SOLUTION OF THE TRUNCATED COMPLEX MOMENT PROBLEM FOR FLAT DATA

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Abstract

We introduce a matricial approach to the truncated complex moment problem, and apply it to the case of moment matrices of *flat* data type, for which the columns corresponding to the homogeneous monomials in z and $\bar{z}$ of highest degree can be written in terms of monomials of lower degree. Necessary and sufficient conditions for the existence and uniqueness of representing measures are obtained in terms of positivity and extension criteria for moment matrices. We discuss the connection between complex moment problems and the subnormal completion problem for 2-variable weighted shifts, and present in detail the construction of solutions for truncated complex moment problems associated with monomials of degrees one and two. Finally, we present generalizations to multivariable moment and completion problems.

For Inés, Carina, Roxanna and Vilsa

and

For Deborah, Velaho, Ruth, Jarmel, Tawana and Atiba

CHAPTER 1

Introduction

Given a doubly indexed finite sequence of complex numbers $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the *truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \leq i + j \leq 2n); \tag{1.1}$$

μ is called a *representing measure* for γ. In this paper we study truncated moment problems (in one or several variables) using an approach based on matrix positivity and extension, combined with a new "functional calculus" for the columns of the associated moment matrix. We show that when the truncated moment problem is of *flat data type* (a notion which generalizes the concept of recursiveness for positive Hankel matrices), a solution always exists; this is compatible with our previous results for measures supported on the real line, nonnegative real line, or some prescribed finite interval (Hamburger, Stieltjes and Hausdorff truncated moment problems [**CF3**]), and for measures supported on the circle (Toeplitz truncated moment problems [**CF3**]). Along the way we develop new machinery for analyzing truncated moment problems in one or several real or complex variables.

The *full* complex moment problem (in which γ is defined as an *infinite moment sequence* $\{\gamma_{ij}\}_{i,j\geq 0}$, $\gamma_{00} > 0$, $\gamma_{ji} = \bar{\gamma}_{ij}$) has been considered by A. Atzmon [**Atz**], M. Putinar [**P1**], K. Schmüdgen [**Sch**], and others ([**Ber**], [**Fug**], [**Sza**]). Atzmon showed that a full moment sequence γ has a representing measure μ supported in the closed unit disk $\bar{\mathbf{D}}$ if and only if two natural positivity conditions hold for a kernel function associated to γ. One positivity condition allows for the construction of an inner product on $\mathbb{C}[z]$, and an additional positivity hypothesis on γ forces the operator M_z, multiplication by the coordinate function z, to be a subnormal contraction on $\mathbb{C}[z]$; the spectral measure of the minimal normal extension of M_z then induces the representing measure μ. On the other hand, Putinar [**P1**] used hyponormal operator theory to solve a closely related moment problem. More generally, the *full K-moment problem* $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu$ $(i, j \geq 0)$, $\operatorname{supp} \mu \subseteq K \subseteq \mathbb{C}$, has been solved for the case when K is a prescribed compact *semi-algebraic* set [**Cas**], [**CP2**], [**McG**], [**P3**], [**Sch2**]. For other sets, including $K = \mathbb{C}$, the full moment problem is unsolved [**Fug**].

A theorem of M. Riesz [**Akh, Theorem 2.6.3, page 71**] for $K \subseteq \mathbb{R}$ shows that the K-moment problem for a sequence $\beta \equiv \{\beta_n\}_{n=0}^{\infty}$ is solvable if and only if the functional $\varphi : p(t) \equiv \sum_k a_k t^k \mapsto \varphi(p) := \sum_k a_k \beta_k$ is *positive,* i.e., $p \mid_K \geq 0 \Rightarrow \varphi(p) \geq 0$. This result was extended to $\mathbb{R}^n$ by Haviland [**Hav1,2**];

however, for many sets, including $K = \mathbb{R}^2$, the description of the positive polynomials is incomplete ([**Sch1,2**], [**P3**], [**McG**], [**BM**]), and the Riesz criterion does not always lead to a concrete criterion for solvability which can be expressed easily in terms of the moments.

For the truncated complex moment problem (1.1), it is only possible to define Atzmon's inner product on the space of polynomials of degree at most n, but since this space is not invariant under M_z, the rest of the argument is obstructed. Perhaps for this reason, there appears to be little literature on the truncated complex moment problem (1.1), although the problem is apparently known to specialists. Apart from its intrinsic interest, the truncated complex moment problem has a direct impact on the full moment problem. Indeed, [**St**] proved that if $K \subseteq \mathbb{C}$ is closed, if $\gamma = \{\gamma_{ij}\}$ is an infinite moment sequence, and if for each $n \geq 1$ there exists a representing measure μ_n for $\{\gamma_{ij}\}_{0 \leq i+j \leq 2n}$ such that $\operatorname{supp} \mu_n \subseteq K$, then by a weak compactness argument, there exists a subsequence of $\{\mu_n\}$ that converges to a representing measure μ for γ with $\operatorname{supp} \mu \subseteq K$ (cf. [**Lan, p. 5**]). (On the other hand, a solution of the full moment problem does *not* imply a solution of the truncated moment problem [**CF3**].)

To illustrate this principle, consider the full moment problem for $K = [0,1]$. A refinement of Riesz's Theorem due to Hausdorff ([**ShT, Theorem 1.5, page 9**], [**Hau**]) shows that for a real sequence $\beta \equiv \{\beta_j\}_{j=0}^{\infty}$, there exists a positive Borel measure μ supported in $[0,1]$ such that $\beta_j = \int t^j \, d\mu \quad (j \geq 0)$ if and only if the functional φ is positive when restricted to the family of polynomials $p_{m,n}(t) \equiv t^m(1-t)^n \quad (m,n \geq 0)$. To obtain a more concrete solution (expressed solely in terms of β) consider the *truncated Hausdorff moment problem* for $\beta^{(n)} := \{\beta_j\}_{j=0}^{2n+1}$. Krein and Nudel'man [**KrN**] proved that $\beta^{(n)}$ admits a representing measure if and only if $L(n) := (\beta_{i+j})_{i,j=0}^n$ and $M(n) := (\beta_{i+j+1})_{i,j=0}^n$ satisfy $L(n) \geq M(n) \geq 0$. It thus follows from [**St**] that β admits a representing measure if and only if $L(n) \geq M(n) \geq 0$ for every $n \geq 0$.

Our goal in this work is to obtain concrete criteria for the existence of representing measures, analogous to those in [**Akh**], [**KrN**] and [**ShT**]. One feature of our study is that in cases in which we are able to solve the truncated moment problem theoretically, we also have algorithms to provide finitely-atomic representing measures whose atoms and densities can be explicitly computed (cf. [**CF3**], [**Fia**]); moreover, these calculations are quite feasible using readily available mathematical software.

Our approach requires a detailed study of the matrix positivity induced by a representing measure, which we now begin. For notational simplicity, our discussion here focuses on the truncated moment problem in one complex variable, but in Chapter 7 we give the appropriate generalizations to several variables. Let μ be a positive Borel measure on $\mathbb{C}$, assume that $\mathbb{C}[z,\bar{z}] \subseteq L^1(\mu)$, and for $i,j \geq 0$ define the (i,j)*-power moment* of μ by $\gamma_{ij} := \int \bar{z}^i z^j \, d\mu(\bar{z},z)$. Given $p \in \mathbb{C}[z,\bar{z}]$,

$p(z,\bar{z}) \equiv \sum_{ij} a_{ij}\bar{z}^i z^j$,

$$(1.2) \quad 0 \le \int |\, p(z,\bar{z})\,|^2 \; d\mu(z,\bar{z}) \\ = \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k}\, d\mu(z,\bar{z}) = \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell}\gamma_{i+\ell,j+k}.$$

To understand the matricial positivity associated with $\gamma := \{\gamma_{ij}\}$ via (1.2), we first introduce the following lexicographic order on the rows and columns of infinite matrices: $\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3, \dots$; e.g., the first column is labeled $\mathit{1}$, the second column is labeled Z, the third $\bar{Z}$, the fourth Z^2, etc.; this order corresponds to the graded homogeneous decomposition of $\mathbb{C}[z,\bar{z}]$. For $m,n \ge 0$, let $M[m,n]$ be the $(m+1)\times(n+1)$ block of Toeplitz form (i.e., with constant diagonals) whose first row has entries given by $\gamma_{m,n}, \gamma_{m+1,n-1}, \dots, \gamma_{m+n,0}$ and whose first column has entries given by $\gamma_{m,n}, \gamma_{m-1,n+1}, \dots, \gamma_{0,n+m}$ (as a consequence, the entry in the lower right-hand corner of $M[m,n]$ is $\gamma_{n,m}$). The *moment matrix* $M \equiv M(\gamma)$ is then built as follows:

$$M := \begin{pmatrix} M[0,0] & M[0,1] & M[0,2] & \dots \\ M[1,0] & M[1,1] & M[1,2] & \dots \\ M[2,0] & M[2,1] & M[2,2] & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

It is now not difficult to see that the positivity condition on γ inherent in (1.2) is equivalent to the condition $M \ge 0$, where M is considered as a quadratic form on $\mathbb{C}^\omega$. The idea of organizing the initial data into moment matrices is perhaps implicit in [**Atz**], [**Fug**] (where the notion of moment sequence is somewhat different from ours), but our plan of studying the truncated complex moment problem from a matricial viewpoint seems to be new.

The *truncated* complex moment problem (1.1) corresponds to the case when only an initial segment of moments is prescribed. It seems natural to assume (as we always do) that such an initial segment has the form $\gamma \equiv \{\gamma_{ij}\}_{0\le i+j\le 2n}$ for some $n \ge 1$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$; thus, the initial data consist of a whole corner of M, namely $M(n)(\gamma) := (M[i,j])_{0\le i,j\le n}$. In the sequel we refer to such a sequence γ as a *truncated moment sequence*, and we refer to $M(n) \equiv M(n)(\gamma)$ as the *moment matrix* associated to γ. We denote the successive columns of $M(n)$ by $\mathit{1}, Z, \bar{Z}, \dots, Z^n, \dots, \bar{Z}^n$. For a polynomial $p(z,\bar{z}) \equiv \sum_{0\le i+j\le n} a_{ij}\bar{z}^i z^j$ we define an element $p(Z,\bar{Z})$ in the column space of $M(n)$ by $p(Z,\bar{Z}) := \sum a_{ij}\bar{Z}^i Z^j$.

The use of functional headings for the columns of $M(n)$ is deliberately suggestive; as we show in the sequel, a necessary condition for the existence of a representing measure is that the columns of $M(n)$ behave like the monomials naming them (in a sense to be made precise); moreover, in a number of cases described herein, such function-like behavior is actually sufficient to insure the existence of representing measures. This perspective, which seems to be new in moment problems, captures the essence of the notion of recursiveness that we introduced in [**CF1,2,3**].

Our approach to the truncated complex moment problem builds on two distinct strategies that we employed in solving single-variable truncated moment problems in [**CF1,3**]; we briefly recall these methods. The *Stieltjes (power) moment problem* asks for a characterization of those sequences of positive numbers $\beta \equiv \{\beta_i\}_{i=0}^{\infty}$ for which there exist positive Borel measures μ supported in $[0, +\infty)$ such that $\beta_i = \int t^i \, d\mu(t) \quad (i \geq 0)$; Stieltjes proved that such a measure μ exists if and only if the Hankel matrices $H(\infty) := (\beta_{i+j})_{i,j \geq 0}$ and $K(\infty) := (\beta_{i+j+1})_{i,j \geq 0}$ are positive as quadratic forms on $\mathbb{C}^{\omega}$ [**ShT**]. In [**CF1**] we solved the *truncated Stieltjes moment problem* corresponding to initial data $\{\beta_i\}_{i=0}^{n}$, $n < \infty$. In the case when n is odd, $n = 2k+1$, a representing measure exists if and only if $H(k) := (\beta_{i+j})_{0 \leq i,j \leq k}$ and $K(k) := (\beta_{i+j+1})_{0 \leq i,j \leq k}$ are positive, and $\mathbf{v}_{k+1} := (\beta_{k+1}, \dots, \beta_{2k+1})^T$ belongs to the range of $H(k)$. The proof of this result in [**CF1**] entails extending $H(k)$ and $K(k)$ to positive Hankel forms $H(\infty)$ and $K(\infty)$, respectively, and in then constructing a representing measure from the spectral measure of a normal operator acting on an infinite dimensional Hilbert space associated to $H(\infty)$.

In the sense that we were using infinite dimensional techniques to solve a finite interpolation problem, we were dissatisfied with the proof in [**CF1**]. In contrast to this approach, in [**CF3**] we developed algorithms for solving truncated moment problems which entail only "finite" techniques, e.g., the solution of finite systems of linear equations, Lagrange interpolation, and finite dimensional operator theory, particularly an extension theory for positive Hankel and Toeplitz matrices. Not only is this approach aesthetically satisfying, but by restricting ourselves to finite extensions we were able to describe *all* finitely atomic solutions of the truncated moment problems of Hamburger, Stieltjes, Hausdorff, and Toeplitz. In particular, for the Hamburger problem (even case), given $n = 2k$ and real numbers β:$\beta_0, \dots, \beta_{2k}$, with $\beta_0 > 0$, there exists a positive Borel measure μ, $\operatorname{supp} \mu \subseteq \mathbb{R}$, such that $\beta_j = \int t^j \, d\mu(t) \;\; (0 \leq j \leq 2k)$ if and only if $H(k)$ admits a positive Hankel extension $H(k+1)$.

The latter condition can be explained in concrete terms using the notion of "recursiveness." Denote the columns of $H(k)$ by $\mathit{1}, T, \dots, T^k$. If $H(k)$ is singular, let $r := \min\{i : T^i \in \langle \mathit{1}, T, \dots, T^{i-1} \rangle\}$; in this case, $1 \leq r \leq k$, and there exist unique scalars $a_0, \dots, a_{r-1}$ such that $T^r = a_0 \mathit{1} + \dots + a_{r-1} T^{r-1}$. $H(k) \geq 0$ admits a positive extension $H(k+1)$ if and only if $H(k)$ is invertible, or $H(k)$ is singular and *recursively generated*, i.e.,

$$T^{r+s} = a_0 T^s + \dots + a_{r-1} T^{r+s-1} \qquad (0 \leq s \leq k-r). \tag{1.3}$$

In this case, we need not invoke the infinite dimensional spectral theorem to construct a representing measure. Instead, we proved in [**CF3**] that the polynomial $t^r - (a_0 + \dots + a_{r-1} t^{r-1})$ has r distinct real roots, $t_0, \dots, t_{r-1}$; these roots may serve as the atoms of a representing measure, with corresponding densities $\rho_0, \dots, \rho_{r-1}$ determined by the Vandermonde equation

$$V(t_0, \dots, t_{r-1})(\rho_0, \dots, \rho_{r-1})^T = (\beta_0, \dots, \beta_{r-1})^T.$$

That the measure $\mu := \sum_{i=0}^{r-1} \rho_i \delta_{t_i}$ fully interpolates β follows from (1.3).

In the present work we introduce an analogue of (1.3) which will serve as our notion of recursiveness for truncated complex moment sequences. We say that γ is *recursively generated* if $M(n)(\gamma)$ satisfies the following property:

(RG) $\quad$ If $p, q \in \mathbb{C}[z, \bar{z}], \deg pq \leq n,$ and $p(Z, \bar{Z}) = 0,$ then $(pq)(Z, \bar{Z}) = 0.$

Our goal is to study the following conjecture.

1.1. Main Conjecture. *Let γ be a truncated moment sequence. The following assertions are equivalent.*

(i) *γ has a representing measure;*
(i′) *γ has a representing measure with moments of all orders;*
(ii) *γ has a compactly supported representing measure;*
(iii) *γ has a finitely atomic representing measure;*
(iv) *γ has a* rank $M(n)$*-atomic representing measure;*
(v) *$M(n)$ admits a positive extension $M(n+1)$;*
(vi) *$M(n) \geq 0$ and $M(n)$ admits a flat (i.e., rank-preserving) extension $M(n+1)$;*
(vii) *$M(n) \geq 0$ and $M(n)$ satisfies property* (RG).

It is clear that (iv)⇒(iii)⇒(ii)⇒(i′)⇒(i), and that (vi)⇒(v) (since flat extensions of positive matrices are positive). In the sequel we establish (i)⇒(vii), (ii)⇒(v)⇒(vii) and (iv)⇔(vi). Thus, to prove the equivalence of (i)–(vii) it suffices to prove (vii)⇒(vi). The main result of this paper, Theorem 5.13, is the equivalence of (iv) and (vi), which results from a combination of the results in Chapters 2, 3 and 4. For the case of the truncated moment problem when $Z = \bar{Z}$ (which is equivalent to the truncated Hamburger moment problem (Theorem 3.19, cf. [**Fia**])), the Main Conjecture is fully established in [**CF3, Section 3**]. In the present work we fully affirm Conjecture 1.1 under the following hypothesis: $M(n) \geq 0$ and every column corresponding to homogeneous monomials in z and $\bar{z}$ of total degree n can be obtained as a linear combination of columns corresponding to monomials of lower degree. (This is equivalent to saying that $M(n)$ is a *flat extension* of $M(n-1)$, i.e., $\operatorname{rank} M(n) = \operatorname{rank} M(n-1)$; cf. the remarks following Proposition 2.2.)

Our method depends heavily on an intrinsic characterization of moment matrices (Theorem 2.1) and on the Structure Theorem for positive moment matrices (Theorem 3.14). The latter result is based on a study in Chapter 3 of "recursiveness" in positive moment matrices, which shows that the conclusion of property (RG) holds whenever $\deg(pq) \leq n-1$. In Chapter 3 we also show that if γ has a representing measure μ, then the linear map from the column space of $M(n)$ into $L^2(\mu)$, induced by $\bar{Z}^i Z^j \mapsto \bar{z}^i z^j$, is an important tool in analyzing the size and location of $\operatorname{supp}\mu$. In Chapter 4 we construct representing measures for finite-rank positive infinite moment matrices. In Theorem 4.7 we use a normal operator construction to prove that a positive infinite moment matrix of finite rank r has a unique representing measure, which is r-atomic.

The flat data case of the truncated complex moment problem is carried out in Chapter 5. The main technical result, Theorem 5.4, states that if $M(n)$ is a flat, positive extension of $M(n-1)$, then $M(n)$ admits a unique flat, positive extension of the form $M(n+1)$; consequently, $M(n)$ also admits a unique positive moment matrix extension of the form $M(\infty)$, and this is a flat extension (Corollary 5.12). Combining these results with Theorem 4.7, we prove in Corollary 5.14 that if γ is flat and $M(n) \geq 0$, then γ has a unique compactly supported representing measure, which is $\operatorname{rank} M(n)$-atomic. Although this result is proved using an extension of $M(n)$ to $M(\infty)$, the extension is carried out in such a way that the corresponding

Atzmon space is finite dimensional; thus we only invoke the spectral theorem in a finite dimensional setting, in keeping with our "finite" philosophy.

As an application of these results we solve the truncated complex moment problem in the case when the *analytic columns* of $M(n)$, $\mathit{1}, Z, \dots, Z^n$, are *dependent.* In this case, there exists a minimal r, $1 \le r \le n$, such that $Z^r = a_0 \mathit{1} + \dots + a_{r-1} Z^{r-1}$. In Corollary 5.15 we prove that, under this hypothesis, γ has a representing measure if and only if $M(n) \ge 0$ and $\{\mathit{1}, Z, \dots, Z^{r-1}\}$ spans the column space of $M(n)$; in this case, γ has a unique representing measure, whose support consists of the r distinct roots of the polynomial $z^r - (a_0 + \dots + a_{r-1} z^{r-1})$.

The case of the truncated complex moment problem in which the analytic columns are *independent* is unsolved and, in particular, the case when $M(n)$ is positive and invertible (in symbols, $M(n) > 0$) is open for $n > 1$. If $M(1) > 0$, then γ has infinitely many 3-atomic representing measures, as we show in Chapter 6. This is part of the quadratic moment problem, which we discuss later in this section.

The main obstruction to obtaining a full proof of Conjecture 1.1 concerns the criteria for flat extensions of positive moment matrices, which we plan to study in detail elsewhere. The difficulty here can best be explained by analogy with the real case. Using the notation preceding (1.3), a positive singular Hankel matrix $H(k)$ *always* satisfies

$$T^{r+s} = a_0 T^s + \dots + a_{r-1} T^{r+s-1} \qquad (0 \le s \le k - r - 1)$$

[**CF3**], so the criterion for a positive Hankel extension $H(k+1)$ is simply

$$T^k = a_0 T^{k-r} + \dots + a_{r-1} T^{k-1}; \tag{1.4}$$

in this case, there is a unique flat extension $H(k+1)$, determined by

$$T^{k+1} := a_0 T^{k-r+1} + \dots + a_{r-1} T^k.$$

In the nonsingular case, we may produce flat extensions by choosing β_{2k+1} arbitrarily, but then β_{2k+2} is uniquely determined from β and β_{2k+1} by the flatness requirement.

For a positive moment matrix $M(n)$ satisfying (RG), to produce a flat extension $M(n+1)$ we must choose an entire *block* $M[n, n+1]$ compatible with positivity, and in such a way that $M[n+1, n+1]$ (which is then uniquely determined by the flatness requirement) is a Toeplitz matrix. The results of Chapter 5 show how to make such a choice under a hypothesis analogous to (1.4): whenever $i + j = n$, $\bar{Z}^i Z^j \in \langle \bar{Z}^r Z^s \rangle_{0 \le r+s \le n-1}$. In Chapter 6 we also show how to produce flat extensions for the quadratic moment problem, which we discuss next.

The *quadratic moment problem* is the case of the truncated complex moment problem when $n = 1$; we start with an initial segment of the form $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$, $\gamma_{00} > 0, \gamma_{ij} = \bar{\gamma}_{ij}$, and we let $r := \operatorname{rank} M(1)$. We show in Theorem 6.1 that the following conditions are equivalent:

(i) γ has a representing measure;

(ii) γ has an r-atomic representing measure;

(iii) $M(1) \ge 0$.

Since property (RG) is vacuously satisfied in the quadratic moment problem, this result is consistent with Conjecture (1.1). Indeed, Theorem 5.13 and the equivalence of (ii) and (iii) imply that if $M(1) \ge 0$, then $M(1)$ admits a flat extension

$M(2)$. When $r = 1$, then $\mu := \rho\delta_w$ ($\rho := \gamma_{00}$, $w := \gamma_{01}/\gamma_{00}$) is the unique representing measure. When $r = 2$, the 2-atomic representing measures are parameterized by the points of a straight line in the complex plane. The existence of infinitely many representing measures for a singular truncated complex moment problem contrasts with the real case, where singular moment problems have unique solutions [**CF3**]. For the case when $r = 3$, flat extensions exist, and we partially parameterize the 3-atomic representing measures by a circle. We conjecture that, more generally, flat extensions always exist in case $M(n)$ is positive and invertible (since property (RG) is satisfied vacuously in this case).

The results of Chapter 6 provide a good illustration of our dualistic approach to moment problems. The cases $r = 1$ and $r = 2$ are proved using the viewpoint of [**CF3**], by directly constructing a polynomial whose roots serve as the support of a representing measure. By contrast, the case $r = 3$ is proved as a corollary of the results of Chapter 5, and hence depends on a flat extension to $M(\infty) \geq 0$.

