal{V}_\beta$ (which has 9 points). Rather than using Theorem 4.2, we will show that $\mathcal{M}(4)$ has a unique, 9-atomic, representing measure by a direct construction. Observe that relations (7.3)-(7.6), together with recursiveness, completely determine any recursively generated extension $\mathcal{M}(5)$ via the following relations: $X^5 = YX^2$, $YX^4 = Y^2X$, $Y^2X^3 = Y^3$, $Y^3X^2 = (x^2q)(X,Y)$, $Y^4X = (yxq)(X,Y)$, $Y^5 = (y^2q)(X,Y)$. A calculation of the degree-5 columns using these relations shows that these columns do fit together to form a moment matrix, which is clearly a flat (i.e., rank-preserving) extension of $\mathcal{M}(4)$. It thus follows from [CuFi2, Corollary 5.14] that $\mathcal{M}(5)$ has a unique representing measure μ, which is 9-atomic. Further, from the recursive definition of $\mathcal{M}(5)$, it follows that μ is the unique representing measure for β. □

References

[AhKr] N.I. Ahiezer and M. Krein, *Some Questions in the Theory of Moments*, Transl. Math. Monographs, vol. 2, American Math. Soc., Providence, 1962.

[Akh] N.I. Akhiezer, *The Classical Moment Problem*, Hafner Publ. Co., New York, 1965.

[Atk] K. Atkinson, *Introduction to Numerical Analysis*, Wiley and Sons, 2nd. Ed. 1989.

[BaTe] C. Bayer and J. Teichmann, The proof of Tchakaloff's Theorem, preprint 2005.

[CLO] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Second Edition, Springer-Verlag, New York, 1992.

[CuFi1] R. Curto and L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, Integral Equations Operator Theory 17(1993), 202-246.

[CuFi2] R. Curto and L. Fialkow, Solution of the truncated complex moment problem with flat data, *Memoirs Amer. Math. Soc.* no. 568, Amer. Math. Soc., Providence, 1996.

[CuFi3] R. Curto and L. Fialkow, Flat extensions of positive moment matrices: Relations in analytic or conjugate terms, *Operator Th.: Adv. Appl.* 104(1998), 59-82.

[CuFi4] R. Curto and L. Fialkow, Flat extensions of positive moment matrices: Recursively generated relations, *Memoirs Amer. Math. Soc.* no. 648, Amer. Math. Soc., Providence, 1998.

[CuFi5] R. Curto and L. Fialkow, The truncated complex K-moment problem, *Trans. Amer. Math. Soc.* 352(2000), 2825-2855.

[CuFi6] R. Curto ad L. Fialkow, The quadratic moment problem for the unit disk and unit circle, *Integral Equations Operator Theory* 38(2000), 377-409.

[CuFi7] R. Curto and L. Fialkow, Solution of the singular quartic moment problem, *J. Operator Theory* 48(2002), 315-354.

[CuFi8] R. Curto and L. Fialkow, Solution of the truncated parabolic moment problem, *Integral Equations Operator Theory* 50(2004), 169-196.

[CuFi9] R. Curto and L. Fialkow, A duality proof of Tchakaloff's theorem, *J. Math. Anal. Appl.* 269(2002), 519-532.

[CuFi10] R. Curto and L. Fialkow, Solution of the truncated hyperbolic moment problem, *Integral Equations Operator Theory* 52(2005), 181-218.

[CuFi11] R. Curto and L. Fialkow, Truncated K-moment problems in several variables, *J. Operator Theory 54(2005), 189-226.*

[Dou] R.G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert spaces, *Proc. Amer. Math. Soc.* 17(1966), 413-415.

[Fia1] L. Fialkow, Positivity, extensions and the truncated complex moment problem, *Contemporary Math.* 185(1995), 133-150.

[Fia2] L. Fialkow, Minimal representing measures arising from rank-increasing moment matrix extensions, *J. Operator Theory* 42(1999), 425-436.

[Fia3] L. Fialkow, Truncated complex moment problems with a $Z\bar{Z}$ relation, *Integral Equations Operator Theory* 45(2003), 405-435.