Moment problems have been traditionally studied using tools and techniques from a variety of subjects, including real analysis, analytic function theory, continued fractions, operator theory, and the extension theory for positive linear functionals on convex cones in function spaces (see for instance [**Akh**], [**AK**], [**BM**], [**KrN**], [**Lan**], [**Sar**], [**St**], [**Sza**]). The above mentioned results of Atzmon, Putinar and Schmüdgen illustrate the use of operator theory in constructing representing measures. The interplay between moment problems and operator theory goes in both directions. Indeed, Atzmon's results were motivated by a reformulation of the invariant subspace problem for subnormal operators; in [**CF1**] we solved Stampfli's subnormal completion problem for unilateral weighted shifts by first solving the truncated Stieltjes moment problem; and in [**CP1,2**] techniques from the theory of moments played a central role in establishing the existence of a non-subnormal polynomially hyponormal operator.

By way of analogy with our strategy in [**CF1**], we shall show in Chapter 6 that the subnormal completion problem for 2-variable weighted shifts is intimately related to the truncated complex moment problem, in such a way that a solution to the latter immediately produces a solution to the former. This can be intuitively anticipated, since the subnormal completion problem ought to be related to the classical bidimensional real moment problem, and this in turn is isomorphic to the complex moment problem.

To study *multidimensional* truncated moment problems in several real or complex variables, we can define moment matrices subordinate to lexicographic orderings of the several variables; in the case of one real variable, such moment matrices are the familiar Hankel matrices $H(k)$. We show in Chapter 7 that many of the results in Chapters 2 through 6 extend to these more general moment matrices; the main difficulty in obtaining these generalizations is notational, not conceptual.

Some of the calculations in Chapter 6, and many of the ideas in this paper, were first obtained through computer experiments using the software tool *Mathematica* [**Wol**].

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CHAPTER 2

Moment Matrices

In this chapter we associate to a moment sequence γ the *moment matrix* $M(n)(\gamma)$, which plays a central role in our analysis of the truncated moment problem. The main result of this chapter is an abstract characterization of moment matrices which will be used in the sequel to construct representing measures.

We begin with some notation. For $m \geq 0$, $M_m(\mathbb{C})$ denotes the algebra of $m \times m$ complex matrices. For $n \geq 0$, let $m \equiv m(n) := (n+1)(n+2)/2$. For $A \in M_m(\mathbb{C})$ we denote the successive rows and columns according to the following lexicographic-functional ordering: $\mathit{1}$, Z, $\bar{Z}$, Z^2, $Z\bar{Z}$, $\bar{Z}^2$, $\ldots$, Z^n, $Z^{n-1}\bar{Z}$, $\ldots$, $Z\bar{Z}^{n-1}$, $\bar{Z}^n$. For $0 \leq i+j \leq n$, $0 \leq \ell + k \leq n$, we denote the entry of A in row $\bar{Z}^\ell Z^k$ and column $\bar{Z}^i Z^j$ by $A_{(\ell,k)(i,j)}$. (In the notation $0 \leq i+j \leq n$ it is always implicit that $i, j \geq 0$.) For $0 \leq i, j \leq n$, $A[i,j]$ denotes the $(i+1) \times (j+1)$ rectangular block in A whose upper left-hand entry is $A_{(0,i)(0,j)}$. We define a basis $\{e_{ij}\}_{0 \leq i+j \leq n}$ for $\mathbb{C}^m$ as follows: $e_{ij} \equiv e_{ij}^{(m)}$ is the vector with 1 in the $\bar{Z}^i Z^j$ entry and 0 in all other positions.

Let $\mathcal{P}_n$ denote the vector space of all complex polynomials in $z, \bar{z}$ of total degree $\leq n$. Each $p \in \mathcal{P}_n$ has a unique representation $p(z,\bar{z}) = \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$ ($a_{ij} \in \mathbb{C}$); $\bar{p}$ then denotes the conjugate function $\sum \bar{a}_{ij} \bar{z}^j z^i$. For $p \in \mathcal{P}_n$, let $\hat{p} = \sum a_{ij} e_{ij} \in \mathbb{C}^m$. We define a sesquilinear form $\langle \cdot, \cdot \rangle_A$ on $\mathcal{P}_n$ by $\langle p, q \rangle_A := \langle A\hat{p}, \hat{q} \rangle$ ($p, q \in \mathcal{P}_n$). In particular, $\langle \bar{z}^i z^j, \bar{z}^k z^\ell \rangle_A = \langle A e_{ij}, e_{k\ell} \rangle = A_{(k,\ell)(i,j)}$. Note that if A is self-adjoint, then $\langle \cdot, \cdot \rangle_A$ is *hermitian*, i.e., $\langle p, q \rangle_A = \overline{\langle q, p \rangle_A}$.

Let γ be a truncated moment sequence: $\gamma = (\gamma_{ij})_{0 \leq i+j \leq 2n}$, $\gamma_{ij} = \bar{\gamma}_{ji}$. We define the *moment matrix* $M(n) \equiv M(n)(\gamma) \in M_m(\mathbb{C})$ as follows: for $0 \leq i+j \leq n$, $0 \leq \ell + k \leq n$, $M(n)_{(\ell,k)(i,j)} = \gamma_{i+k,j+\ell}$; thus $\langle \bar{z}^i z^j, \bar{z}^k z^\ell \rangle_{M(n)} = (M(n) e_{ij}, e_{k\ell}) = M(n)_{(k,\ell)(i,j)} = \gamma_{i+\ell,j+k}$. Since $M(n)^*_{(\ell,k)(i,j)} = \overline{M(n)_{(i,j)(\ell,k)}} = \bar{\gamma}_{j+\ell,i+k} = \gamma_{i+k,j+\ell} = M(n)_{(\ell,k)(i,j)}$, $M(n)$ is self-adjoint. For $0 \leq i, j \leq n$, note that $M(n)[i,j]$ has the form

$$B_{ij} := \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots \\ \vdots & \gamma_{i-1,j+1} & & \vdots \\ \gamma_{0,j+i} & & \cdots & \gamma_{j,i} \end{pmatrix}, \tag{2.1}$$

where B_{ij} has the Toeplitz-like property of being constant on each diagonal; in particular, B_{ii} is a self-adjoint Toeplitz matrix. Now $M(n)$ admits the block decomposition $(B_{ij})_{0 \leq i,j \leq n}$. For $0 \leq i+j \leq 2n$, $i+j$ denotes the *degree* of γ_{ij}; thus B_{ij} contains all of the moments of degree $i+j$, and $M(n)$ has the Hankel-like property that the cross-diagonal blocks $\ldots B_{ij}, B_{i-1,j+1}, B_{i-2,j+2}, \ldots$ each contain the same elements (those of degree $i+j$), though arranged in differing patterns.

We note for future reference that the auxiliary blocks $B_{0,n+1}, \ldots, B_{n-1,n+1}$ may also be defined by (2.1). More generally, if γ is a *full* moment sequence, i.e., with moments γ_{ij} of all orders, we define $M(n)(\gamma)$ via (2.1) for every $n \geq 0$ and set $M(\infty) \equiv M(\infty)(\gamma) := (B_{ij})_{0\leq i,j}$. In particular, if μ is a measure with moments of all orders, and if $\gamma(\mu) := (\gamma_{ij})_{0\leq i,j}$ denotes the moment sequence of μ, then we set $M(n)[\mu] := M(n)(\gamma(\mu))$ and $M(\infty)[\mu] := M(\infty)(\gamma(\mu))$.

The main result of this chapter is the following intrinsic characterization of moment matrices.

THEOREM 2.1. *Let $n \geq 0$ and let $A \in M_{m(n)}(\mathbb{C})$. There exists a truncated moment sequence $\gamma \equiv (\gamma_{ij})_{0\leq i+j\leq 2n}, \gamma_{ij} = \bar{\gamma}_{ji}$, $\gamma_{00} > 0$, such that $A = M(n)(\gamma)$ if and only if*

0) $\langle 1, 1\rangle_A > 0$;
1) $A = A^*$;
2) $\langle p, q\rangle_A = \langle \bar{q}, \bar{p}\rangle_A$ $(p, q \in \mathcal{P}_n)$;
3) $\langle zp, q\rangle_A = \langle p, \bar{z}q\rangle_A$ $(p, q \in \mathcal{P}_{n-1})$;
4) $\langle zp, zq\rangle_A = \langle \bar{z}p, \bar{z}q\rangle_A$ $(p, q \in \mathcal{P}_{n-1})$.

We refer to 2) as the *symmetric* property of $\langle\cdot,\cdot\rangle_A$; 2) is related to establishing that a candidate γ for realizing A as $M(n)(\gamma)$ satisfies $\gamma_{ij} = \bar{\gamma}_{ji}$; 3) is used to establish the above mentioned Hankel-type property for A. A quadratic form satisfying 4) is said to be *normal*; we use normality to establish the Toeplitz-like property of the blocks $A[i, j]$. Note also that if 1)–3) hold, then 3′) $\langle \bar{z}p, q\rangle_A = \langle p, zq\rangle_A$ $(p, q \in \mathcal{P}_{n-1})$: $\langle \bar{z}p, q\rangle_A = \langle \bar{q}, z\bar{p}\rangle_A = \overline{\langle z\bar{p}, \bar{q}\rangle_A} = \overline{\langle \bar{p}, \bar{z}\bar{q}\rangle_A} = \overline{\langle zq, p\rangle_A} = \langle p, zq\rangle_A$.

PROOF OF THEOREM 2.1. Suppose first that $A = M(n)(\gamma)$ for some truncated moment sequence γ. Note that $\langle 1, 1\rangle_A = A_{(0,0)(0,0)} = \gamma_{00} > 0$, so 0) holds. For $0 \leq i+j \leq n$, $0 \leq k+\ell \leq n$, $\bar{A}_{(\ell,k)(i,j)} = \bar{\gamma}_{i+k,j+\ell} = \gamma_{j+\ell,i+k} = A_{(i,j)(\ell,k)}$, so $A = A^*$, establishing (1). Similarly, $\langle \bar{z}^i z^j, \bar{z}^k z^\ell\rangle_A = \gamma_{i+\ell,j+k} = \gamma_{\ell+i,k+j} = \langle \bar{z}^\ell z^k, \bar{z}^j z^i\rangle_A$, so sesquilinearity implies that $\langle p, q\rangle_A = \langle \bar{q}, \bar{p}\rangle_A$ $(p, q \in \mathcal{P}_n)$, proving (2). To establish (3) and (4) it suffices to consider $p = \bar{z}^i z^j$ $(0 \leq i + j \leq n - 1)$ and $q = \bar{z}^\ell z^k$ $(0 \leq k + \ell \leq n - 1)$. Then

$$\begin{aligned}\langle zp, q\rangle_A &= (M(n)e_{i,j+1}, e_{\ell,k}) = \gamma_{i+k,j+\ell+1}\\ &= (M(n)e_{ij}, e_{\ell+1,k}) = \langle p, \bar{z}q\rangle_A,\end{aligned}$$

which establishes (3); similarly,

$$\begin{aligned}\langle zp, zq\rangle_A &= (M(n)e_{i,j+1}, e_{\ell,k+1}) = \gamma_{i+k+1,j+1+\ell}\\ &= (M(n)e_{i+1,j}, e_{\ell+1,k}) = \langle \bar{z}p, \bar{z}q\rangle_A,\end{aligned}$$

proving (4).

For the converse, assume $A \in M_{m(n)}(\mathbb{C})$ satisfies 0)–4); we seek to define a truncated moment sequence $\gamma = (\gamma_{ij})_{0\leq i+j\leq 2n}$, $\gamma_{ij} = \bar{\gamma}_{ji}$, $\gamma_{00} > 0$, such that $A = M(n)(\gamma)$. Note that if $0 \leq i + j \leq 2n$, then there exist $\ell, k, p, q \geq 0$ such that

$$i = \ell + p, \qquad j = k + q, \qquad 0 \leq \ell + k \leq n, \qquad 0 \leq p + q \leq n. \tag{2.2}$$

For,

$$(i,j) = \begin{cases} (i,0)+(0,j) & 0 \le i,j \le n \\ (i,j-n)+(0,n) & 0 \le i \le n,\ n < j \le 2n \\ (i-n,j)+(n,0) & n < i \le 2n,\ 0 \le j \le n \end{cases}.$$

We now define $\gamma_{ij} := A_{(k,\ell)(p,q)}$, and we claim that $\gamma := (\gamma_{ij})_{0 \le i+j \le 2n}$ is a well-defined truncated moment sequence, i.e., γ_{ij} is independent of the decomposition in (2.2), and $\gamma_{ij} = \bar{\gamma}_{ji}$, $\gamma_{00} > 0$. Note that this readily implies that $A = M(n)(\gamma)$: For $0 \le k+\ell \le n$, $0 \le p+q \le n$, $M(n)_{(k,\ell)(p,q)} = \gamma_{\ell+p,k+q} = A_{(k,\ell)(p,q)}$ (using $i = \ell+p$, $j = k+q$ in (2.2)).

To complete the proof of Theorem 2.1, we show that γ is a well-defined moment sequence, and to this end we introduce some notation. For $0 \le i+j \le 2n$, let

$$S_{ij} := \{v = (k,\ell,p,q) \in \mathbb{Z}_+^4 : i = \ell+p,\ j = k+q,\ 0 \le k+\ell \le n,\ 0 \le p+q \le n\}.$$

For $v \in S_{ij}$, let $\alpha(v) := A_{(k,\ell)(p,q)} = \langle \bar{z}^p z^q, \bar{z}^k z^\ell \rangle_A$. We define the *block-type* of v by $\beta(v) := (k+\ell, p+q)$; thus $\alpha(v)$ is an element of the block $A[k+\ell, p+q]$. For $v' := (k',\ell',p',q') \in S_{ij}$, define $\delta(v,v') := |k'-k+\ell'-\ell|$ $(= |q'-q+p'-p|)$; thus $\delta(v,v')$ measures the distance between the blocks of v and v', and $\delta(v,v') = 0 \Leftrightarrow \beta(v) = \beta(v')$.

To show that γ is well-defined we must show that if $v, v' \in S_{ij}$, then $\alpha(v) = \alpha(v')$. We first consider the case when v and v' have the same block type, i.e., $\beta(v) = \beta(v')$. Note that $k = k' \Rightarrow q = q' \Rightarrow p = p' \Rightarrow \ell = \ell' \Rightarrow v = v'$; we may thus assume that $s := k'-k > 0$.

Claim. $\psi(v) := (k+1, \ell-1, p+1, q-1) \in S_{ij}$, $\alpha(\psi(v)) = \alpha(v)$, and $\beta(\psi(v)) = \beta(v)$.

To prove that $\psi(v) \in S_{ij}$, note first that if $q = 0$, then $k = k'+q' > k+q' \ge k$; thus $q > 0$ and $q-1 \ge 0$. Since $k < k'$, then $k+1 \le k' \le n$; also, since $\beta(v) = \beta(v')$, $\ell = k'-k+\ell' > \ell' \ge 0$, whence $\ell-1 \ge 0$; similarly, $p = \ell'-\ell+p' < p' \le n$, so $p+1 \le n$. Thus $\psi(v) \in S_{ij}$; now $\beta(\psi(v)) = ((k+1)+(\ell-1), (p+1)+(q-1)) = (k+\ell, p+q) = \beta(v)$, and $\alpha(\psi(v)) = \langle \bar{z}^{p+1} z^{q-1}, \bar{z}^{k+1} z^{\ell-1} \rangle_A = \langle \bar{z}^p z^q, \bar{z}^k z^\ell \rangle_A = \alpha(v)$ (normality). It follows inductively that $\psi^r(v) \in S_{ij}$ $(0 \le r \le s)$ and that $\beta(\psi^r(v)) = \beta(\psi^{r+1}(v))$, $\alpha(\psi^r(v)) = \alpha(\psi^{r+1}(v))$ $(0 \le r \le s-1)$; thus $\alpha(v) = \alpha(\psi^s(v))$. Denote $\psi^s(v)$ by (K, L, P, Q); then $K = k+s = k'$, $L = \ell - s$, $P = p+s$, $Q = q-s$. Now $k+q = k'+q' = k+s+q' \Rightarrow q' = q-s = Q$; $k+\ell+s = k'+\ell'+s \Rightarrow k'+\ell = k'+\ell'+s \Rightarrow \ell' = \ell - s = L$; $\ell - s + p = \ell' - s + p' \Rightarrow \ell' + p = \ell' - s + p' \Rightarrow p' = p+s = P$. Thus $\psi^s(v) = v'$, whence $\alpha(v) = \alpha(v')$.

We next consider the case when v and v' correspond to different blocks of A $(\beta(v) \ne \beta(v'))$. We will show that there exists $w \in S_{ij}$ such that $\delta(w,v') = \delta(v,v') - 1$ and $\alpha(w) = \alpha(v)$. Using an inductive argument, we can then assume that $\delta(v,v') = 0$, whence it follows from the previous case that $\alpha(v) = \alpha(v')$.

To prove the existence of the desired element w, consider the following maps from S_{ij} to $\mathbb{Z}^4$:

$$\begin{aligned} \rho_1(k,\ell,p,q) &:= (k, \ell-1, p+1, q), \\ \rho_2(k,\ell,p,q) &:= (k-1, \ell, p, q+1), \\ \rho_3(k,\ell,p,q) &:= (k+1, \ell, p, q-1), \\ \rho_4(k,\ell,p,q) &:= (k, \ell+1, p-1, q). \end{aligned}$$

Now $\delta(v, v') > 0$, and we assume first that $d := k' + \ell' - k - \ell > 0$. Since $p' + q' = p + q - d$, we may view the block of v' as below and to the left of the block of v (larger row number, smaller column number). In this case we will use either ρ_3 or ρ_4 to produce an element w of S_{ij} whose block distance to v' is reduced by 1 and for which $\alpha(w) = \alpha(v)$. Note that either $p > 0$ or $q > 0$, for if $p = q = 0$, then $p' + q' = p + q - d = -d < 0$.

Suppose $p > 0$. Then $p - 1 \geq 0$, whence $n \geq p + q - 1 \geq 0$. Since $n \geq k' + \ell' > k + \ell \geq \ell$, then $k + \ell + 1 \leq n$ and $\ell + 1 \leq n$; thus $w \equiv \rho_4(v) \in S_{ij}$ and $\alpha(w) = \alpha(k, \ell + 1, p - 1, q) = \langle \bar{z}^{p-1} z^q, \bar{z}^k z^{\ell+1} \rangle_A = \langle \bar{z}^{(p-1)+1} z^q, \bar{z}^k z^{(\ell+1)-1} \rangle_A$ (property 3′)) $= \alpha(v)$. Now $\delta(w, v') = |k' - k + \ell' - (\ell + 1)| = |(k' + \ell') - (k + \ell + 1)| = \delta(v, v') - 1$, so w satisfies our requirements. If $q > 0$, a similar argument shows that we may use $w := \rho_3(v)$. In the case when $d < 0$, we use either $w := \rho_1(v)$ or $w := \rho_2(v)$. The proof that γ is well-defined is now complete.

We conclude the proof of Theorem 2.1 by verifying that $\gamma_{ij} = \bar{\gamma}_{ji}$ and $\gamma_{00} > 0$. Suppose $i = \ell + p$, $j = k + q$, $0 \leq \ell + k \leq n$, $0 \leq p + q \leq n$, so that $\gamma_{ij} = A_{(k,\ell)(p,q)} = \langle \bar{z}^p z^q, \bar{z}^k z^\ell \rangle_A$. Now set $i' = j$, $j' = i$, $\ell' = k$, $p' = q$, $k' = \ell$, $q' = p$, so that $0 \leq \ell' + k' \leq n$, $0 \leq p' + q' \leq n$. Then $\gamma_{ji} = \gamma_{i'j'} = \langle \bar{z}^{p'} z^{q'}, \bar{z}^{k'} z^{\ell'} \rangle_A = \langle \bar{z}^q z^p, \bar{z}^\ell z^k \rangle_A$, whence $\bar{\gamma}_{ji} = \langle \bar{z}^\ell z^k, \bar{z}^q z^p \rangle_A = \langle \bar{z}^p z^q, \bar{z}^k z^\ell \rangle_A$ (by (2)) $= \gamma_{ij}$. Finally, $\gamma_{00} = A_{(0,0)(0,0)} = \langle 1, 1 \rangle_A > 0$. □

We conclude this chapter with an introduction to the extension problem for positive moment matrices. For $k, \ell \in \mathbb{Z}_+$, let $A \in M_k(\mathbb{C})$, $A = A^*$, $B \in M_{k,\ell}(\mathbb{C})$, $C \in M_\ell(\mathbb{C})$; we refer to any matrix of the form

$$\tilde{A} \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{2.3}$$

as an *extension* of A (this differs from the usual notion of extension for operators). The general theory of flat extensions for (not necessarily positive) Hankel and Toeplitz matrices was developed by Iohvidov [**Ioh**].

Proposition 2.2 [**Smu**]. *For $A \geq 0$, the following are equivalent:*
1) *$\tilde{A} \geq 0$;*
2) *There exists $W \in M_{k,\ell}(\mathbb{C})$ such that $AW = B$ and $C \geq W^* AW$.*

In case $\tilde{A} \geq 0$, the matrix $V^* AV$ is independent of the choice of V satisfying $AV = B$; moreover, $\operatorname{rank} \tilde{A} = \operatorname{rank} A \Leftrightarrow C = W^* AW$ for some W such that $B = AW$. Conversely, if $A \geq 0$, any extension $\tilde{A}$ satisfying $\operatorname{rank} \tilde{A} = \operatorname{rank} A$ is necessarily positive ([**CF1, Proposition 2.3**], [**Smu**]). We refer to a rank-preserving extension of A as a *flat* extension. If $A \geq 0$ and B is prescribed with $\operatorname{Ran} B \subseteq \operatorname{Ran} A$, there is a unique flat extension of the form (2.3), which we denote by $[A; B]$.

Flat extensions of positive Hankel matrices play an important role in the treatment of the truncated Hamburger moment problem in [**CF3**]: Given $\gamma_j \in \mathbb{R}$ $(0 \leq j \leq 2k)$, there exists a positive Borel measure μ, $\operatorname{supp} \mu \subseteq \mathbb{R}$, such that $\int t^j \, d\mu = \gamma_j$ $(0 \leq j \leq 2k)$ if and only if the Hankel matrix $H(k) := (\gamma_{i+j})_{0 \leq i,j \leq k}$ admits a positive flat extension of the form $H(k+1)$. In the sequel we will obtain partial analogues of this result for the truncated complex moment problem. In particular, the next result will prove useful in obtaining flat extensions $M(n+1)$ of positive moment matrices $M(n)$.

Let $A = M(n) \geq 0$ and let

$$B \equiv B(n) = \begin{pmatrix} B_{0,n+1} \\ \vdots \\ B_{n-1,n+1} \\ B_{n,n+1} \end{pmatrix},$$

where $B_{n,n+1}$ is the moment matrix block (2.1) corresponding to a choice of moments of degree $2n+1$. To construct a positive flat extension of the form $M(n+1) = [M(n); B(n)]$ we require a choice of $B_{n,n+1}$ such that $\operatorname{Ran} B \subseteq \operatorname{Ran} A$ (so that $B = AW$ for some W) and such that $C := W^*AW$ is Toeplitz (constant on diagonals). The next result shows that a certain degree of Toeplitz structure is inherited by C from the Toeplitz structures of the blocks B_{ij} $(0 \leq i \leq n,\ 0 \leq j \leq n+1)$ regardless of the choice of $B_{n,n+1}$ satisfying $\operatorname{Ran} B \subseteq \operatorname{Ran} A$.

Proposition 2.3. *If* $\operatorname{Ran} B(n) \subseteq \operatorname{Ran} M(n)$, *then* $M := [M(n); B(n)]$ *satisfies* $M_{(p,q)(r,s)} = M_{(s,r)(q,p)}$ *for all choices of* $p, q, r, s \geq 0$ *such that* $p+q = r+s = n+1$.

Proposition 2.3 can be formulated equivalently as follows: M satisfies the symmetric property $\langle p, q \rangle = \langle \bar{q}, \bar{p} \rangle$ $(p, q \in \mathcal{P}_{n+1})$.

Example 2.4. Consider $M(1) > 0$ (i.e., $M(1)$ is positive and invertible). For each choice of $\gamma_{1,2}$ and $\gamma_{0,3}$ used to define $B_{1,2}$, $\operatorname{Ran} B(1) \subseteq \operatorname{Ran} M(1)$, so by Proposition 2.3, in $C := (c_{ij})_{1 \leq i,j \leq 3} = [M(1); B(1)][2,2]$ we have $c_{11} = c_{33}$, $c_{21} = c_{32}$, $c_{12} = c_{23}$, and $C = C^*$. To construct a flat extension $M(2)$, it is therefore equivalent to find choices for $\gamma_{1,2}$ and $\gamma_{0,3}$ such that $c_{11} = c_{22}$; we resolve this question in Chapter 5.

Proof of Proposition 2.3. Let $A = M(n)$, $B = B(n)$. We denote the columns of the rectangular block matrix (A, B) by $\bar{Z}^i Z^j$ $(0 \leq i + j \leq n+1)$ and we denote the columns of $\tilde{A} := [A; B]$ by $\bar{W}^i W^j$ $(0 \leq i + j \leq n+1)$. For $r, s \geq 0$, $r + s = n+1$, there exist scalars $a_{ij}(r,s)$ $(0 \leq i + j \leq n)$ such that

$$\bar{Z}^r Z^s = \sum_{0 \leq i+j \leq n} a_{ij}(r,s) \bar{Z}^i Z^j \tag{2.4}$$

(equivalently, for $0 \leq p + m \leq n$, $\gamma_{r+p,s+m} = \sum_{0 \leq i+j \leq n} a_{ij}(r,s) \gamma_{i+p,j+m}$).

For $p + q = r + s = n+1$, we seek to show that $\tilde{A}_{(p,q)(r,s)} = \tilde{A}_{(s,r)(q,p)}$. The rank-preserving construction of $[A; B]$ implies that

$$\tilde{A}_{(p,q)(r,s)} = \sum_{0 \leq i+j \leq n} a_{ij}(r,s) \gamma_{i+q,j+p}.$$

(2.4) implies that in $\bar{W}^q W^p$ we have

$$\gamma_{q+i,p+j} = \sum_{0 \leq k+\ell \leq n} a_{k\ell}(q,p) \gamma_{k+i,\ell+j},$$

whence $\tilde{A}_{(p,q)(r,s)} = \sum_{0 \leq i+j \leq n, 0 \leq k+\ell \leq n} a_{ij}(r,s) a_{k\ell}(q,p) \gamma_{k+i,\ell+j}$.

Similarly, $\tilde{A}_{(s,r)(q,p)} = \sum a_{ij}(q,p)a_{k\ell}(r,s)\gamma_{k+i,\ell+j}$; in this sum, we replace i by k, j by ℓ, k by i, ℓ by j and conclude that $\tilde{A}_{(p,q)(r,s)} = \tilde{A}_{(s,r)(q,p)}$. □

CHAPTER 3

Positive Moment Matrices and Representing Measures

The main result of this chapter is a structure theorem for finite positive moment matrices. This result is analogous to the recursive structure theorems for positive Hankel and Toeplitz matrices [**CF3**] and will be used in the sequel in obtaining our existence theorems for representing measures. We begin this chapter by describing relationships between $M(n)(\gamma)$ and representing measures for γ; in particular, we will show that dependence relations in the columns of $M(n)$ correspond to planar algebraic curves containing the support of any representing measure for γ. We use this result to prove that the support of any representing measure for γ contains at least $\operatorname{rank} M(n)$ elements.

The basic connection between $M(n)(\gamma)$ and any representing measure μ for γ is provided by the identity

$$\int f\bar{g}\,d\mu = \langle f,g\rangle_{M(n)} = (M(n)\hat{f},\hat{g}) \qquad (f,g\in\mathcal{P}_n). \tag{3.1}$$

Indeed, for $0\le i+j\le 2n$, $\gamma_{i,j}=\int \bar{z}^i z^j\,d\mu$; thus for $0\le p+q\le n$, $0\le k+\ell\le n$, we have $\langle \bar{z}^p z^q, \bar{z}^k z^\ell\rangle_{M(n)} = \gamma_{p+\ell,q+k} = \int \bar{z}^{p+\ell} z^{q+k}\,d\mu = \int (\bar{z}^p z^q)(\overline{\bar{z}^k z^\ell})\,d\mu$, whence (3.1) follows by sesquilinearity. In particular, $0\le \int |f|^2\,d\mu = \Big(M(n)\hat{f},\hat{f}\Big)$ ($f\in\mathcal{P}_n$), so we have the following fundamental necessary condition for the existence of representing measures:

$$\text{If } \gamma \text{ has a representing measure, then } M(n)(\gamma)\ge 0. \tag{3.2}$$

In $M(n)$, for $0\le i+j\le n$, $\bar{Z}^i Z^j$ denotes the unique column whose initial element is γ_{ij}. Let $\mathcal{C}_{M(n)}$ denote the column space of $M(n)$, i.e., the subspace of $\mathbb{C}^{m(n)}$ spanned by $\{\bar{Z}^i Z^j\}_{0\le i+j\le n}$. For $v\in\mathcal{C}_{M(n)}$, the successive elements will be denoted by $v_{r,s}$ $(0\le r+s\le n)$ in the following order:

$$v_{0,0}, v_{1,0}, v_{0,1}, \dots, v_{n,0}, v_{n-1,1}, \dots, v_{1,n-1}, v_{0,n};$$

we refer to $v_{r,s}$ as the (r,s) element of v; note that this subscript ordering is "conjugate" to the lexicographic ordering of $\{e_{ij}\}_{0\le i+j\le n}$.

Our next result is useful in helping locate the support of a representing measure. For $p\in\mathcal{P}_n$, $p\equiv\sum a_{ij}\bar{z}^i z^j$, let $p(Z,\bar{Z}) := \sum a_{ij}\bar{Z}^i Z^j \in \mathcal{C}_{M(n)}$, and let $\mathcal{Z}(p) := \{z\in\mathbb{C} : p(z,\bar{z})=0\}$. Note that for $0\le r+s\le n$, the (r,s) element of $v := p(Z,\bar{Z})$ is equal to $v_{r,s} = \langle v, e_{s,r}\rangle = \langle p, \bar{z}^s z^r\rangle_{M(n)} = \sum_{0\le i+j\le n} a_{ij}\langle \bar{z}^i z^j, \bar{z}^s z^r\rangle_{M(n)} = \sum_{0\le i+j\le n} a_{ij}\gamma_{i+r,j+s}$.

Proposition 3.1 (Cf. [**StSz, Proposition 1**]). *Suppose μ is a representing measure for γ. For $p \in \mathcal{P}_n$, $\operatorname{supp}\mu \subseteq \mathcal{Z}(p) \Leftrightarrow p(Z,\bar{Z}) = \mathbf{0}$.*

Proof. Suppose first that $p(Z,\bar{Z}) = \mathbf{0}$, i.e.,

$$\mathbf{0} = \sum_{0\le i+j\le n} a_{ij}\bar{Z}^i Z^j. \tag{3.3}$$

Since μ is a positive Borel measure, to prove $\operatorname{supp}\mu \subseteq \mathcal{Z}(p)$, it suffices to show that $\int |p|^2\, d\mu = 0$. Now

$$|p|^2 = \sum_{0\le i+j\le n, 0\le r+s\le n} a_{ij}\bar{a}_{rs}\bar{z}^{i+s}z^{j+r}, \tag{3.4}$$

so

$$\int |p|^2\, d\mu = \sum_{0\le i+j\le n, 0\le r+s\le n} a_{ij}\bar{a}_{rs}\gamma_{i+s,j+r}. \tag{3.5}$$

For r, s fixed, $0 \le r+s \le n$, the (s,r) element in (3.3) is $\sum_{0\le i+j\le n} a_{ij}\gamma_{i+s,j+r} = 0$, whence

$$\int |p|^2\, d\mu = \sum_{0\le r+s\le n} \bar{a}_{rs} \sum_{0\le i+j\le n} a_{ij}\gamma_{i+s,j+r} = 0.$$

For the converse, suppose $p \equiv 0$ on $\operatorname{supp}\mu$. For $0 \le r+s \le n$, the (s,r) element of $p(Z,\bar{Z})$ is $\sum_{0\le i+j\le n} a_{ij}\gamma_{i+s,j+r} = \int \bar{z}^s z^r p(z,\bar{z})\, d\mu = \int 0\, d\mu = 0$, so $p(Z,\bar{Z}) = \mathbf{0}$. □

Example 3.2. Consider the moment problem for

$$M(2) \quad := \quad \begin{array}{c} \begin{matrix} \mathit{1} & Z & \bar{Z} & Z^2 & Z\bar{Z} & \bar{Z}^2 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Note that the $\bar{Z} = Z^2$ and $Z\bar{Z} = \mathit{1}$. If there exists a representing measure μ, then Proposition 3.1 implies that on $\operatorname{supp}\mu$, $z^3 = z(z^2) = z\bar{z} = 1$. If z_0, z_1, z_2 denote the distinct cube roots of unity, then $\operatorname{supp}\mu \subseteq \{z_0, z_1, z_2\}$, and it is easy to verify that $v := (1/3)(\delta_{z_0} + \delta_{z_1} + \delta_{z_2})$ is a representing measure. We will show below that v is actually the unique representing measure.

Suppose μ is a representing measure for γ: (3.5) implies that $\mathcal{P}_n \subseteq L^2(\mu)$. Let $p \in \mathcal{P}_n$; since p is *continuous* on $\operatorname{supp}\mu$, it follows that $p = 0$ as an element of $L^2(\mu)$ if and only if $p\,|_{\operatorname{supp}\mu} \equiv 0$. We next define mappings $\psi \equiv \psi(\gamma) : \mathcal{C}_{M(n)} \to L^2(\mu)$, $\rho : \mathcal{C}_{M(n)} \to \mathcal{P}_n\,|_{\operatorname{supp}\mu}$, and $\iota : \mathcal{P}_n\,|_{\operatorname{supp}\mu} \to L^2(\mu)$ by

$$\psi(p(Z,\bar{Z})) := p(z,\bar{z}),$$
$$\rho(p(Z,\bar{Z})) := p(z,\bar{z})\,|_{\operatorname{supp}\mu}$$

and

$$\iota(p(z,\bar{z})\,|_{\operatorname{supp}\mu}) := p(z,\bar{z})$$

($p \in \mathcal{P}_n$), respectively.

PROPOSITION 3.3.

i) *ψ, ρ and ι are well-defined, linear, and one-to-one, ρ is an isomorphism, and $\psi = \iota \circ \rho$;*

ii) *If $f, g, fg \in \mathcal{P}_n$, then $\psi((fg)(Z, \bar{Z})) = \psi(f(Z, \bar{Z}))\psi(g(Z, \bar{Z}))$;*

iii) *If $f \in \mathcal{P}_n$, then $\psi(\bar{f}(Z, \bar{Z})) = \overline{\psi(f(Z, \bar{Z}))}$.*

PROOF. i) follows directly from Proposition 3.1; ii) and iii) are immediate consequences of i). □

COROLLARY 3.4. *Let μ be a representing measure for γ. If $f, g, fg \in \mathcal{P}_n$ and $f(Z, \bar{Z}) = \mathbf{0}$, then $(fg)(Z, \bar{Z}) = \mathbf{0}$.*

PROOF. Since $f(Z, \bar{Z}) = \mathbf{0}$, then $\psi(f(Z, \bar{Z})) = 0$ (Proposition 3.3-i). Now $\psi((fg)(Z, \bar{Z})) = \psi(f(Z, \bar{Z}))\psi(g(Z, \bar{Z})) = 0$, whence $(fg)(Z, \bar{Z}) = \mathbf{0}$ by the injectivity of ψ. □

COROLLARY 3.5. $\dim L^2(\mu) \geq \operatorname{rank} M(n)$.

PROOF. From Proposition 3.3-i), $\operatorname{rank} M(n) = \dim \mathcal{C}_{M(n)} \leq \dim L^2(\mu)$. □

LEMMA 3.6. *Let μ be a finitely atomic positive measure on $\mathbb{C}$, with $k :=$ card supp μ. Then $\{1, z, \ldots, z^{k-1}\}$ is a basis for $L^2(\mu)$.*

PROOF. Lagrange interpolation implies that $\{1, \ldots, z^{k-1}\}$ spans $L^2(\mu)$. If there exist scalars $c_0, \ldots, c_{k-1}$ such that $p(z) \equiv c_0 + \cdots + c_{k-1}z^{k-1}$ satisfies $p = 0$ in $L^2(\mu)$, then $\operatorname{supp} \mu \subseteq \mathcal{Z}(p)$, whence card supp $\mu \leq k - 1$; this contradiction implies that $\{1, \ldots, z^{k-1}\}$ is a basis for $L^2(\mu)$. □

COROLLARY 3.7. *If μ is a representing measure for γ, then* card supp $\mu \geq$ rank $M(n)$.

PROOF. Straightforward from Corollary 3.5 and Lemma 3.6. □

Returning to Example 3.2, we know that for any representing measure μ, $\operatorname{supp} \mu \subseteq \{z_0, z_1, z_2\}$; on the other hand, Corollary 3.7 shows that card supp $\mu \geq \operatorname{rank} M(2) = 3$. Thus $\operatorname{supp} \mu = \{z_0, z_1, z_2\}$, and it follows readily that $\mu := (1/3)(\delta_{z_0} + \delta_{z_1} + \delta_{z_2})$ is the unique representing measure. Since $M(2)$ is clearly a flat extension of $M(1)$, this example is an illustration of Corollary 5.14 (below).

By an *interpolating measure* for γ we mean a (not necessarily positive) Borel measure μ such that $\int \bar{z}^i z^j \, d\mu = \gamma_{i,j}$ for $0 \leq i + j \leq 2n$.

PROPOSITION 3.8. *Let μ be a k-atomic interpolating measure for γ, $k \leq n+1$. If $M(n) \geq 0$, then $\mu \geq 0$.*

PROOF. An interpolating measure satisfies (3.1). Let $\mu = \sum_{1 \leq i \leq k} \rho_i \delta_{w_i}$, where $\operatorname{supp} \mu = \{w_i\}_{i=1}^k$. For $1 \leq j \leq k$, there exists an analytic polynomial f_j of degree $k - 1 (\leq n)$ such that $f_j(w_j) = 1$ and $f_j(w_i) = 0$ $(i \neq j)$. Then by (3.1) and the positivity of $M(n)$, $0 \leq (M(n)\hat{f}_j, \hat{f}_j) = \int |f_j|^2 \, d\mu = \rho_j$; thus $\mu \geq 0$. □

We next present several preliminary results leading to the structure theorem for positive moment matrices. For $A \in M_{k+1}(\mathbb{C})$, $A := (a_{ij})_{0 \le i,j \le k}$, let $v_j := (a_{ij})_{0 \le i \le k}$, denote the j-th column vector $(0 \le j \le k)$. For $0 \le \ell \le k$, let $A(\ell) := (a_{ij})_{0 \le i,j \le \ell} \in M_{\ell+1}(\mathbb{C})$ and let $v(j,\ell)$ denote the j-th column vector of $A(\ell)$ $(0 \le j \le \ell)$. The following result is closely related to the structure of positive 2×2 operator matrices [**Smu**] (cf. Proposition 2.2).

Proposition 3.9 (Extension Principle) [**Fia, Proposition 2.4**]. *Let $A \in M_{k+1}(\mathbb{C})$, $A \ge 0$. If there exist p, $0 \le p \le k$, and scalars $c_0, \ldots, c_p$ such that $c_0 v(0,p) + \cdots + c_p v(p,p) = 0$, then $c_0 v_0 + \cdots + c_p v_p = 0$.*

The referee has pointed out that a more direct proof of Proposition 3.9 can be based on the fact that if $A \ge 0$ and $(Ax, x) = 0$, then $Ax = 0$.

Lemma 3.10. *Let $M(n)$ be a moment matrix and let $p \in \mathcal{P}_n$. If $p(Z, \bar{Z}) = \mathbf{0}$, then $\bar{p}(Z, \bar{Z}) = \mathbf{0}$.*

Proof. Let $p(z, \bar{z}) = \sum_{0 \le i+j \le n} a_{ij} \bar{z}^i z^j$. Now $p(Z, \bar{Z}) = \mathbf{0} \Leftrightarrow \sum a_{ij} \bar{Z}^i Z^j = \mathbf{0} \Leftrightarrow$ for all r, s, $0 \le r+s \le n$, $\sum_{0 \le i+j \le n} a_{ij} \gamma_{i+r,j+s} = 0 \Leftrightarrow$ for all r, s, $0 \le r+s \le n$, $\sum_{0 \le i+j \le n} \bar{a}_{ij} \gamma_{j+s,i+r} = 0 \Leftrightarrow \sum_{0 \le i+j \le n} \bar{a}_{ij} \bar{Z}^j Z^i = \mathbf{0} \Leftrightarrow \bar{p}(Z, \bar{Z}) = \mathbf{0}$. □

Lemma 3.11. *Let $M(n) \ge 0$. If $p \in \mathcal{P}_{n-2}$ and $p(Z,\bar{Z}) = \mathbf{0}$, then $(zp)(Z, \bar{Z}) = \mathbf{0}$.*

Proof. We have $p(z, \bar{z}) = \sum_{0 \le i+j \le n-2} a_{ij} \bar{z}^i z^j$, so

$$\mathbf{0} = Y := \sum_{0 \le i+j \le n-2} a_{ij} \bar{Z}^i Z^j;$$

thus for $0 \le r+s \le n$, the (r,s) entry of Y is $\sum_{0 \le i+j \le n-2} a_{ij} \gamma_{i+r,j+s} = 0$. Let $W := (zp)(Z, \bar{Z}) = \sum_{0 \le i+j \le n-2} a_{ij} \bar{Z}^i Z^{j+1}$. Since $i + j + 1 \le n - 1$, each of the columns of $M(n)$ used in computing W has degree $\le n-1$ and thus extends a column of $M(n-1)$. The entries of W corresponding to degree $\le n-1$ may be described as follows: for $0 \le r+s \le n-1$, the (r,s) entry of W is

$$\sum_{0 \le i+j \le n-2} a_{ij} \gamma_{i+r,j+1+s}. \tag{3.5}$$

This expression coincides with the $(r, s+1)$ entry of Y, whence $W_{(r,s)} = 0$ for $0 \le r+s \le n-1$. In $\mathcal{C}_{M(n-1)}$ we thus have $(zp)(Z, \bar{Z}) = \mathbf{0}$, and since $M(n) \ge 0$, it follows from Proposition 3.9 that $(zp)(Z, \bar{Z}) = \mathbf{0}$ in $\mathcal{C}_{M(n)}$. □

Lemma 3.12. *If $M(n) \ge 0$ and $p \in \mathcal{P}_{n-2}$ satisfies $p(Z, \bar{Z}) = \mathbf{0}$, then $(\bar{z}p)(Z, \bar{Z}) = \mathbf{0}$.*

Proof. We have, successively, $\bar{p}(Z, \bar{Z}) = \mathbf{0}$ (Lemma 3.10), $(z\bar{p})(Z, \bar{Z}) = \mathbf{0}$ (Lemma 3.11), and $(\bar{z}p)(Z, \bar{Z}) = \mathbf{0}$ (Lemma 3.10). □

Lemma 3.13. *If $M(n) \ge 0$, $p \in \mathcal{P}_n$ satisfies $p(Z, \bar{Z}) = \mathbf{0}$ and $r, s \ge 0$ satisfy $r + s + \deg p \le n - 1$, then $(\bar{z}^r z^s p)(Z, \bar{Z}) = \mathbf{0}$.*

PROOF. Successive application of Lemmas 3.11 and 3.12. □

Repeated application of Lemma 3.13 now yields the structure theorem for positive moment matrices.

THEOREM 3.14 (Structure Theorem). *Let* $M(n) \geq 0$. *If* $f, g, fg \in \mathcal{P}_{n-1}$ *and* $f(Z, \bar{Z}) = \mathbf{0}$, *then* $(fg)(Z, \bar{Z}) = \mathbf{0}$.

REMARK 3.15.

i) If γ has a representing measure μ, the conclusion of Theorem 3.14 holds if we merely assume that $fg \in \mathcal{P}_n$. Indeed, under this hypothesis, $f(Z, \bar{Z}) = \mathbf{0} \Rightarrow f \equiv 0 \in L^2(\mu)$ (Proposition 3.3-i) $\Rightarrow fg \equiv 0 \in L^2(\mu) \Rightarrow (fg)(Z, \bar{Z}) = \mathbf{0}$ (Proposition 3.3-i). This establishes (i)⇒(vii) in Conjecture 1.1.

ii) Using Theorem 3.14, we can show that the following condition is necessary for the existence of a positive extension $M(n+1)$ of $M(n)$:

(RG) $$f, g, fg \in \mathcal{P}_n, \quad f(Z, \bar{Z}) = \mathbf{0} \Rightarrow (fg)(Z, \bar{Z}) = \mathbf{0}.$$

Indeed, suppose $M(n+1) \geq 0$ exists and let f, g, fg be in $\mathcal{P}_n$ with $f(Z, \bar{Z}) = \mathbf{0}$ in $C_{M(n)}$. The Extension Principle implies that $f(Z, \bar{Z}) = \mathbf{0}$ in $C_{M(n+1)}$. Since $\deg(fg) \leq n < n+1$, Theorem 3.14 (applied to $M(n+1)$) implies that $(fg)(Z, \bar{Z}) = \mathbf{0}$ in $C_{M(n+1)}$, whence $(fg)(Z, \bar{Z}) = \mathbf{0}$ in $C_{M(n)}$.

Note that the preceding argument proves the implication (v)⇒(vii) of Conjecture 1.1; note also that condition (RG) requires that $\mathrm{J}_\gamma := \{p \in \mathcal{P}_n : p(Z, \bar{Z}) = \mathbf{0}\}$ behave like an "ideal" in the polynomial space $\mathcal{P}_n$; thus (RG) imposes restrictions on the possible values of $\operatorname{rank} M(n)$.

CONJECTURE 3.16 [**Fia**]. *For* $M(n)(\gamma) \geq 0$, *the following are equivalent:*

i) *There exists an extension* $M(n+1) \geq 0$;
ii) *There exists a flat extension* $M(n+1)$;
iii) *There exist flat extensions* $M(n+k)$ *for every* $k \geq 1$;
iv) $M(n)$ *satisfies condition* (RG).

An affirmation of Conjecture 3.16 would provide (RG) as a "concrete" condition equivalent to (v) of the Main Conjecture; when combined with the results of Chapter 4, a proof of Conjecture 3.16 would yield the equivalence of (ii)-(vii) of the Main Conjecture. As we outline below, results of [**Fia**] show that Conjecture 3.16 is true in case $Z = \bar{Z}$. The results of Chapter 5 show that Conjecture 3.16 is also true if $\operatorname{rank} M(n) = \operatorname{rank} M(n-1)$; moreover, Chapter 6 (together with Theorem 5.13) shows that Conjecture 3.16 is true for $n = 1$.

Using Theorem 3.14 it is possible to recover the recursive structure theorems for positive singular Hankel and Toeplitz matrices and to thereby solve the truncated complex moment problem in two important special cases. This development will be presented in detail elsewhere [**Fia**], so here we merely discuss the results.