[Fia4] L. Fialkow, Extensions and varieties of moment matrices, in preparation.

[FiPe] L. Fialkow and S. Petrovic, A moment matrix approach to multivariable cubature, *Integral Equations Operator Theory* 52(2005), 85-124.

[Ful] W. Fulton, *Algebraic curves. An introduction to algebraic geometry,* Mathematics Lecture Notes Series. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[KrNu] M.G. Krein and A.A. Nudel'man, *The Markov Moment Problem and Extremal Problems*, *Transl. Math. Monographs,* vol. 50, American Mathematical Society, Providence, R.I., 1977.

[KuMa] S. Kuhlmann and M. Marshall, Positivity, sums of squares and the multidimensional moment problem, *Trans. Amer. Math. Soc.* 354(2002), 4285-4301.

[Lau1] M. Laurent, Semidefinite representations for finite varieties, preprint 2002.

[Lau2] M. Laurent, Revisiting two theorems of Curto and Fialkow on moment matrices, *Proc. Amer. Math. Soc.* 133 (2005), 2965-2976.

[MMM] M.G. Marinari, H.M. Möller, and T. Mora, On multiplicities in polynomial system solving, *Trans. Amer. Math. Soc.* 348(1996), 3283 – 3321.

[Moe1] H.M. Möller, An inverse problem for cubature formulae, *Computat. Technol.* 9 (2004), 13 – 20.

[Moe2] H.M. Möller, On square positive extensions and cubature formulas, to appear in *J. Comput. Applied Math.*, Special issue ed. by W. zu Castell and F. Filbir.

[MoSa] H.M. Möller and T. Sauer, H-bases for polynomial interpolation and system solving. Multivariate polynomial interpolation. *Adv. Comput. Math.* 12 (2000), 335–362.

[Mys] I.P. Mysovskikh, On Chakalov's theorem, *U.S.S.R. Comput. Math. and Math. Phys.* 15 (1975), 221–227; (translation in *Zh. Vychisl. Mat. i Mat. Fiz.* 15 (1975), 1589–1593).

[PoSc] V. Powers and C. Scheiderer, The moment problem for non-compact semialgebraic sets, *Advances in Geometry* 1(2001), 71-88.

[Put] M. Putinar, On Tchakaloff's Theorem, *Proc. Amer. Math. Soc.* 125(1997), 2409-2414.

[PuVa] M. Putinar and F.-H. Vasilescu, Solving moment problems by dimensional extension, *Ann. of Math.* (2) 149(1999), no. 3, 1087-1107.

[Sche] C. Scheiderer, Sums of squares of regular functions on real algebraic varieties, *Trans. Amer. Math. Soc.* 352(2000), 1039-1069.

[Schm1] K. Schmüdgen, The K-moment problem for semi-algebraic sets, *Math. Ann.* 289(1991), 203-206.

[Schm2] K. Schmüdgen, On the moment problem of closed semi-algebraic sets, *J. Reine Angew. Math.* 558 (2003), 225–234.

[ShTa] J.A. Shohat and J.D. Tamarkin, *The Problem of Moments*, Math. Surveys I, American Math. Soc., Providence, 1943.

[Smu] J.L. Smul'jan, An operator Hellinger integral (Russian), *Mat. Sb.* 91(1959), 381-430.

[Sto1] J. Stochel, Moment functions on real algebraic sets, *Ark. Mat.* 30(1992), 133-148.

[Sto2] J. Stochel, Solving the truncated moment problem solves the moment problem, *Glasgow J. Math.* 43(2001), 335-341.

[StSz1] J. Stochel and F.H. Szafraniec, Algebraic operators and moments on algebraic sets, *Portugal. Math.* 51(1994), 25-45.

[StSz2] J. Stochel and F.H. Szafraniec, The complex moment problem and subnormality: A polar decomposition approach, *J. Funct. Anal.* 159(1998), 432-491.

[Tch] V. Tchakaloff, Formules de cubatures mécaniques à coefficients non négatifs, *Bull. Sc. Math.* 81(1957).

[Wol] Wolfram Research, Inc., *Mathematica*, Version 4.2, Wolfram Research, Inc., Champaign, IL, 2002.

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