In the first case, given the truncated moment sequence γ, we assume that $Z = \bar{Z}$; equivalently, there exists a sequence $\beta_0, \ldots, \beta_{2n}$ (of necessarily *real* scalars) such that $\gamma_{ij} = \beta_{i+j}$ $(0 \leq i+j \leq 2n)$. This is the case in which, for each block B_{ij} of $M(n)$, the entries of B_{ij} are all equal to $\gamma_{ij} (= \beta_{i+j})$. We will show that this case is equivalent to the truncated Hamburger moment problem for $\beta_0, \ldots, \beta_{2n}$. Let

$H(n) \equiv H(n)(\beta)$ denote the Hankel matrix $(\beta_{i+j})_{0\le i,j\le n}$ and let $HM(n)$ denote $M(n)(\gamma)$ in this case.

PROPOSITION 3.17 [**Fia**]. $H(n) \ge 0 \Leftrightarrow HM(n) \ge 0$.

Let $1, T, \dots, T^n$ denote the successive columns of $H(n)$. Assume $H(n)$ is positive and singular and let $r = \min\{j : T^j \in \langle 1, T, \dots, T^{j-1}\rangle\}$. Then $1 \le r \le n$ and there exist unique scalars $c_0, \dots, c_{r-1}$ such that $T^r = c_0 1 + \dots + c_{r-1}T^{r-1}$. Using Theorem 3.14 and Proposition 3.17 we can recover the structure theorem for positive singular Hankel matrices.

PROPOSITION 3.18 [**CF3**]. *If $H(n)$ is positive and singular, then*

$$T^{r+s} = c_0 T^s + \dots + c_{r-1}T^{r-1+s} \qquad (0 \le s \le n-r-1);$$

equivalently, $\beta_j = c_0\beta_{j-r} + \dots + c_{r-1}\beta_{j-1}$ $(r \le j \le 2n-1)$.

We can now formulate a solution to the $Z = \bar{Z}$ case of the truncated complex moment problem as follows; this result and Proposition 3.17 readily imply that Conjectures 1.1 and 3.16 are true in the case $Z = \bar{Z}$.

THEOREM 3.19 [**Fia**]. *Suppose γ is a truncated complex moment sequence and $\gamma_{ij} = \beta_{i+j}$ $(0 \le i+j \le 2n)$. The following are equivalent.*

i) *γ has a representing measure;*
ii) *$HM(n)$ admits a positive extension $HM(n+1)$;*
iii) *$H(n)$ admits a positive extension $H(n+1)$;*
iv) *$H(n) \ge 0$, and either $H(n)$ is invertible or $T^n = c_0T^{n-r} + \dots + c_{r-1}T^{n-1}$ (i.e., $\beta_{2n} = c_0\beta_{2n-r} + \dots + c_{r-1}\beta_{2n-1}$).*

In this case, there exists a rank $H(n)$-atomic representing measure; if $H(n)$ is singular, there is a unique representing measure, which has support equal to the r distinct real roots of $t^r - (c_0 + \dots + c_{r-1}t^{r-1})$.

For $n > 1$, we next consider a moment sequence γ for which $Z\bar{Z} = 1$; equivalently, there exists a sequence $\{\beta_k\}_{-2n\le k\le 2n}$ such that $\gamma_{ij} = \beta_{j-i}$ $(0 \le i+j \le 2n)$. In this case we denote $M(n)(\gamma)$ by $TM(n)$ and we let $T(2n) \equiv T(2n)(\beta)$ denote the Toeplitz matrix $(\beta_{j-i})_{0\le i,j\le 2n}$. For $n > 1$, the next result reduces the moment problem for γ to the following *truncated trigonometric moment problem*: $\beta_k = \int z^k\, d\mu$ $(0 \le k \le 2n)$, $\mu \ge 0$, $\operatorname{supp}\mu \subseteq \{z : |z| = 1\}$. Indeed, since $Z\bar{Z} = 1$, any representing measure must have support contained in the unit circle, as required in the trigonometric moment problem [**CF3, Section 6**].

PROPOSITION 3.20 [**Fia**]. $TM(n) \ge 0 \Leftrightarrow T(2n) \ge 0$.

Let $1, Z, \dots, Z^n$ denote the successive columns of $T(2n)$. Assume $T(2n)$ is positive and singular and let $r = \min\{j : Z^j \in \langle 1, \dots, Z^{j-1}\rangle\}$; there exist unique scalars $c_0, \dots, c_{r-1}$ such that $Z^r = c_0 1 + \dots + c_{r-1}Z^{r-1}$. The recursive structure theorem for $T(2n)$ [**CF3**] now assumes the form: $Z^{r+s} = c_0Z^s + \dots + c_{r-1}Z^{r+s-1}$ $(0 \le s \le n-r)$. We can thus formulate a solution of the $Z\bar{Z} = 1$ case of the truncated complex moment problem which establishes the equivalence of i), ii), iii) and v) of the Main Conjecture for the case $Z\bar{Z} = 1$.

PROPOSITION 3.21 [**Fia**]. *Let $n > 1$ and suppose the moment sequence γ satisfies $\gamma_{ij} = \beta_{j-i}$ $(0 \le i+j \le 2n)$. The following are equivalent:*

i) γ *has a representing measure;*
ii) $TM(n) \geq 0$*;*
iii) $T(2n) \geq 0$*;*
iv) γ *has a rank* $T(2n)$*-atomic representing measure.*

Under the conditions of Proposition 3.21, there exists a *unique* representing measure for γ if and only if $T(2n)$ is singular, in which case the support of the representing measure consists of the r distinct roots of $z^r - \left(c_0 + \cdots + c_{r-1}z^{r-1}\right)$. For $n = 1$, the preceding analysis does not apply; indeed $TM(1)$ is an arbitrary moment matrix of the form $M(1)$ and representing measures need not have support on the unit circle in this case. The $M(1)$ moment problem is solved in Chapter 6.

CHAPTER 4

Existence of Representing Measures

The purpose of this chapter is to prove that every positive finite-rank infinite moment matrix admits a finitely atomic representing measure.

Let M be an infinite matrix, and let $\mathcal{C}_M$ denote its associated column space, generated by columns labeled $\mathit{1}$, Z, $\bar{Z}$, Z^2, $Z\bar{Z}$, $\bar{Z}^2$, $\dots$. We regard M as a linear map on $\mathbb{C}_0^\omega := \{v \equiv (v_{ij})_{i,j\geq 0} \in \mathbb{C}^\omega : \ v_{ij} = 0 \text{ for all but finitely many pairs } (i,j)\}$, so that $\mathcal{C}_M = \operatorname{Ran} M$. For $k \geq 1$, let P_k denote the projection of $\mathbb{C}_0^\omega$ onto the first k coordinates. We define $\operatorname{rank} M = \dim \mathcal{C}_M$. We state without proof the following result.

Lemma 4.1. *Let $n_1 < n_2 < \cdots$ be an increasing sequence of nonnegative integers. Then* $\operatorname{rank} M = \sup\limits_k \operatorname{rank} P_{n_k} M P_{n_k}$.

The map $\varphi : \mathbb{C}[z,\bar{z}] \to \mathcal{C}_M$ is defined by $\varphi(\bar{z}^i z^j) := \bar{Z}^i Z^j$, i, $j \geq 0$. It is straightforward to check that $\varphi(p) = M\hat{p} = p(Z,\bar{Z}) = \sum a_{ij} \bar{Z}^i Z^j$, where as usual $\hat{p} := (a_{ij})$ for a given polynomial $p(z,\bar{z}) = \sum_{i,j} a_{ij} \bar{z}^i z^j$. Let $\mathcal{N} := \{p \in \mathbb{C}[z,\bar{z}] : \langle M\hat{p},\hat{p}\rangle = 0\}$, and let $\ker\varphi := \{p \in \mathbb{C}[z,\bar{z}] : \varphi(p) = \mathbf{0}\}$. It is not difficult to see that $\ker\varphi \subseteq \mathcal{N}$. Our next lemma shows that this containment is indeed an equality.

Lemma 4.2. *Let M be a positive infinite matrix. Then $\mathcal{N} = \ker\varphi$.*

Proof. Let $p \in \mathcal{N}$, i.e., $\langle M\hat{p},\hat{p}\rangle = 0$. To show that $M\hat{p} = 0$, it suffices to prove that for every $q \in \mathbb{C}[z,\bar{z}]$, $\langle M\hat{p},\hat{q}\rangle = 0$. Let $q \in \mathbb{C}[z,\bar{z}]$. Choose $k \equiv k(p,q)$ such that $\hat{p},\hat{q} \in \operatorname{Ran} P_k$, and set $M_k := P_k M P_k$. Since $\langle M\hat{p},\hat{p}\rangle = 0$, then

$$\|\sqrt{M_k}\hat{p}\|^2 = \langle M_k\hat{p},\hat{p}\rangle = \langle M P_k\hat{p}, P_k\hat{p}\rangle = \langle M\hat{p},\hat{p}\rangle = 0,$$

so that $M_k\hat{p} = 0$. Now $\langle M\hat{p},\hat{q}\rangle = \langle M P_k\hat{p}, P_k\hat{q}\rangle = \langle M_k\hat{p},\hat{q}\rangle = 0$. □

We now proceed to consider the quotient space $\mathbb{C}[z,\bar{z}]/\mathcal{N}$, in case M is a positive infinite *moment* matrix. We define a sesquilinear form on $\mathbb{C}[z,\bar{z}]$ by $\langle p,q\rangle_M = \langle M\hat{p},\hat{q}\rangle$ $(p,q \in \mathbb{C}[z,\bar{z}])$; thus $\gamma_{ij} = \langle z^j, z^i\rangle_M$.

The following analogue of Theorem 3.14 gives the structure of a positive infinite moment matrix.

Proposition 4.3. *Let M be a positive infinite moment matrix. Then $\ker\varphi$ is an ideal of $\mathbb{C}[z,\bar{z}]$.*

Proof. Let $p \in \ker\varphi$, $q \in \mathbb{C}[z,\bar{z}]$, let $K := \deg pq$ and let $k \geq K$. In $\mathcal{C}_{M(k+1)}$, $p(Z,\bar{Z}) = \mathbf{0}$, so Theorem 3.14 implies that $(pq)(Z,\bar{Z}) = \mathbf{0}$ in $\mathcal{C}_{M(k+1)}$. Since $k \geq K$ is arbitrary, then $(pq)(Z,\bar{Z}) = \mathbf{0}$ in $\mathcal{C}_M$, i.e., $pq \in \ker\varphi$. □

Lemma 4.2 and Proposition 4.3 imply that $\mathcal{N}$ is an ideal of $\mathbb{C}[z,\bar{z}]$, so the operator of multiplication by z acting on $\mathbb{C}[z,\bar{z}]$ factors through $\mathcal{N}$ to give rise to an induced multiplication on $\mathbb{C}[z,\bar{z}]/\mathcal{N}$, which we will denote by M_z. Moreover, $\mathbb{C}[z,\bar{z}]/\mathcal{N}$ admits a natural inner product, namely $\langle f+\mathcal{N}, g+\mathcal{N}\rangle := \langle f,g\rangle_M = \langle M\hat{f},\hat{g}\rangle$, for $f,g \in \mathbb{C}[z,\bar{z}]$. Using Lemma 4.2, it is straightforward to verify that $\langle\cdot,\cdot\rangle$ is well-defined, sesquilinear, and positive semi-definite, and clearly $\langle f+\mathcal{N}, f+\mathcal{N}\rangle = 0$ implies $f \in \mathcal{N}$.

Lemma 4.4. *Let M be a finite-rank positive infinite moment matrix. Then $\mathbb{C}[z,\bar{z}]/\mathcal{N}$ is a finite dimensional Hilbert space and* $\dim \mathbb{C}[z,\bar{z}]/\mathcal{N} = \operatorname{rank} M$.

Proof. Consider the map $\Phi : \mathcal{C}_M \to \mathbb{C}[z,\bar{z}]/\mathcal{N}$ defined by $\Phi(p(Z,\bar{Z})) := p+\mathcal{N}$, $p \in \mathbb{C}[z,\bar{z}]$. If $q \in \mathbb{C}[z,\bar{z}]$ and $q(Z,\bar{Z}) = p(Z,\bar{Z})$, then $q-p \in \ker\varphi = \mathcal{N}$, so $q+\mathcal{N} = p+\mathcal{N}$; thus Φ is well-defined. Φ is clearly linear and surjective; if $\Phi(p(Z,\bar{Z})) = 0$, then $p \in \mathcal{N} = \ker\varphi$, so $p(Z,\bar{Z}) = \varphi(p) = 0$. Thus Φ is an isomorphism, so $\dim \mathbb{C}[z,\bar{z}]/\mathcal{N} = \dim C_M = \operatorname{rank} M < \infty$, and therefore $\mathbb{C}[z,\bar{z}]/\mathcal{N}$ is a complete pre-Hilbert space. □

In the next result we use the elementary fact (valid for arbitrary matrices M) that if $p,q \in \mathbb{C}[z,\bar{z}]$ and $\hat{p},\hat{q} \in \operatorname{Ran} P_k$, then $\langle p,q\rangle_M = \langle p,q\rangle_{P_k M P_k|\operatorname{Ran} P_k}$.

Lemma 4.5. *Let M be a finite-rank positive infinite moment matrix. Then M_z, acting on $\mathbb{C}[z,\bar{z}]/\mathcal{N}$, is normal.*

Proof. We shall first verify that $M_z^* = M_{\bar{z}}$. Let $f,g \in \mathbb{C}[z,\bar{z}]$ and let $k := 1+\max\{\deg f, \deg g\}$. Then

$$\begin{aligned}
&\langle M_z^*(f+\mathcal{N}), g+\mathcal{N}\rangle \\
&\quad = \langle f+\mathcal{N}, z(g+\mathcal{N})\rangle = \langle f+\mathcal{N}, zg+\mathcal{N}\rangle \\
&\quad = \langle f, zg\rangle_M = \langle f, zg\rangle_{M(k)} = \langle \bar{z}f, g\rangle_{M(k)} \qquad \text{(by Theorem 2.1(iii))} \\
&\quad = \langle \bar{z}f, g\rangle_M = \langle \bar{z}f+\mathcal{N}, g+\mathcal{N}\rangle = \langle \bar{z}(f+\mathcal{N}), g+\mathcal{N}\rangle.
\end{aligned}$$

A similar argument using Theorem 2.1(iv) implies that $\langle zf, zg\rangle_M = \langle \bar{z}f, \bar{z}g\rangle_M$ for all $f,g \in \mathbb{C}[z,\bar{z}]$, from which it follows that

$$\begin{aligned}
\langle z(f+\mathcal{N}), z(g+\mathcal{N})\rangle &= \langle zf+\mathcal{N}, zg+\mathcal{N}\rangle = \langle zf, zg\rangle_M \\
&= \langle \bar{z}f, \bar{z}g\rangle_M = \langle \bar{z}f+\mathcal{N}, \bar{z}g+\mathcal{N}\rangle \\
&= \langle \bar{z}(f+\mathcal{N}), \bar{z}(g+\mathcal{N})\rangle
\end{aligned}$$

$(f,g \in \mathbb{C}[z,\bar{z}])$. Therefore, $\|M_z(f+\mathcal{N})\|^2 = \|M_{\bar{z}}(f+\mathcal{N})\|^2 = \|M_z^*(f+\mathcal{N})\|^2$, i.e., M_z is normal. □

Proposition 4.6. *Let M be an infinite moment matrix with representing measure μ. Then* $\operatorname{card}\operatorname{supp}\mu = \operatorname{rank} M$.

Proof. For each $n > 0$, since μ is a representing measure for $M(n)$, Corollary 3.7 implies that $k := \operatorname{card}\operatorname{supp}\mu \geq \operatorname{rank} M(n)$. Lemma 4.1 implies that $r := \operatorname{rank} M = \sup \operatorname{rank} M(n)$, so $k \geq r$. Assume $k > r$, let m be an integer such that $k \geq m > r$, and let $w_1, \ldots, w_m$ be m distinct points in $\operatorname{supp}\mu$. By Lagrange interpolation, there exist analytic polynomials $f_1, \ldots, f_m$ (of degree $m-1$) such that $f_i(w_j) = \delta_{ij}$ $(1 \le i, j \le m)$. Since $\{f_1, \ldots, f_m\}$ is linearly independent in $\mathcal{P}_{m-1} \mid_{\operatorname{supp}\mu}$, Proposition 3.3-i) implies that $\{f_1(Z, \bar{Z}), \ldots, f_m(Z, \bar{Z})\}$ is linearly independent in $\mathcal{C}_{M(m-1)}$. Therefore $\operatorname{rank} M(m-1) \ge m$, which forces $r \ge m$, a contradiction; thus, $k = r$. □

THEOREM 4.7. *Let M be a finite-rank positive infinite moment matrix. Then M has a unique representing measure, which is $\operatorname{rank} M$-atomic. In this case, let $r := \operatorname{rank} M$; there exist unique scalars $\alpha_0, \ldots, \alpha_{r-1}$ such that $Z^r = \alpha_0 1 + \cdots + \alpha_{r-1} Z^{r-1}$. The unique representing measure for M has support equal to the r distinct roots $z_0, \ldots, z_{r-1}$ of the polynomial $z^r - (\alpha_0 + \cdots + \alpha_{r-1} z^{r-1})$, and densities $\rho_0, \ldots, \rho_{r-1}$ determined by the Vandermonde equation*

$$V(z_0, \ldots, z_{r-1})(\rho_0, \ldots, \rho_{r-1})^T = (\gamma_{00}, \ldots, \gamma_{0,r-1})^T.$$

PROOF. Lemma 4.5 shows that M_z, acting on $\mathbb{C}[z, \bar{z}]/\mathcal{N}$, is normal. By the Spectral Theorem, $C^*(M_z) \cong C(\sigma(M_z))$, and the linear functional $\eta(f) := \langle f(M_z)(1+\mathcal{N}), 1 + \mathcal{N}\rangle$ $(f \in C(\sigma(M_z)))$ is positive. Thus, the Riesz Representation Theorem implies that there exists a positive Borel measure μ, with $\operatorname{supp}\mu \subseteq \sigma(M_z)$, such that $\eta(f) = \int f d\mu$. Then

$$\begin{aligned} \int \bar{z}^i z^j d\mu = \eta(\bar{z}^i z^j) &= \langle M_z^{*i} M_z^j (1 + \mathcal{N}), 1 + \mathcal{N}\rangle \\ &= \langle z^j + \mathcal{N}, z^i + \mathcal{N}\rangle = \langle z^j, z^i\rangle_M = \gamma_{ij}. \end{aligned}$$

Thus μ is a representing measure for M. Proposition 4.6 implies that

$$\operatorname{card}\operatorname{supp}\mu = \operatorname{rank} M = r < \infty.$$

By Lemma 3.6, $\{1, z, \ldots, z^{r-1}\}$ is a basis for $L^2(\mu)$, and hence, by Proposition 3.3-i), $\{1, Z, \ldots, Z^{r-1}\}$ is independent in $\mathcal{C}_{M(r-1)}$; thus $\{1, Z, \ldots, Z^{r-1}\}$ is independent in $\mathcal{C}_M$. Since $r = \operatorname{rank} M = \dim \mathcal{C}_M$, then $\{1, Z, \ldots, Z^{r-1}\}$ is a basis for $\mathcal{C}_M$. Thus there exist unique scalars $\alpha_0, \ldots, \alpha_{r-1}$ such that $Z^r = \alpha_0 1 + \cdots + \alpha_{r-1} Z^{r-1}$ in $\mathcal{C}_M$. In $\mathcal{C}_{M(r)}$ we thus have the same relation, so Proposition 3.1 implies that $\operatorname{supp}\mu \subseteq \mathcal{Z}(p)$, where $p(z) := z^r - (\alpha_0 + \cdots + \alpha_{r-1} z^{r-1})$. Now $r = \operatorname{card}\operatorname{supp}\mu \le \operatorname{card}\mathcal{Z}(p) \le r$, so p has exactly r distinct roots, say $z_0, \ldots, z_{r-1}$, and $\operatorname{supp}\mu = \mathcal{Z}(p)$. Thus μ is of the form $\mu = \sum_{i=0}^{r-1} \rho_i \delta_{z_i}$, and since μ interpolates γ_{0i} $(0 \le i \le r-1)$, then $\rho_0, \ldots, \rho_{r-1}$ are uniquely determined from the Vandermonde equation $V(z_0, \ldots, z_{r-1})(\rho_0, \ldots, \rho_{r-1})^T = (\gamma_{00}, \ldots, \gamma_{0,r-1})^T$. □

We remark that, in the last proof, $\sigma(M_z) = \mathcal{Z}(p)$. Indeed, $\mathcal{Z}(p) = \operatorname{supp}\mu \subseteq \sigma(M_z)$ and $r = \operatorname{card}\mathcal{Z}(p) = \operatorname{card}\operatorname{supp}\mu \le \operatorname{card}\sigma(M_z) \le \dim \mathbb{C}[z, \bar{z}]/\mathcal{N} = \operatorname{rank} M = r$. Note also that $r = \min\{j : Z^j \in \langle 1, \ldots, Z^{j-1}\rangle\}$.

We also note that an *infinite-rank* positive moment matrix need not have a representing measure. Indeed, for the full moment problem in two real variables, Berg-Christensen-Jensen [**BCJ**] and Schmüdgen [**Sch1**] independently proved the existence of a *positive definite multisequence* β for which there is no representing measure [**Fug**], [**Ber**]; since the 2-dimensional real moment problem is equivalent to the complex moment problem (see Chapter 6), β gives rise to a complex moment sequence γ such that $M(\infty)(\gamma) \ge 0$ and such that γ has no representing measure.

This pathology is a genuinely multidimensional phenomenon; indeed, Hamburger's Theorem implies that if $M(\infty)(\gamma) \geq 0$ and $Z = \bar{Z}$, then γ *does* have a representing measure.

CHAPTER 5

Extension of Flat Positive Moment Matrices

In this chapter, we shall see that every finite flat positive moment matrix can be extended in a unique way to an infinite positive moment matrix, which has the same rank. By combining this result with those in the preceding chapter, we derive that for such matrices, the truncated moment problem always admits a solution.

DEFINITION 5.1. Let $M(n) \equiv M(n)(\gamma)$ be a finite moment matrix, with columns labeled $1, Z, \bar{Z}, \ldots, Z^n, \ldots, \bar{Z}^n$. We say that $M(n)$ (and therefore γ) is *flat* if there exist polynomials $p_{ij} \in \mathcal{P}_{n-1}$ such that $\bar{z}^i z^j - p_{ij}(z, \bar{z}) \in \ker \varphi$ for all i, j with $i + j = n$, where φ is the map defined in Chapter 4 (i.e., $\bar{Z}^i Z^j = p_{ij}(Z, \bar{Z})$ in $\mathcal{C}_{M(n)}$).

The remarks following Proposition 2.2 make it clear that if $M(n) \geq 0$, flatness is equivalent to the condition $\operatorname{rank} M(n) = \operatorname{rank} M(n-1)$ (i.e., $M(n)$ is a flat extension of $M(n-1)$). In order to construct a positive extension $M(n+1)$ whose rank equals that of $M(n)$, we shall make use of Smul'jan's Theorem (Proposition 2.2). Thus, we will be searching for a moment matrix $M(n+1)$ of the form

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where $A = M(n)$, $B = AW$, and $C = W^*AW$. The next lemma, however, does not require that A be $M(n)$.

LEMMA 5.2. *Let A, B, C and $\tilde{A}$ be as above, let $V_1, \ldots, V_m$ be the columns of A, let $V_{m+1}, \ldots, V_{m+p}$ be the columns of B, and let $\tilde{V}_1, \ldots, \tilde{V}_m, \tilde{V}_{m+1}, \ldots, \tilde{V}_{m+p}$ be the columns of $\tilde{A}$. Assume that $\tilde{A} \geq 0$.*

(i) *If there exist scalars $a_1, \ldots, a_m$ such that $\sum_{i=1}^m a_i V_i = 0$, then $\sum_{i=1}^m a_i \tilde{V}_i = 0$.*

(ii) *If $\tilde{A}$ is a flat extension of A and $\sum_{i=1}^{m+p} a_i V_i = 0$, then $\sum_{i=1}^{m+p} a_i \tilde{V}_i = 0$.*

PROOF. (i) follows from the Extension Principle (Proposition 3.9).

(ii) Let $\mathbf{a} := (a_1, \ldots a_n)^T$, $\mathbf{b} := (a_{m+1}, \ldots, a_{m+p})^T$, and $\mathbf{x} := \binom{\mathbf{a}}{\mathbf{b}}$. Then $A\mathbf{a} + B\mathbf{b} = \mathbf{0}$, and by the positivity and flatness of $\tilde{A}$ there exists W such that $B = AW$ and $C = W^*B$. Then $B^*\mathbf{a} + C\mathbf{b} = W^*A\mathbf{a} + W^*B\mathbf{b} = 0$, which readily implies that $\tilde{A}\mathbf{x} = \mathbf{0}$, and the result follows. □

When $A = M(n)(\gamma)$, we shall, as usual, denote the columns of A by $\{\bar{Z}^i Z^j\}_{0 \leq i+j \leq n}$; we shall also use $\{\bar{Z}^i Z^j\}_{0 \leq i+j \leq n+1}$ to denote the columns of $(A\ B)$, and use $\{\tilde{\bar{Z}}^i \tilde{Z}^j\}_{0 \leq i+j \leq n+1}$ to denote the columns of $\tilde{A}$. If V is in the column space of $\tilde{A}$, we denote by $[V]_p$ the truncation of V through monomials of degree p in the lexicographic ordering of the rows of $\tilde{A}$; we use similar notation for the truncation of general vectors in $\mathbb{C}^{m(n)}$ and $\mathbb{C}^{m(n+1)}$.

Let $[M(n)]_{n-1} := (B_{ij})_{0\le i\le n-1, 0\le j\le n}$, let $[B]_{n-1} := (B_{i,n+1})_{0\le i\le n-1}$, and let $\{\bar{V}^i V^j\}_{0\le i+j\le n+1}$ denote the columns of the block $S := ([M(n)]_{n-1}\ [B]_{n-1})$. An obvious adaptation of the proof of Lemma 3.10 shows:

$$\text{If } p \in \mathcal{P}_{n+1} \text{ and } p(V,\bar{V}) = \mathbf{0} \text{ in } \mathcal{C}_S, \text{ then } \bar{p}(V,\bar{V}) = \mathbf{0}. \tag{5.1}$$

Similarly, an adaptation of the proof of Theorem 3.14 shows:

$$\begin{gathered}\text{If } p,q \in \mathcal{P}_n,\ pq \in \mathcal{P}_{n+1}, \text{ then} \\ p(Z,\bar{Z}) = \mathbf{0} \text{ in } \mathcal{C}_{M(n)} \Rightarrow (pq)(V,\bar{V}) = \mathbf{0} \text{ in } \mathcal{C}_S.\end{gathered} \tag{5.2}$$

We shall assume in the sequel that γ satisfies the following flatness requirement:

$$\begin{gathered}\text{For all } i,j \ge 0 \text{ with } i+j=n, \\ \text{there exists } p_{ij} \in \mathcal{P}_{n-1} \text{ such that } \bar{Z}^i Z^j = p_{ij}(Z,\bar{Z}).\end{gathered} \tag{5.3}$$

Lemma 5.3. *Assume that $\tilde{A} \ge 0$.*

(i) $p_{ij}(\widetilde{Z},\overline{\widetilde{Z}}) = \overline{\widetilde{Z}}^i \widetilde{Z}^j$

(ii) $\bar{p}_{ij}(\widetilde{Z},\overline{\widetilde{Z}}) = \overline{\widetilde{Z}}^j \widetilde{Z}^i$.

Proof. (i) Use Lemma 5.2(i). (ii) Combine Lemma 5.2(i) and Lemma 3.10.□

Assume that $A \equiv M(n)$ is positive and satisfies (5.3). We now proceed to define the matrix B for the proposed flat extension $\tilde{A} \equiv M(n+1)$; we denote the columns of B by $Z^{n+1}, \dots, \bar{Z}^{n+1}$. Let k,ℓ be such that $k+\ell = n+1$.

Case 1. $k \ge 1$
$\bar{Z}^k Z^\ell := \varphi(\bar{z}p_{k-1,\ell})$.
To check that $\bar{Z}^k Z^\ell$ is well-defined, suppose $p \in \mathcal{P}_{n-1}$ satisfies $\bar{Z}^{k-1}Z^\ell = p(Z,\bar{Z})$; (5.2) implies $(\bar{z}p_{k-1,\ell})(V,\bar{V}) = (\bar{z}p)(V,\bar{V})$. Since $\bar{z}p_{k-1,\ell}, \bar{z}p \in \mathcal{P}_n$ and $M(n)(\ge 0)$ is a flat extension of $M(n-1)$, Lemma 5.2-ii) implies $(\bar{z}p_{k-1,\ell})(Z,\bar{Z}) = (\bar{z}p)(Z,\bar{Z})$.

Case 2. $k = 0$ ($\Rightarrow \ell = n+1$)
$Z^{n+1} := \varphi(zp_{0,n})$ (the proof that Z^{n+1} is well-defined is similar to the argument used in Case 1).

The argument used to prove that B is well-defined can also be used to verify that $M(n)$ satisfies property (RG). Having constructed B, which is obviously of the form AW for some W, we let $C := W^*AW$ and $\tilde{A} := [A;B]$. We note for future reference that for $p,q \in \mathcal{P}_{n+1}$,

$$\langle p,q\rangle_{\tilde{A}} \equiv \langle \tilde{A}\hat{p},\hat{q}\rangle = \langle p(\widetilde{Z},\overline{\widetilde{Z}}),\hat{q}\rangle = \langle \hat{p}, q(\widetilde{Z},\overline{\widetilde{Z}})\rangle$$

(since $\tilde{A}$ is self-adjoint). Moreover, if $p,q \in \mathcal{P}_n$, then $\langle p(\widetilde{Z},\overline{\widetilde{Z}}),\hat{q}\rangle = \langle p(Z,\bar{Z}),[\hat{q}]_n\rangle = \langle p,q\rangle_A$; thus

$$\langle p,q\rangle_{\tilde{A}} = \langle p,q\rangle_A \qquad (p,q \in \mathcal{P}_n). \tag{5.4}$$

THEOREM 5.4. *If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.*

To prove Theorem 5.4, we will establish that
(i) $\tilde{A}$ is a moment matrix;
(ii) $\tilde{A}$ is an extension of $M(n)$;
(iii) $\tilde{A}$ is flat;
(iv) $\tilde{A}$ is the unique flat extension of $M(n)$ of the form $M(n+1)$.

The proof of (i) will be the result of a series of lemmas and propositions, aimed at establishing four conditions in Theorem 2.1, namely:
(a) $\tilde{A}$ is self-adjoint;
(b) $\langle p, q\rangle_{\tilde{A}} = \langle \bar{q}, \bar{p}\rangle_{\tilde{A}}$ $(p, q \in \mathcal{P}_{n+1})$;
(c) $\langle zp, q\rangle_{\tilde{A}} = \langle p, \bar{z}q\rangle_{\tilde{A}}$ $(p, q \in \mathcal{P}_n)$;
(d) $\langle zp, zq\rangle_{\tilde{A}} = \langle \bar{z}p, \bar{z}q\rangle_{\tilde{A}}$ $(p, q \in \mathcal{P}_n)$.

(ii) Since $\tilde{A}_{(i,j)(k,\ell)} = \langle \bar{z}^k z^\ell, \bar{z}^i z^j\rangle_{\tilde{A}}$, we need to check that $\langle \bar{z}^k z^\ell, \bar{z}^i z^j\rangle_{\tilde{A}} = \gamma_{k+j,\ell+i}$ for $k+\ell = n+1$ and $i+j \leq n-1$. As before, there are two cases, $k \geq 1$ and $k = 0$. When $k \geq 1$ and $i+j \leq n-1$,

$$\begin{aligned}
\langle \bar{z}^k z^\ell, \bar{z}^i z^j\rangle_{\tilde{A}} &= \langle \tilde{A}\widehat{\bar{z}^k z^\ell}, \widehat{\bar{z}^i z^j}\rangle = \langle [\tilde{A}\widehat{\bar{z}^k z^\ell}]_n, [\widehat{\bar{z}^i z^j}]_n\rangle && \text{(since } i+j < n\text{)}\\
&= \langle (\bar{z}p_{k-1,\ell})(\bar{Z}, Z), [\widehat{\bar{z}^i z^j}]_n\rangle\\
&= \langle A(\widehat{\bar{z}p_{k-1,\ell}}), \widehat{\bar{z}^i z^j}\rangle\\
&= \langle \bar{z}p_{k-1,\ell}, \bar{z}^i z^j\rangle_A = \langle p_{k-1,\ell}, \bar{z}^i z^{j+1}\rangle_A && \text{(by Theorem 2.1)}\\
&= \langle \bar{z}^{k-1} z^\ell, \bar{z}^i z^{j+1}\rangle_A = \gamma_{(k-1)+(j+1),\ell+i} && \text{(since } A = M(n)\text{)}\\
&= \gamma_{k+j,\ell+i}.
\end{aligned}$$

For $k = 0, \ell = n+1, i+j \leq n-1$,

$$\begin{aligned}
\langle z^{n+1}, \bar{z}^i z^j\rangle_{\tilde{A}} &= \langle [\widetilde{Z}^{n+1}]_n, [\widehat{\bar{z}^i z^j}]_n\rangle = \langle zp_{0,n}, \bar{z}^i z^j\rangle_A\\
&= \langle p_{0,n}, \bar{z}^{i+1} z^j\rangle_A = \langle z^n, \bar{z}^{i+1} z^j\rangle_A = \gamma_{j,n+i+1} = \gamma_{k+j,\ell+i}.
\end{aligned}$$

(iii) $\tilde{A}$ is flat by construction.

(iv) will be proved after (i) is completed.

For (i) we must consider the four above-mentioned conditions in Theorem 2.1; (a) is clear from the definition of $\tilde{A}$, but the other three conditions require some arguments involving the flatness of γ. In verifying these conditions, the sesquilinearity of the inner product will allow us to restrict attention to monomials of the form $\bar{z}^k z^\ell$. We begin with condition (b). It is clear that there is no loss of generality in assuming that the degrees of p and q are not simultaneously less than $n+1$ (see the remark immediately preceding Theorem 5.4). Thus, the proof will be divided into two cases, one corresponding to the top degrees (to be given later) and one to the case when exactly one of p and q has degree $n+1$. This case, in turn, will be split into two subcases: first we establish that if $\langle \bar{z}^k z^\ell, \bar{z}^i z^j\rangle_{\tilde{A}} = \langle z^i \bar{z}^j, z^k \bar{z}^\ell\rangle_{\tilde{A}}$ for $k+\ell = n+1$ and $0 \leq i+j \leq n-1$, then the same is true when $k+\ell = n+1$ and $i+j = n$. We then show that $\langle \bar{z}^k z^\ell, \bar{z}^i z^j\rangle_{\tilde{A}} = \langle z^i \bar{z}^j, z^k \bar{z}^\ell\rangle_{\tilde{A}}$ does indeed hold for $k+\ell = n+1$ and $0 \leq i+j \leq n-1$.

Lemma 5.5 Condition (b), Case $\deg p = n+1$, $\deg q \le n$, First Subcase). *Assume that* $\langle \bar z^k z^\ell, \bar z^i z^j\rangle_{\tilde A} = \langle z^i \bar z^j, z^k \bar z^\ell\rangle_{\tilde A}$ *whenever* $k+\ell = n+1$ *and* $i+j \le n-1$. *Then* $\langle \bar z^k z^\ell, \bar z^i z^j\rangle_{\tilde A} = \langle z^i \bar z^j, z^k \bar z^\ell\rangle_{\tilde A}$ *holds for* $k+\ell = n+1$ *and* $i+j=n$.

Proof. We have

$$\langle \bar z^k z^\ell, \bar z^i z^j\rangle_{\tilde A} = \langle \widehat{\bar z^k z^\ell}, \overline{\widetilde{Z}}^i \widetilde{Z}^j\rangle = \langle \widehat{\bar z^k z^\ell}, p_{ij}(\widetilde{Z}, \overline{\widetilde{Z}})\rangle = \langle \bar z^k z^\ell, p_{ij}(z,\bar z)\rangle_{\tilde A}$$

(where we have used the definition of $\langle\cdot,\cdot\rangle_{\tilde A}$ and Lemma 5.3–(i))

$$\begin{aligned} &= \langle \bar p_{ij}(z,\bar z), z^k \bar z^\ell\rangle_{\tilde A} = \langle \bar p_{ij}(\widetilde{Z}, \overline{\widetilde{Z}}), \widehat{z^k \bar z^\ell}\rangle \\ &= \langle \overline{\widetilde{Z}}^j \widetilde{Z}^i, \widehat{z^k \bar z^\ell}\rangle \text{ (by Lemma 5.3–(ii))} = \langle z^i \bar z^j, z^k \bar z^\ell\rangle_{\tilde A}. \square \end{aligned}$$

Lemma 5.6 (Condition (b), Case $\deg p = n+1$, $\deg q \le n$, Second Subcase). *Let* $k+\ell = n+1$ *and* $i+j \le n-1$. *Then* $\langle \bar z^k z^\ell, \bar z^i z^j\rangle_{\tilde A} = \langle z^i \bar z^j, z^k \bar z^\ell\rangle_{\tilde A}$.

Proof. Assume first that $k \ge 1$. Since $\bar Z^{k-1} Z^\ell = p_{k-1,\ell}(Z,\bar Z)$, then (5.2) implies $\bar V^k V^\ell = (\bar z p_{k-1,\ell})(V,\bar V)$. Also, Lemma 5.2-(ii) and (ii) (above) imply $[\overline{\widetilde{Z}}^k \widetilde{Z}^\ell]_{n-1} = [\bar Z^k Z^\ell]_{n-1} = \bar V^k V^\ell$; thus $[\overline{\widetilde{Z}}^k \widetilde{Z}^\ell]_{n-1} = (\bar z p_{k-1,\ell})(V,\bar V)$. From (5.1) we have $\bar V^\ell V^k = (z \bar p_{k-1,\ell})(V,\bar V)$ and (as just before) $[\overline{\widetilde{Z}}^\ell \widetilde{Z}^k]_{n-1} = \bar V^\ell V^k$, so $[\overline{\widetilde{Z}}^\ell \widetilde{Z}^k]_{n-1} = (z\bar p_{k-1,\ell})(V,\bar V)$. Now,

$$\begin{aligned} \langle \bar z^k z^\ell, \bar z^i z^j\rangle_{\tilde A} &= \langle \overline{\widetilde{Z}}^k \widetilde{Z}^\ell, \widehat{\bar z^i z^j}\rangle = \langle [\overline{\widetilde{Z}}^k \widetilde{Z}^\ell]_{n-1}, [\widehat{\bar z^i z^j}]_{n-1}\rangle \\ &= \langle (\bar z p_{k-1,\ell})(V,\bar V), [\widehat{\bar z^i z^j}]_{n-1}\rangle = \langle [(\bar z p_{k-1,\ell})(Z,\bar Z)]_{n-1}, [\widehat{\bar z^i z^j}]_{n-1}\rangle \\ &= \langle \bar z p_{k-1,\ell}, \bar z^i z^j\rangle_A = \langle \bar z^j z^i, z\bar p_{k-1,\ell}\rangle_A \\ &= \langle [\widehat{\bar z^j z^i}]_{n-1}, (z\bar p_{k-1,\ell})(V,\bar V)\rangle \\ &= \langle [\widehat{\bar z^j z^i}]_{n-1}, [\overline{\widetilde{Z}}^\ell \widetilde{Z}^k]_{n-1}\rangle = \langle \widehat{\bar z^j z^i}, \overline{\widetilde{Z}}^\ell \widetilde{Z}^k\rangle = \langle \bar z^j z^i, \bar z^\ell z^k\rangle_{\tilde A}. \end{aligned}$$

The case when $k=0, \ell = n+1$ is somewhat simpler; we leave the details to the reader. $\square$

Proposition 5.7 (Condition (b), Case $\deg p = \deg q = n+1$). *If* $k+\ell = i+j = n+1$, *then* $\langle \bar z^k z^\ell, \bar z^i z^j\rangle_{\tilde A} = \langle z^i \bar z^j, z^k \bar z^\ell\rangle_{\tilde A}$.

We shall need the following lemma.

Lemma 5.8. *If* $k+\ell = n+1$, *then there exists a polynomial* $r_{k,\ell} \in \mathcal{P}_{n-1}$ *such that* $\overline{\widetilde{Z}}^k \widetilde{Z}^\ell = r_{k,\ell}(\widetilde{Z}, \overline{\widetilde{Z}})$ *and* $\overline{\widetilde{Z}}^\ell \widetilde{Z}^k = \bar r_{k,\ell}(\widetilde{Z}, \overline{\widetilde{Z}})$.

Proof. Assume first that $k,\ell \ge 1$. From (5.3), there exists $p_{k-1,\ell} \in \mathcal{P}_{n-1}$ such that $\bar Z^{k-1} Z^\ell = p_{k-1,\ell}(Z,\bar Z)$, so $\bar Z^k Z^\ell$ in block B is defined by $\bar Z^k Z^\ell := (\bar z p_{k-1,\ell})(Z,\bar Z)$. Since $\bar z p_{k-1,\ell} \in \mathcal{P}_n$, (5.3) implies that there exists $r_{k,\ell} \in \mathcal{P}_{n-1}$ such that $\bar Z^k Z^\ell = r_{k,\ell}(Z,\bar Z)$, whence $\bar V^k V^\ell = r_{k,\ell}(V,\bar V)$; (5.1) now implies that $\bar V^\ell V^k = \bar r_{k,\ell}(V,\bar V)$. Similarly, since $\ell \ge 1$, there exists $s_{\ell,k} \in \mathcal{P}_{n-1}$ such that $\bar Z^\ell Z^k = s_{\ell,k}(Z,\bar Z)$, whence $s_{\ell,k}(V,\bar V) = \bar V^\ell V^k = \bar r_{k,\ell}(V,\bar V)$. Thus $s_{\ell,k}(V,\bar V) =$

$\bar{r}_{k,\ell}(V,\bar{V})$ in $\mathcal{C}_{M(n-1)}$; since $M(n) \geq 0$, the Extension Principle implies that $\bar{Z}^\ell Z^k = s_{\ell,k}(Z,\bar{Z}) = \bar{r}_{k,\ell}(Z,\bar{Z})$. Since $\tilde{A}$ is a flat extension of $M(n)$, it now follows from Lemma 5.2-(ii) that $\bar{\tilde{Z}}^k \tilde{Z}^\ell = r_{k,\ell}(\tilde{Z},\bar{\tilde{Z}})$ and $\bar{\tilde{Z}}^\ell \tilde{Z}^k = \bar{r}_{k,\ell}(\tilde{Z},\bar{\tilde{Z}})$.

Let $k = 0$, $\ell = n+1$. There exists $p_{0,n} \in \mathcal{P}_{n-1}$ such that $Z^n = p_{0,n}(Z,\bar{Z})$, whence $\bar{Z}^n = \bar{p}_{0,n}(Z,\bar{Z})$ (Lemma 3.10). Now, by definition, $Z^{n+1} = (zp_{0,n})(Z,\bar{Z})$ and $\bar{Z}^{n+1} = (\bar{z}\bar{p}_{0,n})(Z,\bar{Z})$. Since $zp_{0,n} \in \mathcal{P}_n$, (5.3) implies that there exists $r_{0,n} \in \mathcal{P}_{n-1}$ such that $r_{0,n}(Z,\bar{Z}) = (zp_{0,n})(Z,\bar{Z})$, whence Lemma 3.10 implies $\bar{r}_{0,n}(Z,\bar{Z}) = (\bar{z}\bar{p}_{0,n})(Z,\bar{Z})$; thus $Z^{n+1} = r_{0,n}(Z,\bar{Z})$ and $\bar{Z}^{n+1} = \bar{r}_{0,n}(Z,\bar{Z})$.

The case when $k = n+1$, $n = 0$ is handled similarly; we omit the details. □

Proof of Proposition 5.7.

$$\begin{aligned}
\langle \bar{z}^k z^\ell, \bar{z}^i z^j \rangle_{\tilde{A}} &= \langle \bar{\tilde{Z}}^k \tilde{Z}^\ell, \widehat{\bar{z}^i z^j} \rangle = \langle r_{k,\ell}(\tilde{Z},\bar{\tilde{Z}}), \widehat{\bar{z}^i z^j} \rangle && \text{(by Lemma 5.8)}\\
&= \langle r_{k,\ell}, \bar{z}^i z^j \rangle_{\tilde{A}} = \langle \bar{z}^j z^i, \bar{r}_{k,\ell} \rangle_{\tilde{A}} && \text{(by Lemma 5.6)}\\
&= \langle \widehat{\bar{z}^j z^i}, \bar{r}_{k,\ell}(\tilde{Z},\bar{\tilde{Z}}) \rangle = \langle \widehat{\bar{z}^j z^i}, \bar{\tilde{Z}}^\ell \tilde{Z}^k \rangle_{\tilde{A}} && \text{(by Lemma 5.8)}\\
&= \langle z^i \bar{z}^j, z^k \bar{z}^\ell \rangle_{\tilde{A}}. \square
\end{aligned}$$

The proof of Condition (b) is now complete. We turn to Condition (c); we require an auxiliary result.

Lemma 5.9. *Assume that $i+j = n$ and that $\bar{Z}^i Z^j = p(Z,\bar{Z})$ for some $p \in \mathcal{P}_{n-1}$. Then $\bar{Z}^i Z^{j+1} = (zp)(Z,\bar{Z})$ and $\bar{\tilde{Z}}^i \tilde{Z}^{j+1} = (zp)(\tilde{Z},\bar{\tilde{Z}})$.*

Proof. Assume first that $i \geq 1$. Observe that $\bar{Z}^i Z^j = p(Z,\bar{Z})$ readily implies $\bar{Z}^j Z^i = \bar{p}(Z,\bar{Z})$ (Lemma 3.10), and by the definition of $\bar{Z}^{j+1} Z^i$ in block B, we have $\bar{Z}^{j+1} Z^i = (\bar{z}\bar{p})(Z,\bar{Z})$, whence $\bar{V}^{j+1} V^i = (\bar{z}\bar{p})(V,\bar{V})$. Thus (5.1) implies that $\bar{V}^i V^{j+1} = (zp)(V,\bar{V})$. Now $zp \in \mathcal{P}_n$, so there exists $r \in \mathcal{P}_{n-1}$ such that $(zp)(Z,\bar{Z}) = r(Z,\bar{Z})$. Also, $\bar{Z}^{i-1} Z^{j+1} = s(Z,\bar{Z})$ for some $s \in \mathcal{P}_{n-1}$, so $\bar{Z}^i Z^{j+1} = (\bar{z}s)(Z,\bar{Z})$; moreover, there exists $t \in \mathcal{P}_{n-1}$ such that $(\bar{z}s)(Z,\bar{Z}) = t(Z,\bar{Z})$. Now $t(V,\bar{V}) = \bar{V}^i V^{j+1} = (zp)(V,\bar{V}) = r(V,\bar{V})$ in $\mathcal{C}_{M(n-1)}$. The Extension Principle thus implies $t(Z,\bar{Z}) = r(Z,\bar{Z})$, whence $\bar{Z}^i Z^{j+1} = (\bar{z}s)(Z,\bar{Z}) = t(Z,\bar{Z}) = r(Z,\bar{Z}) = (zp)(Z,\bar{Z})$; Lemma 5.2-(ii) now implies $\bar{\tilde{Z}}^i \tilde{Z}^{j+1} = (zp)(\tilde{Z},\bar{\tilde{Z}})$.

For the case $i = 0$, $j = n$, $Z^n = p(Z,\bar{Z})$ for some $p \in \mathcal{P}_{n-1}$, so by definition $Z^{n+1} = (zp)(Z,\bar{Z})$, whence $\tilde{Z}^{n+1} = (zp)(\tilde{Z},\bar{\tilde{Z}})$. We leave the case $i = n$, $j = 0$ to the reader. □

Proposition 5.10 (Condition (c)). *For $k + \ell \leq n$ and $i + j \leq n$,*

$$\langle z(\bar{z}^k z^\ell), \bar{z}^i z^j \rangle_{\tilde{A}} = \langle \bar{z}^k z^\ell, \bar{z}(\bar{z}^i z^j) \rangle_{\tilde{A}}.$$

Proof. First observe that $\langle \bar{z}p, q \rangle_{\tilde{A}} = \langle p, zq \rangle_{\tilde{A}}$ for all $p, q \in P_n$ implies that $\langle zp, q \rangle_{\tilde{A}} = \langle p, \bar{z}q \rangle_{\tilde{A}}$ for all $p, q \in \mathcal{P}_n$. For,

$$\begin{aligned}
\langle zp, q \rangle_{\tilde{A}} &= \langle \bar{q}, \bar{z}\bar{p} \rangle_{\tilde{A}} && \text{(by Condition (b))}\\
&= \overline{\langle \bar{z}\bar{p}, \bar{q} \rangle_{\tilde{A}}} = \overline{\langle \bar{p}, z\bar{q} \rangle_{\tilde{A}}} && \text{(by the above mentioned assumption)}\\
&= \langle z\bar{q}, \bar{p} \rangle_{\tilde{A}} = \langle p, \bar{z}q \rangle && \text{(again by Condition (b)).}
\end{aligned}$$

We shall therefore establish that $\langle \bar{z}(\bar{z}^k z^\ell), \bar{z}^i z^j \rangle_{\tilde{A}} = \langle \bar{z}^k z^\ell, z(\bar{z}^i z^j) \rangle_{\tilde{A}}$. For $k+\ell, i+j \le n-1$,

$$\langle \bar{z}^{k+1} z^\ell, \bar{z}^i z^j \rangle_{\tilde{A}} = \langle \bar{z}^{k+1} z^\ell, \bar{z}^i z^j \rangle_A = \langle \bar{z}^k z^\ell, \bar{z}^i z^{j+1} \rangle_A = \langle \bar{z}^k z^\ell, \bar{z}^i z^{j+1} \rangle_{\tilde{A}}.$$

For $i+j \le n-1$, $k+\ell = n$,

$$\begin{aligned} \langle \bar{z}^{k+1} z^\ell, \bar{z}^i z^j \rangle_{\tilde{A}} &= \langle \bar{z} p_{k,\ell}, \bar{z}^i z^j \rangle_{\tilde{A}} = \langle p_{k,\ell}, \bar{z}^i z^{j+1} \rangle_{\tilde{A}} \qquad \text{(by the previous case)} \\ &= \langle \bar{z}^k z^\ell, \bar{z}^i z^{j+1} \rangle_{\tilde{A}}. \end{aligned}$$

Assume now that $k+\ell \le n$ and $i+j = n$. We have

$$\begin{aligned} &\langle \bar{z}(\bar{z}^k z^\ell), \bar{z}^i z^j \rangle_{\tilde{A}} \\ &= \langle \bar{z}^{k+1} z^\ell, p_{i,j} \rangle_{\tilde{A}} = \langle \bar{z}^k z^\ell, z p_{i,j} \rangle_{\tilde{A}} \qquad \text{(by the previous cases)} \\ &= \langle \widehat{\bar{z}^k z^\ell}, (z p_{i,j})(\widetilde{Z}, \overline{\widetilde{Z}}) \rangle = \langle \widehat{\bar{z}^k z^\ell}, \overline{\widetilde{Z}}^i \widetilde{Z}^{j+1} \rangle \qquad \text{(by Lemma 5.9)} \\ &= \langle \bar{z}^k z^\ell, z(\bar{z}^i z^j) \rangle_{\tilde{A}}. \square \end{aligned}$$

Proposition 5.11 (Condition (d)). *For $k+\ell \le n$ and $i+j \le n$,*

$$\langle z(\bar{z}^k z^\ell), z(\bar{z}^i z^j) \rangle_{\tilde{A}} = \langle \bar{z}(\bar{z}^k z^\ell), \bar{z}(\bar{z}^i z^j) \rangle_{\tilde{A}}.$$

Proof. It is easy to see that the result holds when $k+\ell, i+j \le n-1$. Assume now that $k+\ell = n$, $i+j \le n-1$, and write $\bar{Z}^k Z^\ell$ as $p_{k,\ell}(Z, \bar{Z})$. By Lemma 5.9, $\overline{\widetilde{Z}}^k \widetilde{Z}^{\ell+1} = (z p_{k,\ell})(\widetilde{Z}, \overline{\widetilde{Z}})$. Then

$$\begin{aligned} \langle z(\bar{z}^k z^\ell), z(\bar{z}^i z^j) \rangle_{\tilde{A}} &= \langle \bar{z}^k z^{\ell+1}, \bar{z}^i z^{j+1} \rangle_{\tilde{A}} = \langle \overline{\widetilde{Z}}^k \widetilde{Z}^{\ell+1}, \widehat{\bar{z}^i z^{j+1}} \rangle \\ &= \langle (z p_{k,\ell})(\widetilde{Z}, \overline{\widetilde{Z}}), \widehat{\bar{z}^i z^{j+1}} \rangle = \langle z p_{k,\ell}, \bar{z}^i z^{j+1} \rangle_{\tilde{A}} \\ &= \langle z p_{k,\ell}, z(\bar{z}^i z^j) \rangle_A \qquad \text{(by (5.4))} \\ &= \langle \bar{z} p_{k,\ell}, \bar{z}(\bar{z}^i z^j) \rangle_A \qquad \text{(by the "normality" of } A) \\ &= \langle \bar{z} p_{k,\ell}, \bar{z}^{i+1} z^j \rangle_{\tilde{A}} = \langle (\bar{z} p_{k,\ell})(\widetilde{Z}, \overline{\widetilde{Z}}), \widehat{\bar{z}^{i+1} z^j} \rangle \\ &= \langle \overline{\widetilde{Z}}^{k+1} \widetilde{Z}^\ell, \widehat{\bar{z}^{i+1} z^j} \rangle \\ &= \langle \bar{z}^{k+1} z^\ell, \bar{z}^{i+1} z^j \rangle_{\tilde{A}} = \langle \bar{z}(\bar{z}^k z^\ell), \bar{z}(\bar{z}^i z^j) \rangle_{\tilde{A}}. \end{aligned}$$

Finally, for $k+\ell = n$, $i+j = n$ we have $\langle z(\bar{z}^k z^\ell), z(\bar{z}^i z^j) \rangle_{\tilde{A}} = \langle \bar{z}^k z^{\ell+1}, z p_{ij} \rangle_{\tilde{A}}$ (by Lemma 5.9)$= \langle \bar{z}^{k+1} z^\ell, \bar{z} p_{ij} \rangle_{\tilde{A}}$ (by the previous case)$= \langle \bar{z}^{k+1} z^\ell, \bar{z}^{i+1} z^j \rangle_{\tilde{A}}$ (by the definition of $\bar{Z}^{i+1} Z^j$ and Lemma 5.2-(ii)).

(iv) For uniqueness, suppose

$$\tilde{A}' \equiv \begin{pmatrix} M(n) & B' \\ B'^* & C' \end{pmatrix}$$

is a flat extension of $M(n)$ of the form $M(n+1)$. Let $\{\bar{Y}^i Y^j\}_{0 \le i+j \le n+1}$ denote the columns of $(M(n)\ B')$ and let $\{\overline{\tilde{Y}}^i \tilde{Y}^j\}_{0 \le i+j \le n+1}$ denote the columns of $\tilde{A}'$. For $i+j = n$, in $\mathcal{C}_{M(n)}$ we have $\bar{Y}^i Y^j = \bar{Z}^i Z^j = p_{ij}(Z, \bar{Z}) = p_{ij}(Y, \bar{Y})$; thus the Extension Principle implies $\overline{\tilde{Y}}^i \tilde{Y}^j = p_{ij}(\tilde{Y}, \overline{\tilde{Y}})$. An adaptation of the proof of Lemma 3.11 now implies that $(\bar{z}p_{ij})(Y, \bar{Y}) = \bar{Y}^{i+1}Y^j$. Since $\bar{Z}^{i+1}Z^j$ in B is defined to equal $(\bar{z}p_{ij})(Z, \bar{Z})$, it follows that $\bar{Y}^{i+1}Y^j = \bar{Z}^{i+1}Z^j$. Similarly, $Y^n \equiv p_{0,n}(Y, \bar{Y}) \Rightarrow \tilde{Y}^n = p_{0,n}(\tilde{Y}, \overline{\tilde{Y}}) \Rightarrow Y^{n+1} = (zp_{0,n})(Y, \bar{Y}) \Rightarrow Z^{n+1} \equiv (zp_{0,n})(Z, \bar{Z}) = (zp_{0,n})(Y, \bar{Y}) = Y^{n+1}$. Thus $B' = B$ and so $\tilde{A}' = [M(n); B'] = [M(n); B] = \tilde{A}$. □

The proof of Theorem 5.4 is now complete. We conclude this chapter by establishing a number of corollaries to Theorem 5.4.

COROLLARY 5.12. *If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique positive extension of the form $M(\infty)$, and this is a flat extension of $M(n)$.*

PROOF. The unique flat extension of the form $M(\infty)$ may be constructed by successive application of Theorem 5.4, which yields unique flat extensions $M(n+1), M(n+2), \ldots$. It thus suffices to prove that if $M \equiv M(\infty)$ is a positive infinite moment matrix extension of $M(n)$, then M is a flat extension; however, this follows readily from the hypothesis that γ is flat via Proposition 4.3. □

We are now able to prove the equivalence of (iv) and (vi) of Conjecture 1.1.

THEOREM 5.13. *The truncated moment sequence γ has a $\operatorname{rank} M(n)$-atomic representing measure if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

PROOF. Suppose $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$. Corollary 5.12 implies that $M(n+1)$ (and hence $M(n)$) admits a flat extension $M \equiv M(\infty)$. Theorem 4.7 implies that M has a $\operatorname{rank} M$-atomic representing measure μ, and μ is clearly also a $\operatorname{rank} M(n)$-atomic representing measure for γ. Conversely, suppose μ is a $\operatorname{rank} M(n)$-atomic representing measure for γ. Consider $M(n+1)[\mu]$; then $\operatorname{rank} M(n) = \operatorname{card} \operatorname{supp} \mu \geq \operatorname{rank} M(n+1)[\mu]$ (by Corollary 3.7, since μ is a representing measure for $M(n+1)[\mu]$) $\geq \operatorname{rank} M(n)$, and thus $M(n+1)[\mu]$ is a flat extension of $M(n)$. □

We remark that there is no uniqueness in the preceding result; in Chapter 6 we show that if $M(1) \geq 0$ and $\operatorname{rank} M(1) = 2$, then there exist infinitely many 2-atomic representing measures. By contrast, in the presence of flatness, we do have uniqueness.

COROLLARY 5.14. *If γ is flat and $M(n) \geq 0$, then there exists a unique representing measure having moments of all orders, and this measure is $\operatorname{rank} M(n)$-atomic.*

PROOF. Corollary 5.12 implies that $M(n)$ has a unique positive moment matrix extension $M \equiv M(\infty)$ and that $\operatorname{rank} M = \operatorname{rank} M(n)$. Theorem 4.7 implies that M has a unique representing measure μ, which is $\operatorname{rank} M$-atomic; thus $M = M(\infty)[\mu]$ and μ is a $\operatorname{rank} M(n)$-atomic representing measure for γ. Suppose ν is a representing measure for γ with moments of all orders, and let $N := M(\infty)[\nu]$. Corollary 5.12 implies that $N = M$, whence Theorem 4.7 implies $\nu = \mu$. □

Corollary 5.15. *Assume $M(n) \geq 0$ and that the analytic columns of $M(n)$ are linearly dependent. Let $r := \min\{k \geq 1 : Z^k \in \langle 1, \dots, Z^{k-1} \rangle\}$. Then γ has a representing measure if and only if $\{1, \dots, Z^{r-1}\}$ spans $\mathcal{C}_{M(n)}$. In this case, write $Z^r = a_0 1 + \cdots + a_{r-1} Z^{r-1}$. Then $p(z) := z^r - (a_0 + \cdots + a_{r-1} z^{r-1})$ has r distinct roots, $z_0, \dots, z_{r-1}$, and γ has a unique representing measure, which is of the form $\mu = \sum_{i=0}^{r-1} \rho_i \delta_{z_i}$, where $V(z_0, \dots, z_{r-1})(\rho_0, \dots, \rho_{r-1})^T = (\gamma_{00}, \dots, \gamma_{0,r-1})^T$.*

Proof. Suppose $\{1, \dots, Z^{r-1}\}$ spans $\mathcal{C}_{M(n)}$; then γ is flat, so by Corollary 5.12, $M(n)$ admits a flat extension $M(\infty)$, and $\operatorname{rank} M(\infty) = r$. The existence of the desired measure now follows from Theorem 4.7. Conversely, suppose μ is a representing measure. Since $p(Z, \bar{Z}) = 0$, Proposition 3.1 implies $\operatorname{supp} \mu \subseteq \mathcal{Z}(p)$, so from Corollary 3.7, $r \leq \operatorname{rank} M(n) \leq \operatorname{card} \operatorname{supp} \mu \leq \operatorname{card} \mathcal{Z}(p) \leq \deg p = r$. Thus $\operatorname{card} \operatorname{supp} \mu = r$, and $\{1, \dots, z^{r-1}\}$ is a basis for $L^2(\mu)$, by Lemma 3.6. Now the injectivity of $\psi : \mathcal{C}_{M(n)} \to L^2(\mu)$ implies that $\{1, \dots, Z^{r-1}\}$ spans $\mathcal{C}_{M(n)}$ (Proposition 3.3–(i)). For uniqueness, the preceding argument implies that if μ is a representing measure, then $\operatorname{supp} \mu = \mathcal{Z}(p)$, whence μ is uniquely determined by the Vandermonde equation. □

We pause to describe an algorithm for explicitly computing the unique representing measure for a flat positive moment matrix $M(n)$. Let $r = \operatorname{rank} M(n)$. If $r \leq n$, then $\{1, Z, \dots, Z^n\}$ is dependent, so the unique representing measure may be computed as in Corollary 5.15. If $r > n$, use linear algebra to compute $p_{ij} \in \mathcal{P}_{n-1}$ such that $\bar{Z}^i Z^j = p_{ij}(Z, \bar{Z})$ for $i + j = n$. Use the p_{ij}s and the construction of this chapter to compute the unique flat extension $M(n+1)$, and note that $n + 1 \leq r = \operatorname{rank} M(n+1)$. We may thus use the flat extension method repeatedly to extend $M(n+1)$ up to $M(r)$. Since $\operatorname{rank} M(r) = r$, Propositions 3.1 and 3.3 and Corollary 3.5 imply that $\{1, \dots, Z^{r-1}\}$ is a basis for $\mathcal{C}_{M(r)}$. Thus $Z^r = a_0 1 + \cdots + a_{r-1} Z^{r-1}$ for unique scalars $a_0, \dots, a_{r-1}$, and the polynomial $p(z) := z^r - (a_0 + \cdots + a_{r-1} z^{r-1})$ has r distinct roots. The unique representing measure is now determined as in Corollary 5.15 by the Vandermonde equation.

Corollary 5.16. *Assume $M(n) \geq 0, M(2) > 0$, $M(n)$ satisfies condition* (RG), *and $\{1, Z, \dots, Z^n\}$ spans $\mathcal{C}_{M(n)}$. Then γ has a* rank $M(n)$*-atomic representing measure, which is the unique representing measure having moments of all orders.*

Proof. Since $M(2) > 0$, we must have $n \geq 2$. Write $\bar{Z} = a_0 1 + \ldots + a_n Z^n$. Assume first that $a_n \neq 0$. Then $Z^n \in \langle 1, \dots, Z^{n-1}, \bar{Z} \rangle$, and since $\{1, \dots, Z^n\}$ spans $\mathcal{C}_{M(n)}$, it follows that γ is flat; thus γ has a rank $M(n)$-atomic representing measure by Corollary 5.14. Suppose $a_n = 0$ and let p denote the smallest index such that $a_j = 0$ for $p \leq j \leq n$. Since $M(2) > 0$, then $n > p - 1 \geq 2$, whence $\bar{Z} = a_0 1 + \cdots + a_{p-1} Z^{p-1}$ with $a_{p-1} \neq 0$. Since $M(n)$ satisfies (RG), then $\bar{Z} Z^{n-p+1} = a_0 Z^{n-p+1} + \cdots + a_{p-1} Z^n$. Now $a_{p-1} \neq 0$ and $n - p + 2 \leq n - 1$, so it follows that $Z^n \in \langle \bar{Z}^i Z^j \rangle_{0 \leq i+j \leq n-1}$. Thus γ is flat and so has a rank $M(n)$-atomic representing measure by Corollary 5.14. Uniqueness also follows from Corollary 5.14. □

CHAPTER 6

Applications

6.1. The Quadratic Moment Problem. We first present existence-uniqueness criteria for the quadratic moment problem with data γ: γ_{00}, γ_{01}, γ_{10}, γ_{02}, γ_{11}, γ_{20}. Let $M(1)$ be the corresponding moment matrix, and let $r := \operatorname{rank} M(1)$.

THEOREM 6.1. *The following are equivalent:*
i) *γ has a representing measure;*
ii) *γ has an r-atomic representing measure;*
iii) $M(1) \geq 0$.
In this case, if $r = 1$ there exists a unique representing measure; if $r = 2$ the 2-atomic representing measures are parameterized by a line; if $r = 3$ the 3-atomic representing measures contain a sub-parameterization by a circle.

Positivity of $M(1)$ is clearly a necessary condition for the existence of a representing measure, so in the sequel we assume $M(1) \geq 0$. We divide the proof of the existence of representing measures according to the value of r.

PROPOSITION 6.2. *If $M(1) \geq 0$ and $r = 1$, then $\mu := \rho\delta_w$ ($\rho := \gamma_{00}$, $w := \gamma_{01}/\gamma_{00}$) is the unique representing measure of γ.*

PROOF. Since $r = 1$, there exists $\alpha \in \mathbb{C}$ such that $Z = \alpha 1$, whence $\gamma_{01} = \alpha\gamma_{00}$ (so $\alpha = \gamma_{01}/\gamma_{00}$), $\gamma_{11} = \alpha\gamma_{10}$, $\gamma_{02} = \alpha\gamma_{01}$. We evaluate the moments of μ:

$$\begin{aligned}
&\int 1\, d\mu = \rho = \gamma_{00};\\
&\int z\, d\mu = \gamma_{00}(\gamma_{01}/\gamma_{00}) = \gamma_{01};\\
&\int z^2\, d\mu = \gamma_{00}\gamma_{01}^2/\gamma_{00}^2 = \gamma_{00}\alpha^2 = \gamma_{01}\alpha = \gamma_{02};\\
&\int z\bar{z}\, d\mu = \gamma_{00}(\gamma_{01}/\gamma_{00})(\gamma_{10}/\gamma_{00}) = \alpha\gamma_{10} = \gamma_{11}.
\end{aligned}$$

Thus μ is a 1-atomic representing measure for γ. If ν is any representing measure for γ, then the relation $Z = \alpha 1$ and Proposition 3.1 imply $\operatorname{supp}\nu = \{w\}$, whence $\nu = \mu$. □

PROPOSITION 6.3. *If $M(1) \geq 0$ and $r = 2$, then γ has a representing measure. The 2-atomic representing measures of γ are parameterized by a line.*

Proof. Note that in $\mathcal{C} := \mathcal{C}_{M(1)}$, $\bar{Z} = \alpha 1 + \beta Z \Leftrightarrow Z = \bar{\alpha} 1 + \bar{\beta}\bar{Z}$; we may thus assume that $\{1, Z\}$ is a basis for $\mathcal{C}$ and that $\bar{Z} = \alpha 1 + \beta Z$ for certain α, $\beta \in \mathbb{C}$, $\beta \neq 0$. To construct a 2-atomic representing measure for γ we will define a particular quadratic polynomial $z^2 - (a + bz)$, whose distinct roots, which will form the support of the measure, lie on the line ℓ: $\bar{z} = \alpha + \beta z$. To this end, we need the precise values of α and β. Since $M(1) \geq 0$ and $\{1, Z\}$ is independent,

$$A := \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix}$$

is positive and invertible, so $\delta := \det(A) > 0$ and

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A^{-1} \begin{pmatrix} \gamma_{10} \\ \gamma_{20} \end{pmatrix} = \begin{pmatrix} \frac{1}{\delta} \end{pmatrix} \begin{pmatrix} \gamma_{11}\gamma_{10} - \gamma_{01}\gamma_{20} \\ -\gamma_{10}\gamma_{10} + \gamma_{00}\gamma_{20} \end{pmatrix}. \tag{6.1}$$

For future reference we note also that $|\beta| = 1$. Indeed,

$$|\beta|^2 = \frac{\left(|\gamma_{01}|^4 - \gamma_{10}^2\gamma_{00}\gamma_{02} - \gamma_{00}\gamma_{20}\gamma_{01}^2 + \gamma_{00}^2|\gamma_{02}|^2\right)}{\left(\gamma_{00}\gamma_{11} - |\gamma_{01}|^2\right)^2},$$

so $|\beta| = 1 \Leftrightarrow |\beta|^2 = 1 \Leftrightarrow$

$$\theta := \gamma_{10}^2\gamma_{00}\gamma_{02} + \gamma_{00}\gamma_{20}\gamma_{01}^2 - \gamma_{00}^2|\gamma_{02}|^2 + \gamma_{00}^2\gamma_{11}^2 - 2\gamma_{00}\gamma_{11}|\gamma_{01}|^2 = 0.$$

Now $\theta = \gamma_{00} \det M(1)$, so $r = 2 \Rightarrow \det M(1) = 0 \Rightarrow \theta = 0 \Rightarrow |\beta| = 1$.

Claim A. There exist $a, b, z_0, z_1 \in \mathbb{C}$, $z_0 \neq z_1$, such that

$$a\gamma_{00} + b\gamma_{01} = \gamma_{02}; \tag{6.2}$$

$$z_i^2 = a + bz_i \qquad (i = 0, 1); \tag{6.3}$$

$$\bar{z}_i = \alpha + \beta z_i \qquad (i = 0, 1). \tag{6.4}$$

We defer the proof of Claim A and first show how to use (6.2)–(6.4) to construct a representing measure. Define $\rho_1 := (\gamma_{00}z_0 - \gamma_{01})/(z_0 - z_1)$, $\rho_0 := \gamma_{00} - \rho_1$, and $\mu := \rho_0\delta_{z_0} + \rho_1\delta_{z_1}$. Note that ρ_1 is real:

$$\begin{aligned} \bar{\rho}_1 &= \frac{\gamma_{00}(\alpha + \beta z_0) - \gamma_{10}}{\beta(z_0 - z_1)} \\ &= \frac{\gamma_{00}(\alpha + \beta z_0) - (\alpha\gamma_{00} + \beta\gamma_{01})}{\beta(z_0 - z_1)} = \frac{\gamma_{00}z_0 - \gamma_{01}}{z_0 - z_1} = \rho_1. \end{aligned}$$

We next check the moments of μ:

$$\begin{aligned} \int 1\, d\mu &= \rho_0 + \rho_1 = \gamma_{00}; \\ \int z\, d\mu &= \rho_0 z_0 + \rho_1 z_1 = \rho_0 z_0 + \rho_1 z_0 - \gamma_{00}z_0 + \gamma_{01} = \gamma_{01}; \\ \int z^2\, d\mu &= \rho_0 z_0^2 + \rho_1 z_1^2 = \rho_0(a + bz_0) + \rho_1(a + bz_1) \\ &= a(\rho_0 + \rho_1) + b(\rho_0 z_0 + \rho_1 z_1) = a\gamma_{00} + b\gamma_{01} = \gamma_{02}. \end{aligned}$$

Since $\rho_0, \rho_1 \in \mathbb{R}$, it follows that $\int \bar{z}\, d\mu = \bar{\gamma}_{01} = \gamma_{10}$ and $\int \bar{z}^2\, d\mu = \bar{\gamma}_{02} = \gamma_{20}$. Finally,

$$\begin{aligned}\int z\bar{z}\, d\mu &= \rho_0 z_0(\alpha + \beta z_0) + \rho_1 z_1(\alpha + \beta z_1)\\ &= \alpha\rho_0 z_0 + \beta\rho_0(a + bz_0) + \alpha\rho_1 z_1 + \beta\rho_1(a + bz_1)\\ &= a\beta\gamma_{00} + \alpha(\rho_0 z_0 + \rho_1 z_1) + b\beta(\rho_0 z_0 + \rho_1 z_1)\\ &= a\beta\gamma_{00} + \alpha\gamma_{01} + b\beta\gamma_{01} = \alpha\gamma_{01} + \beta(a\gamma_{00} + b\gamma_{01}) = \alpha\gamma_{01} + \beta\gamma_{02} = \gamma_{11}.\end{aligned}$$

Thus μ interpolates γ. Since $M(1) \geq 0$ and $\operatorname{card}\operatorname{supp}\mu \leq 2$, Proposition 3.8 implies that $\mu \geq 0$. Thus μ is a representing measure; since $r = 2$, Corollary 3.7 implies $\operatorname{card}\operatorname{supp}\mu = 2$, i.e., $\rho_0 > 0$, $\rho_1 > 0$.

It remains to prove Claim A. Choose z_0 on the line $\bar{z} = \alpha + \beta z$, $z_0 \neq \gamma_{01}/\gamma_{00}$, and define

$$z_1 := \frac{(\gamma_{02} - z_0\gamma_{01})}{(\gamma_{01} - z_0\gamma_{00})}. \tag{6.5}$$

We first show that $z_0 \neq z_1$. Note that $\tau(z) := (\gamma_{02} - z\gamma_{01})/(\gamma_{01} - z\gamma_{00})$ satisfies $\tau(z) = z$ if and only if $z = \gamma_{01}/\gamma_{00} \pm (\gamma_{01}^2 - \gamma_{02}\gamma_{00})^{1/2}/\gamma_{00}$. If $\tau(z_0) = z_0$, then since z_0 and γ_{01}/γ_{00} lie on $\bar{z} = \alpha + \beta z$, it follows that $\lambda := \pm(\overline{\gamma_{01}^2 - \gamma_{02}\gamma_{00}})^{1/2}$ satisfies $\bar{\lambda} = \beta\lambda$. Now from (6.1), $\gamma_{01}^2 - \gamma_{02}\gamma_{00} = -\delta\bar{\beta}$, so $\bar{\lambda} = \beta\lambda \Leftrightarrow \overline{((-\delta\bar{\beta})^{1/2})} = \beta(-\delta\bar{\beta})^{1/2}$ $\Leftrightarrow -i\beta^{1/2} = \overline{\beta i((\beta)^{1/2})} \Leftrightarrow -\beta^{1/2} = \beta(1/\beta^{1/2})$ (since $|\beta| = 1$) $\Leftrightarrow -\beta = \beta$. This contradiction implies that with z_0 chosen as above, $z_1 = \tau(z_0) \neq z_0$.

Let $p(z) = (z - z_0)(z - z_1) = z^2 - (z_0 + z_1)z + z_0 z_1$, let $a := -z_0 z_1$, $b := z_0 + z_1$; thus (6.3) holds. Now (6.5) implies that $a\gamma_{00} + b\gamma_{01} = -z_0 z_1\gamma_{00} + z_0\gamma_{01} + z_1\gamma_{01} = \gamma_{02}$, so (6.2) holds. To prove (6.4) it suffices to check that $\bar{z}_1 = \alpha + \beta z_1$. Now

$$\bar{z}_1 = (\gamma_{20} - (\alpha + \beta z_0)\gamma_{10})/(\gamma_{10} - (\alpha + \beta z_0)\gamma_{00})$$

and

$$\alpha + \beta z_1 = (\alpha\gamma_{01} - \alpha z_0\gamma_{00} + \beta(\gamma_{02} - z_0\gamma_{01}))/(\gamma_{01} - z_0\gamma_{00}).$$

Subtracting these expressions, adding the fractions, and simplifying the resulting numerator yields the expression

$$\gamma_{01}\gamma_{20} - z_0\gamma_{00}\gamma_{20} - \gamma_{10}\gamma_{11} + z_0\gamma_{10}^2 + \delta(\alpha + \beta z_0),$$

which equals 0 by (6.1). This completes the proof of Claim A.

The proof shows that to each choice of z_0 on $\bar{z} = \alpha + \beta z$ ($z_0 \neq \gamma_{01}/\gamma_{00}$), there corresponds a distinct 2-atomic representing measure for γ. Conversely, the relation $\bar{Z} = \alpha 1 + \beta Z$, shows that every representing measure for γ is supported on ℓ; thus the representing measures we have constructed from the points of ℓ give a complete parameterization of the 2-atomic representing measures for γ. □

Proposition 6.4. *If $M(1) > 0$, there exist flat extensions $M(2)$.*

PROOF. Let $\gamma_{00} = 1$ and let $w = \gamma_{01}$, $u = \gamma_{02}$, $x = \gamma_{11}$. Since $A := M(1) > 0$, then $x > |w|^2$ and $x^2 > |u|^2$. We will exhibit flat extensions $M(2)$ with $\gamma_{12} = 0$, $\gamma_{03} = y$. In $B(1)$ we have $Z^2 = \langle u, 0, y\rangle^T$ and $\bar{Z}Z = \langle x, 0, 0\rangle^T$. Let

$$\begin{aligned}(a,b,c)^T &:= (\det A)A^{-1}Z^2 \\ &= ((x^2 - |u|^2)u + (w\bar{u} - x\bar{w})y, -(\bar{w}x - \bar{u}w)u - (\bar{u} - \bar{w}^2)y, \\ &\qquad (\bar{w}u - xw)u + (x - |w|^2)y)^T\end{aligned}$$

and let

$$(\alpha, \beta, \gamma)^T := (\det A)A^{-1}\bar{Z}Z = ((x^2 - |u|^2)x, -(\bar{w}x - \bar{u}w)x, (\bar{w}u - xw)x)^T.$$

It follows from Proposition 2.3 and Example 2.4 that a flat extension of $[M(1); B(1)]$ corresponding to y will be of the form of a moment matrix $M(2)$ if and only if the proposed $B_{2,2}$ block, which we denote by $(c_{ij})_{1\le i,j\le 3}$ satisfies $c_{11} = c_{22}$. This is equivalent to the requirement

$$a\gamma_{20} + b\gamma_{21} + c\gamma_{30} = \alpha\gamma_{11} + \beta\gamma_{12} + \gamma\gamma_{21},$$

or (upon simplification)

$$2\operatorname{Re}((w\bar{u} - x\bar{w})\bar{u}y) + (x - |w|^2)|y|^2 = (x^2 - |u|^2)^2. \tag{6.6}$$

Let $t := (x - |w|^2)^{1/2}$ (> 0), $s := (\bar{w}u - xw)u/t$, $p := x^2 - |u|^2$ (> 0). Thus (6.6) is equivalent to $|ty + s|^2 = p^2 + |s|^2$, so the solutions y are the points of the circle $C(\gamma)$ centered at $-s/t$ with radius $(p^2 + |s|^2)^{1/2}/t > 0$. □

PROPOSITION 6.5. *If $M(1) > 0$, the circle $C(\gamma)$ is a sub-parameterization of the 3-atomic representing measures for γ.*

PROOF. For each $y \in C(\gamma)$, the flat extension $M(2) := M(2)(y)$ described in Proposition 6.4 satisfies $\operatorname{rank} M(2) = 3$, $\mathcal{C}_{M(2)} = \langle 1, Z, \bar{Z}\rangle$. Thus Corollary 5.14 implies that $M(2)(y)$ has a 3-atomic representing measure $\mu(y)$. It follows from Proposition 3.1 that $c \neq 0$, so Theorem 3.14 and Proposition 3.1 imply that the atoms of $\mu(y)$ are the 3 distinct roots of

$$(\det A)^2 z^3 = c\alpha - a\gamma + ((\det A)a + c\beta - b\gamma)z + (\det A)(b + \gamma)z^2.$$

The measures $\mu(y)$ thus provide a partial parameterization of the 3-atomic representing measures of γ. □

Theorem 6.1 results by combining Propositions 6.2, 6.3, and 6.5.

Proposition 6.4 provides positive evidence for the validity of the following

CONJECTURE 6.6. *Assume that $M(n)$ is positive and invertible. Then there exists a flat extension $M(n+1)$.*

6.2. 2-Variable Weighted Shifts. We conclude this chapter with an application to the multivariable subnormal completion problem.

For $\alpha = (\alpha_1, \alpha_2) \in (\ell^\infty(\mathbb{Z}_+^2))^2$, let $W_\alpha \equiv (W_{\alpha_1}, W_{\alpha_2})$ be the associated 2-variable weighted shift, acting on $\ell^2(\mathbb{Z}_+^2)$ as follows:

$$W_{\alpha_i} e_k := \alpha_i(k) e_{k+\eta_i} \qquad (k \in \mathbb{Z}_+^2,\ i = 1, 2),$$

where $\{e_k\}_{k\in\mathbb{Z}_+^2}$ is the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+^2)$, $\eta_1 := (1, 0)$ and $\eta_2 := (0, 1)$. Assume that $\alpha_i(k+\eta_j)\alpha_j(k) = \alpha_j(k+\eta_i)\alpha_i(k)$ for all $k \in \mathbb{Z}_+^2$, $i, j = 1, 2$, so that W_α is commuting. The generalized Berger Theorem [JL, Proposition 23] says that W_α is subnormal if and only if there exists a compactly supported positive Borel measure μ on $\mathbb{R}_+^2$ such that

$$\int t^k \, d\mu(t) := \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) = \tilde{\gamma}_k \qquad (k \in \mathbb{Z}_+^2),$$

where
(6.7)

$$\tilde{\gamma}_k := \begin{cases} 1 & k = (0,0) \\ \alpha_1^2(0,0) \cdot \ldots \cdot \alpha_1^2(k_1 - 1, 0) & k_1 \geq 1, k_2 = 0 \\ \alpha_2^2(0,0) \cdot \ldots \cdot \alpha_2^2(0, k_2 - 1) & k_1 = 0, k_2 \geq 1 \\ \alpha_1^2(0,0) \cdot \ldots \cdot \alpha_1^2(k_1 - 1, 0)\alpha_2^2(k_1, 0) \cdot \ldots \cdot \alpha_2^2(k_1, k_2 - 1) & k_1, k_2 \geq 1. \end{cases}$$

2-variable Subnormal Completion Problem. Given $m \geq 0$ and a finite collection of pairs of positive numbers $C = \{\alpha(k) \equiv (\alpha_1(k), \alpha_2(k))\}_{|k|\leq m}$ ($|k| := k_1 + k_2$), find necessary and sufficient conditions to guarantee the existence of a subnormal 2-variable weighted shift whose initial weights are given by C.

We shall see now that a solution to the complex truncated moment problem immediately provides a solution to the 2-variable subnormal completion problem. First, let $\mathbb{C}[t_1, t_2]_{m+1}$ be the set of complex polynomials in t_1 and t_2 of total degree at most $m + 1$, and let $\tilde{\varphi}$ be the *complex* linear functional on $\mathbb{C}[t_1, t_2]_{m+1}$ induced by $\tilde{\gamma} := \{\tilde{\gamma}_k\}_{|k|\leq m+1}$, i.e., $\tilde{\varphi}(t_1^{k_1} t_2^{k_2}) := \tilde{\gamma}_{(k_1,k_2)}$, $0 \leq k_1 + k_2 \leq m + 1$, where $\tilde{\gamma}_{(k_1,k_2)}$ is defined as in (6.7).

For $0 \leq i + j \leq m + 1$ define

$$\gamma_{ij} := \tilde{\varphi}((t_1 - it_2)^i (t_1 + it_2)^j).$$

We pause briefly to exhibit $\{\gamma_{ij}\}$ when $m = 1$. Here

$$\begin{aligned}
\gamma_{00} &= \tilde{\varphi}(1) = \widetilde{\gamma}_{(0,0)} = 1, \\
\gamma_{01} &= \tilde{\varphi}(t_1 + it_2) = \tilde{\varphi}(t_1) + i\tilde{\varphi}(t_2) = \widetilde{\gamma}_{(1,0)} + i\widetilde{\gamma}_{(0,1)}, \\
\gamma_{10} &= \bar{\gamma}_{01} = \widetilde{\gamma}_{(1,0)} - i\widetilde{\gamma}_{(0,1)}, \\
\gamma_{02} &= \tilde{\varphi}(t_1^2 + 2it_1t_2 - t_2^2) = \tilde{\varphi}(t_1^2) + 2i\tilde{\varphi}(t_1t_2) - \tilde{\varphi}(t_2^2) = \widetilde{\gamma}_{(2,0)} + 2i\widetilde{\gamma}_{(1,1)} - \widetilde{\gamma}_{(0,2)}, \\
\gamma_{11} &= \tilde{\varphi}(t_1^2 + t_2^2) = \tilde{\varphi}(t_1^2) + \tilde{\varphi}(t_2^2) = \widetilde{\gamma}_{(2,0)} + \widetilde{\gamma}_{(0,2)},
\end{aligned}$$

and

$$\gamma_{20} = \bar{\gamma}_{02} = \widetilde{\gamma}_{(2,0)} - 2i\widetilde{\gamma}_{(1,1)} - \widetilde{\gamma}_{(0,2)}.$$

Using $\{\gamma_{ij}\}_{0\le i+j\le m+1}$ as data, assume now that a compactly supported representing measure ν has been found, i.e.,

$$\int \bar{z}^i z^j \, d\nu(z,\bar{z}) = \gamma_{ij} \qquad (0 \le i+j \le m+1).$$

Let $d\mu(t_1,t_2) := d\nu(t_1+it_2, t_1-it_2)$. It follows easily that μ is a compactly supported positive Borel measure on $\mathbb{R}^2_+$ which interpolates $\tilde{\gamma}$, so that the associated subnormal 2-variable weighted shift provides a solution to the subnormal completion problem for C. To illustrate this, consider again the case when $m=1$; we shall verify that μ interpolates $\widetilde{\gamma}_{11}$ correctly:

$$\begin{aligned}\int t_1 t_2 \, d\mu(t_1,t_2) &= \int \frac{z+\bar{z}}{2}\frac{z-\bar{z}}{2i}\, d\nu(z,\bar{z}) \\ &= \frac{1}{4i}\int (z^2-\bar{z}^2)\, d\nu(z,\bar{z}) = \frac{1}{4i}(\gamma_{02}-\gamma_{20}) = \frac{1}{4i}(4i\widetilde{\gamma}_{11}) = \widetilde{\gamma}_{11}.\end{aligned}$$

Conversely, if there exists a subnormal completion for C, then (via [JL]) the associated truncated complex moment problem for $\{\gamma_k\}_{|k|\le m+1}$ admits a solution.

Example 6.7. Consider the subnormal completion problem for $m=1$, where $C = \{(\alpha_1(0,0),\alpha_2(0,0)),(\alpha_1(0,1),\alpha_2(0,1)),(\alpha_1(1,0),\alpha_2(1,0))\}$, with the condition $\alpha_2(0,0)\alpha_1(0,1) = \alpha_1(0,0)\alpha_2(1,0)$. Compatible with the previous discussion, we define

$$\begin{aligned}\gamma_{00} &:= 1, \\ \gamma_{01} &:= \alpha_1^2(0,0) + i\alpha_2^2(0,0), \\ \gamma_{10} &:= \bar{\gamma}_{01}, \\ \gamma_{02} &:= \alpha_1^2(0,0)\alpha_1^2(1,0) - \alpha_2^2(0,0)\alpha_2^2(0,1) + 2i\alpha_1^2(0,0)\alpha_2^2(1,0), \\ \gamma_{20} &:= \bar{\gamma}_{02},\end{aligned}$$

and

$$\gamma_{11} := \alpha_1^2(0,0)\alpha_1^2(1,0) + \alpha_2^2(0,0)\alpha_2^2(0,1).$$

Theorem 6.1 now implies that C has a subnormal completion if and only if $M_1(\gamma) \ge 0$, where $\gamma := \{\gamma_{ij}\}_{0\le i+j\le 2}$.

CHAPTER 7

Generalizations to Several Variables

In this chapter we extend many of the results of Chapters 2–6 to truncated moment problems in $r > 1$ complex variables. For $\mathbf{z} \equiv (z_1, \dots, z_r) \in \mathbb{C}^r$ and multi-indices $\mathbf{i} \equiv (i_1, \dots, i_r)$, $\mathbf{j} \equiv (j_1, \dots, j_r) \in \mathbb{Z}_+^r$, $\bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}$ denotes the monomial $\bar{z}_1^{i_1} \cdots \bar{z}_r^{i_r} z_1^{j_1} \cdots z_r^{j_r}$ and $|\mathbf{i}|$ denotes $i_1 + \cdots + i_r$. Given $n \geq 0$ and $\gamma_{\mathbf{ij}} \in \mathbb{C}$ ($\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^r$, $0 \leq |\mathbf{i}| + |\mathbf{j}| \leq 2n$), $\gamma_{\mathbf{ji}} = \bar{\gamma}_{\mathbf{ij}}$, $\gamma_{\mathbf{00}} > 0$, the *r-dimensional truncated complex moment problem* entails find a positive Borel measure μ on $\mathbb{C}^r$ such that

$$\gamma_{\mathbf{ij}} = \int \bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}\, d\mu \qquad (0 \leq |\mathbf{i}| + |\mathbf{j}| \leq 2n).$$

Let $M(n,r)$ denote the complex matrices whose rows and columns are denoted by the lexicographic ordering of $\{\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}}\}_{0 \leq |\mathbf{i}|+|\mathbf{j}| \leq n}$. For example, with $n = 2$, $r = 2$ this ordering is

$$1, Z_1, Z_2, \bar{Z}_1, \bar{Z}_2, Z_1^2, Z_1Z_2, Z_1\bar{Z}_1, Z_1\bar{Z}_2, Z_2^2, Z_2\bar{Z}_1, Z_2\bar{Z}_2, \bar{Z}_1^2, \bar{Z}_1\bar{Z}_2, \bar{Z}_2^2,$$

so $M(2,2)$ corresponds to a special labeling of the rows and columns of $M_{15}(\mathbb{C})$. (Note that we do not distinguish between commutative rearrangements of variables.) For $A \in M(n,r)$ and $0 \leq i,j \leq n$, $A[i,j]$ denotes the rectangular block in A consisting of the intersection of rows of total degree i with columns of total degree j. For $0 \leq |\mathbf{p}| + |\mathbf{q}| \leq n$, $0 \leq |\ell| + |\mathbf{k}| \leq n$, $A_{(\mathbf{k},\ell)(\mathbf{p},\mathbf{q})}$ denotes the entry of A in row $\bar{\mathbf{Z}}^{\mathbf{k}}\mathbf{Z}^{\ell}$ and column $\bar{\mathbf{Z}}^{\mathbf{p}}\mathbf{Z}^{\mathbf{q}}$. Let $d \equiv d(n,r)$ denote the number of rows (or columns) of a matrix in $M(n,r)$.

For $0 \leq |\mathbf{i}| + |\mathbf{j}| \leq n$, let $e_{\mathbf{ij}}$ denote the vector in $\mathbb{C}^d$ with 1 in the $\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}}$ position and 0 elsewhere; then $E \equiv \{e_{\mathbf{ij}}\}_{0 \leq |\mathbf{i}|+|\mathbf{j}| \leq n}$ is a basis for $\mathbb{C}^d$. Let $\mathcal{P}(n,r)$ denote the space of complex polynomials in $z_1, \dots, z_r, \bar{z}_1, \dots, \bar{z}_r$ of total degree n; each $p \in \mathcal{P}(n,r)$ admits a unique representation $p \equiv p(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{0 \leq |\mathbf{i}|+|\mathbf{j}| \leq n} a_{\mathbf{ij}} \bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}$ ($a_{\mathbf{ij}} \in \mathbb{C}$); we define $\hat{p}$ as the coordinate vector $(a_{\mathbf{ij}}) \in \mathbb{C}^d$ relative to E. Given $A \in M(n,r)$, we define a sesquilinear form on $\mathcal{P}(n,r)$ via $\langle p, q \rangle_A = \langle A\hat{p}, \hat{q} \rangle$ ($p, q \in \mathcal{P}(n,r)$). Given a *truncated moment sequence* $\gamma \equiv \{\gamma_{\mathbf{ij}}\}_{0 \leq |\mathbf{i}|+|\mathbf{j}| \leq 2n}$, $\gamma_{\mathbf{ij}} = \bar{\gamma}_{\mathbf{ji}}$, $\gamma_{\mathbf{0},\mathbf{0}} > 0$, we define the moment matrix $M \equiv M_{n,r}(\gamma)$ by

$$M_{(\mathbf{k},\ell)(\mathbf{p},\mathbf{q})} := \langle \bar{\mathbf{z}}^{\mathbf{p}}\mathbf{z}^{\mathbf{q}}, \bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell} \rangle_M = \gamma_{\ell+\mathbf{p},\mathbf{k}+\mathbf{q}}. \tag{7.1}$$

($0 \leq |\mathbf{k}|+|\ell| \leq n$, $0 \leq |\mathbf{p}|+|\mathbf{q}| \leq n$) (note that $|\ell+\mathbf{p}|+|\mathbf{k}+\mathbf{q}| = |\ell|+|\mathbf{k}|+|\mathbf{p}|+|\mathbf{q}| \leq 2n$). To obtain our generalization of Conjecture 1.1, it is now necessary merely to replace $\mathcal{P}_n$ by $\mathcal{P}(n,r)$ and $M(n)(\gamma)$ by $M_{n,r}(\gamma)$ both in Conjecture 1.1 and in the formulation of property (RG).

We begin with a characterization of moment matrices which is analogous to Theorem 2.1.

THEOREM 7.1. *Let* $n \geq 0$, $r \geq 1$ *and let* $A \in M(n,r)$. *There exists a truncated moment sequence* $\gamma \equiv (\gamma_{\mathbf{ij}})_{0\leq|\mathbf{i}|+|\mathbf{j}|\leq 2n}$, $\gamma_{\mathbf{ij}} = \bar{\gamma}_{\mathbf{ji}}$, $\gamma_{\mathbf{00}} > 0$, *such that* $A = M_{n,r}(\gamma) \Longleftrightarrow$

0) $\langle 1,1\rangle_A > 0$
1) $A = A^*$
2) $\langle p,q\rangle_A = \langle \bar{q},\bar{p}\rangle_A \quad (p,q \in \mathcal{P}(n,r))$
3) $\langle z_t p,q\rangle_A = \langle p,\bar{z}_t q\rangle_A$ $(p,q \in \mathcal{P}(n-1,r),\ 1 \leq t \leq r)$
4) $\langle z_s p,z_t q\rangle_A = \langle \bar{z}_t p,\bar{z}_s q\rangle_A$ $(p,q \in \mathcal{P}(n-1,r),\ 1 \leq s,t \leq r)$
5) $\langle z_s p,\bar{z}_t q\rangle_A = \langle z_t p,\bar{z}_s q\rangle_A$ $(p,q \in \mathcal{P}(n-1,r),\ 1 \leq s,t \leq r)$.

PROOF. Suppose $A = M_{n,r}(\gamma)$ for a truncated moment sequence γ. The proofs of 0), 1), and 2) proceed exactly as in the proof of Theorem 2.1. To prove 3)–5) we need some extra notation; for $0 \leq s \leq r$, let $\eta_s \in \mathbb{Z}_+^r$ denote the element with 1 in position s and 0 elsewhere. The proofs of 3), 4), and 5) are very similar, so we include only the proof of 5). We may assume $p = \bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}$ and $q = \bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell}$, where $0 \leq |\mathbf{i}|+|\mathbf{j}| \leq n-1$ and $0 \leq |\mathbf{k}|+|\ell| \leq n-1$; then (7.1) implies

$$\begin{aligned}\langle z_s p,\bar{z}_t q\rangle_A &= \langle \bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}+\eta_s},\bar{\mathbf{z}}^{\mathbf{k}+\eta_t}\mathbf{z}^{\ell}\rangle_A \\ &= \gamma_{\mathbf{i}+\ell,(\mathbf{j}+\eta_s)+(\mathbf{k}+\eta_t)} = \gamma_{\mathbf{i}+\ell,(\mathbf{j}+\eta_t)+(\mathbf{k}+\eta_s)} \\ &= \langle \bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}+\eta_t},\bar{\mathbf{z}}^{\mathbf{k}+\eta_s}\mathbf{z}^{\ell}\rangle_A = \langle z_t p,\bar{z}_s q\rangle_A.\end{aligned}$$

For the converse, suppose $A \in M(n,r)$ satisfies 0)–5). We seek to define $\gamma_{\mathbf{ij}}$ $(0 \leq |\mathbf{i}|+|\mathbf{j}| \leq 2n)$ such that $A = M_{n,r}(\gamma)$. Let $0 \leq |\mathbf{i}|+|\mathbf{j}| \leq 2n$.

Claim. There exists $\mathbf{v} \equiv (\mathbf{p},\mathbf{q},\mathbf{k},\ell) \in (\mathbb{Z}_+^r)^4$ such that

$$\mathbf{i} = \ell + \mathbf{p}, \qquad \mathbf{j} = \mathbf{k}+\mathbf{q}, \qquad |\ell|+|\mathbf{k}| \leq n, \qquad |\mathbf{p}|+|\mathbf{q}| \leq n. \tag{7.2}$$

Case 1. $|\mathbf{i}|,|\mathbf{j}| \leq n$. Let $\ell = \mathbf{i},\ \mathbf{p} = \mathbf{0},\ \mathbf{q} = \mathbf{j},\ \mathbf{k} = \mathbf{0}$.

Case 2. $|\mathbf{i}| \leq n,\ |\mathbf{j}| > n$. Choose $\mathbf{s} \in \mathbb{Z}_+^r$ such that $\mathbf{s} \leq \mathbf{j}$ and $|\mathbf{s}| = n$. Then let $\ell = \mathbf{i},\ \mathbf{p} = \mathbf{0},\ \mathbf{q} = \mathbf{s},\ \mathbf{k} = \mathbf{j}-s$.

Case 3. $|\mathbf{i}| > n,\ |\mathbf{j}| \leq n$. Exchange the roles of $\mathbf{i}$ and $\mathbf{j}$ in Case 2.

We now define $\gamma_{\mathbf{ij}} = A_{(\mathbf{k},\ell)(\mathbf{p},\mathbf{q})}$. Our aim is to prove that $\gamma_{\mathbf{ij}}$ is well-defined, i.e., $\gamma_{\mathbf{ij}}$ is independent of the decomposition in (7.2); from this it will follow (as in the proof of Theorem 2.1) that γ is a truncated moment sequence and that $A = M_{n,r}(\gamma)$.

For $\mathbf{v} \equiv (\mathbf{p},\mathbf{q},\mathbf{k},\ell)$ as in (7.2), let $\alpha(\mathbf{v}) := A_{(\mathbf{k},\ell)(\mathbf{p},\mathbf{q})}$ and let $\beta(\mathbf{v}) := (|\mathbf{k}|+|\ell|,|\mathbf{p}|+|\mathbf{q}|)$ (the *block type* of $\mathbf{v}$).

For $\mathbf{v}' \equiv (\mathbf{p}',\mathbf{q}',\mathbf{k}',\ell')$ satisfying (7.2) we seek to show that $\alpha(\mathbf{v}') = \alpha(\mathbf{v})$. We first consider the case when $\beta(\mathbf{v}') = \beta(\mathbf{v})$.

Suppose that $k_1' > k_1$; then $q_1 = (k_1'-k_1)+q_1' > 0$.

Subcase a. Suppose $\ell_m > 0$ for some m, $1 \leq m \leq r$. Consider

$$\widetilde{\mathbf{v}} \equiv (k_1+1,\ldots,k_r,\ell_1,\ldots,\ell_m-1,\ldots,\ell_r,p_1,\ldots,p_m+1,\ldots,p_r,q_1-1,\ldots,q_r).$$

Then $\widetilde{\mathbf{v}}$ satisfies (7.2), $\beta(\widetilde{\mathbf{v}}) = \beta(\mathbf{v})$, and

$$\begin{aligned}\alpha(\widetilde{\mathbf{v}}) &= \langle \bar{z}_m(\bar{\mathbf{z}}^{\mathbf{p}}\mathbf{z}^{\mathbf{q}-\eta_1}), \bar{z}_1(\bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell-\eta_m})\rangle_A \\ &= \langle z_1(\bar{\mathbf{z}}^{\mathbf{p}}\mathbf{z}^{\mathbf{q}-\eta_1}), z_m(\bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell-\eta_m})\rangle_A \qquad \text{(by property 4))} \\ &= \langle \bar{\mathbf{z}}^{\mathbf{p}}\mathbf{z}^{\mathbf{q}}, \bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell}\rangle_A = \alpha(\mathbf{v}).\end{aligned}$$

Subcase b. $\ell_t = 0$ $(1 \le t \le r)$. We claim that for some m, $1 < m \le r$, $k_m \ne k'_m$ and $k_m > 0$; for otherwise,

$$\begin{aligned}|\mathbf{k}| + |\ell| = |\mathbf{k}'| + |\ell'| &\Rightarrow k_1 + \cdots + k_r = k'_1 + \cdots + k'_r + |\ell'| \\ &\Rightarrow k_1 = k'_1 + K + |\ell'| \quad \text{for some } K \ge 0 \\ &\Rightarrow k_1 \ge k'_1, \quad \text{a contradiction.}\end{aligned}$$

Consider

$$\widetilde{\mathbf{v}} = (k_1 + 1, \ldots, k_m - 1, \ldots, k_r, \ell_1, \ldots, \ell_r, p_1, \ldots, p_r, q_1 - 1, \ldots, q_m + 1, \ldots, q_r).$$

Then $\widetilde{\mathbf{v}}$ satisfies (7.2), $\beta(\widetilde{\mathbf{v}}) = \beta(\mathbf{v})$, and

$$\begin{aligned}\alpha(\widetilde{\mathbf{v}}) &= \langle z_m(\bar{\mathbf{z}}^{\mathbf{p}}\mathbf{z}^{\mathbf{q}-\eta_1}), \bar{z}_1(\bar{\mathbf{z}}^{\mathbf{k}-\eta_m}\mathbf{z}^{\ell})\rangle_A \\ &= \langle z_1(\bar{\mathbf{z}}^{\mathbf{p}}\mathbf{z}^{\mathbf{q}-\eta_1}), \bar{z}_m(\bar{\mathbf{z}}^{\mathbf{k}-\eta_m}\mathbf{z}^{\ell})\rangle_A \qquad \text{(by property 5))} \\ &= \langle \bar{\mathbf{z}}^{\mathbf{p}}\mathbf{z}^{\mathbf{q}}, \bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell}\rangle_A = \alpha(\mathbf{v}).\end{aligned}$$

Note that both in Case a and in Case b, in constructing $\widetilde{\mathbf{v}}$ we do not change the value of k_t if $k_t = k'_t$. It follows that by applying the preceding argument finitely many times we may assume that $\mathbf{v}'$ satisfies $\mathbf{k} = \mathbf{k}'$ $(\Rightarrow \mathbf{q} = \mathbf{q}')$. Under this assumption, assume $\ell'_1 > \ell_1$ $(\Rightarrow p_1 > 0)$. Since $|\ell| = |\ell'|$ it follows that for some $m > 1$, $\ell_m > 0$. Consider

$$\widetilde{\mathbf{v}} := (k_1, \ldots, k_r, \ell_1 + 1, \ldots, \ell_m - 1, \ldots, \ell_r, p_1 - 1, \ldots, p_m + 1, \ldots, p_r, q_1, \ldots, q_r);$$

as before, $\widetilde{\mathbf{v}}$ satisfies (7.2), $\beta(\widetilde{\mathbf{v}}) = \beta(\mathbf{v})$ and $\alpha(\widetilde{\mathbf{v}}) = \alpha(\mathbf{v})$. Thus, by an inductive argument we may conclude that $\ell = \ell'$ $(\Rightarrow \mathbf{p} = \mathbf{p}')$, whence $\mathbf{v} = \mathbf{v}'$ and $\alpha(\mathbf{v}) = \alpha(\mathbf{v}')$.

For the case in which $\mathbf{v}$ and $\mathbf{v}'$ are in different blocks, we may assume that $d := |\mathbf{k}'| + |\ell'| - (|\mathbf{k}| + |\ell|) > 0$. Since $d = |\mathbf{p}| + |\mathbf{q}| - (|\mathbf{p}'| + |\mathbf{q}'|)$, we may assume $|\mathbf{p}| > 0$ or $|\mathbf{q}| > 0$. We consider here the case $|\mathbf{p}| > 0$; thus $p_j > 0$ for some j, $1 \le j \le r$. Note that $n \ge |\mathbf{k}'| + |\ell'| > |\mathbf{k}| + |\ell|$ and consider

$$\widetilde{\mathbf{v}} := (k_1, \ldots, k_r, \ell_1, \ldots, \ell_j + 1, \ldots, \ell_r, p_1, \ldots, p_j - 1, \ldots, p_r, q_1, \ldots, q_r).$$

Now $k_1 + \cdots + k_r + \ell_1 + \cdots + \ell_j + 1 + \cdots + \ell_r \le n$ and $p_1 + \cdots + p_j - 1 + \cdots + p_r + q_1 + \cdots + q_r \ge 0$; thus $\widetilde{\mathbf{v}}$ satisfies (7.2). Moreover,

$$\begin{aligned}\alpha(\widetilde{\mathbf{v}}) &= \langle \bar{\mathbf{z}}^{\mathbf{p}-\eta_j}\mathbf{z}^{\mathbf{q}}, z_j(\bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell})\rangle_A \\ &= \langle \bar{z}_j\bar{\mathbf{z}}^{\mathbf{p}-\eta_j}\mathbf{z}^{\mathbf{q}}, \bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell}\rangle_A \qquad \text{(by property 3))} \\ &= \langle \bar{\mathbf{z}}^{\mathbf{p}}\mathbf{z}^{\mathbf{q}}, \bar{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\ell}\rangle_A = \alpha(\mathbf{v}).\end{aligned}$$

The case when $|\mathbf{q}| > 0$ is treated similarly; we omit the details. By repeating the preceding argument finitely many times, we may assume that $\mathbf{v}$ and $\mathbf{v}'$ have the same block type, whence $\alpha(\mathbf{v}) = \alpha(\mathbf{v}')$ by the first part of the proof. $\square$

The results of Chapter 3 extend to $r > 1$ variables in a completely straightforward way; we state the main results, but omit the proofs, which are virtually identical to those in Chapter 3. If μ is a representing measure for γ, then

$$\int f\bar{g}\,d\mu = \langle f, g\rangle_{M_{n,r}(\gamma)} \qquad (f, g \in \mathcal{P}(n,r)). \tag{7.3}$$

(7.3) readily implies that $M_{n,r}(\gamma) \geq 0$ and that $\mathcal{P}(n,r) \subseteq L^2(\mu)$. For $p \in \mathcal{P}(n,r)$, $p \equiv \sum_{0\leq|\mathbf{i}|+|\mathbf{j}|\leq n} a_{\mathbf{ij}}\bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}$, let $p(\mathbf{Z},\bar{\mathbf{Z}}) := \sum a_{\mathbf{ij}}\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}} \in \mathcal{C}_{M_{n,r}(\gamma)}$.

$$\text{Let } p \in \mathcal{P}(n,r);\ \text{then } \operatorname{supp}\mu \subseteq \mathcal{Z}(p) \Leftrightarrow p(\mathbf{Z},\bar{\mathbf{Z}}) = \mathbf{0}. \tag{7.4}$$

Define the map $\psi \equiv \psi(\gamma) : \mathcal{C}_{M_{n,r}(\gamma)} \to L^2(\mu)$ by

$$\psi\left(\sum a_{\mathbf{ij}}\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}}\right) := \sum a_{\mathbf{ij}}\bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}.$$

(7.4) implies that ψ is well-defined, linear, and one-to-one; in particular

$$\operatorname{rank} M_{n,r}(\gamma) \leq \dim L^2(\mu). \tag{7.5}$$

If μ is k-atomic ($k < \infty$), then clearly $L^2(\mu) \cong \mathbb{C}^k$, so if μ is any representing measure, then (7.5) implies

$$\operatorname{rank} M_{n,r}(\gamma) \leq \operatorname{card}\operatorname{supp}\mu. \tag{7.6}$$

Note that if μ is k-atomic, with atoms

$$\mathbf{w}_i = (w_{i1}, \ldots, w_{ir}) \qquad (1 \leq i \leq k),$$

then the polynomials

$$f_j(z_1, \ldots, z_r) \equiv \prod_{i\neq j} \frac{\| \mathbf{z} - \mathbf{w}_i \|^2}{\| \mathbf{w}_j - \mathbf{w}_i \|^2} \qquad (1 \leq j \leq k) \tag{7.7}$$

form a basis for $L^2(\mu)$. Note that $\deg f_j = 2(k-1)$; we thus have the following analogue of Proposition 3.8 (with the same proof).

Proposition 7.2. *Let μ be a k-atomic interpolating measure for γ, $k \leq \frac{n}{2}+1$. If $M_{n,r}(\gamma) \geq 0$, then $\mu \geq 0$.*

Proposition 7.3. *Let $M_{n,r}(\gamma)$ be a moment matrix and let $p \in \mathcal{P}(n,r)$. If $p(\mathbf{Z},\bar{\mathbf{Z}}) = \mathbf{0}$, then $\bar{p}(\mathbf{Z},\bar{\mathbf{Z}}) = \mathbf{0}$.*

Lemma 7.4. *Let $M_{n,r}(\gamma) \geq 0$. If $p \in \mathcal{P}(n-2,r)$ and $p(\mathbf{Z},\bar{\mathbf{Z}}) = \mathbf{0}$, then $(z_s p)(\mathbf{Z},\bar{\mathbf{Z}}) = \mathbf{0}$ $(1 \leq s \leq r)$.*

Theorem 7.5 (Structure Theorem). *Let $M_{n,r}(\gamma) \geq 0$. If $f, g, fg \in \mathcal{P}(n-1,r)$ and $f(\mathbf{Z},\bar{\mathbf{Z}}) = \mathbf{0}$, then $(fg)(\mathbf{Z},\bar{\mathbf{Z}}) = \mathbf{0}$.*

Concerning generalizations of the results of Chapter 4, let $M \equiv M_{\infty,r}$ denote an infinite moment matrix in r variables. We consider again the linear map $\varphi : \mathbb{C}[\mathbf{z},\bar{\mathbf{z}}] \to \mathcal{C}_M$ defined by $\varphi(\bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}) := \bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}}$, $(\mathbf{i},\mathbf{j} \geq \mathbf{0})$, and we let $\mathcal{N} := \{p \in \mathbb{C}[\mathbf{z},\bar{\mathbf{z}}] : \langle M\widehat{p},\widehat{p}\rangle = 0\}$ and $\ker\varphi := \{p \in \mathbb{C}[\mathbf{z},\bar{\mathbf{z}}] : \varphi(p) = 0\}$. As in Chapter 4, $\mathcal{N} = \ker\varphi$, and $\ker\varphi$ is an ideal in $\mathbb{C}[\mathbf{z},\bar{\mathbf{z}}]$. Assume now that $\operatorname{rank} M$ is finite. A straightforward generalization of Lemmas 4.4 and 4.5 shows that $\mathbb{C}[\mathbf{z},\bar{\mathbf{z}}]/\mathcal{N}$ is a finite dimensional Hilbert space, that $\dim \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]/\mathcal{N} = \operatorname{rank} M$, and that the r-tuple $M_{\mathbf{z}} \equiv (M_{z_1}, \ldots, M_{z_r})$ acting on $\mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]/\mathcal{N}$ is normal. Moreover, we can use the polynomials (in $\mathbf{z}$ and $\bar{\mathbf{z}}$) given by (7.7) to see that m distinct points in $\operatorname{supp} \mu$ give rise to m linearly independent elements of $P(n,r)\,|_{\operatorname{supp}\mu}$ $(\cong \mathcal{C}_{M_{n,r}})$. Thus, we obtain the following analogue of Proposition 4.6.

Proposition 7.6. *Let M be an infinite moment matrix in $r \geq 1$ variables, with representing measure μ. Then* $\operatorname{card} \operatorname{supp} \mu = \operatorname{rank} M$.

Proof. Analogous to the proof of Proposition 4.6, except that in view of the fact that the $\deg f_j = 2(k-1)$ in (7.7), the proper corner of M to use for the rank estimate is not $M(m-1)$ but $M(2m-2)$. □

Since the zero set of a non-constant multivariable polynomial is never finite, part of the proof of Theorem 4.7 cannot be generalized. We do, however, obtain the existence portion of that result. (A similar problem will arise when we try to extend the results of Chapter 5.)

Theorem 7.7. *Let M be a finite-rank positive infinite moment matrix in $r \geq 1$ variables. Then M admits a representing measure, and every representing measure is* $\operatorname{rank} M$*-atomic.*

Proof. As in the proof of Theorem 4.7, $C^*(M_{\mathbf{z}}) \cong C(\sigma_T(M_{\mathbf{z}}))$ (where σ_T denotes Taylor spectrum) [**Cu, Proposition 7.2**], and the linear functional $\eta(f) := \langle f(M_{\mathbf{z}})(1+\mathcal{N}), 1+\mathcal{N}\rangle$ $(f \in C(\sigma_T(M_{\mathbf{z}})))$ is positive. The Riesz Representation Theorem again gives a positive Borel measure μ with $\operatorname{supp} \mu \subseteq \sigma_T(M_{\mathbf{z}})$, such that $\eta(f) = \int f d\mu$ for every $f \in C(\sigma_T(M_{\mathbf{z}}))$. That μ interpolates γ follows as before, and Proposition 7.6 gives the cardinality of $\operatorname{supp} \mu$. □

Proposition 7.6 and Theorem 7.7 together show that an infinite moment matrix in r variables has a k-atomic representing measure if and only if $M \geq 0$ and $k = \operatorname{rank} M$. This statement generalizes to several variables a well known fact about the classical Hamburger moment problem in the case when the associated Hankel matrix H is positive and finite-rank; that is, there exists a k-atomic representing measure if and only if $\det H(j) > 0$ $(j = 0, \ldots, k-1)$ and $\det H(j) = 0$ $(j \geq k)$ [**ShT, Theorem I.1.2**]. Indeed, by [**CF1, Proposition 2.14(ii)**] and Lemma 4.1, the latter condition is equivalent to the hypothesis that $H \geq 0$ and $\operatorname{rank} H = k$.

The definition of a flat moment matrix in the multivariable case is almost identical to that used in Chapter 5: $M_{n,r}(\gamma)$ is *flat* if there exist polynomials $p_{\mathbf{ij}} \in \mathcal{P}(n-1, r)$ such that $\bar{\mathbf{z}}^{\mathbf{i}}\mathbf{z}^{\mathbf{j}} - p_{\mathbf{ij}} \in \ker \varphi$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{Z}_+^r$ with $|\mathbf{i}| + |\mathbf{j}| = n$. The results of Chapter 5 extend almost verbatim to the multivariable case, provided we make the following adjustment in the definition of $\bar{\mathbf{Z}}^{\mathbf{k}}\mathbf{Z}^{\ell}$ for $|\mathbf{k}| + |\ell| = n+1$. We proceed as follows.

Case 1. $|\mathbf{k}| \geq 1$
Assume that i is the first index for which $k_i \geq 1$. Then we let

$$\bar{\mathbf{Z}}^{\mathbf{k}}\mathbf{Z}^{\ell} := \varphi(\bar{z}_i p_{\mathbf{k}-\eta_i, \ell})$$

Case 2. $\mathbf{k} = \mathbf{0}$ $(\Rightarrow |\ell| = n+1)$

Again, let i be the first index for which $\ell_i \geq 1$. Then we let

$$\bar{\mathbf{Z}}^{\mathbf{k}}\mathbf{Z}^{\ell} := \varphi(z_i p_{\mathbf{0},\ell-\eta_i}).$$

Recall the ingredients of the proof of Theorem 5.4: Theorem 2.1, Theorem 3.14 and its attendant lemmas, the Extension Principle, and a series of lemmas and propositions concerning matrix blocks corresponding to monomials of degrees $n-1$, n, or $n+1$. We have already seen that the results of Chapters 2 and 3 extend to several variables, and a careful reading of Chapter 5 shows that the block matrix results also generalize; for example, we state the analogues of Lemmas 5.8 and 5.9:

- If $|\mathbf{k}| + |\ell| = n+1$, then there exists a polynomial $R_{\mathbf{k},\ell} \in \mathcal{P}(n-1,r)$ such that $\widetilde{\bar{\mathbf{Z}}}^{\mathbf{k}}\widetilde{\mathbf{Z}}^{\ell} = R_{\mathbf{k},\ell}(\widetilde{\mathbf{Z}}, \widetilde{\bar{\mathbf{Z}}})$ and $\widetilde{\bar{\mathbf{Z}}}^{\ell}\widetilde{\mathbf{Z}}^{\mathbf{k}} = \bar{R}_{\mathbf{k},\ell}(\widetilde{\mathbf{Z}}, \widetilde{\bar{\mathbf{Z}}})$.
- Assume that $|\mathbf{i}| + |\mathbf{j}| = n$ and that $\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}} = p(\mathbf{Z}, \bar{\mathbf{Z}})$ for some $p \in \mathcal{P}(n-1,r)$. Then for each s, $1 \leq s \leq r \mathbf{Z}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}+\eta_s} = (z_s p)(\mathbf{Z}, \bar{\mathbf{Z}})$ and $\widetilde{\bar{\mathbf{Z}}}^{\mathbf{i}}\widetilde{\mathbf{Z}}^{\mathbf{j}+\eta_s} = (z_s p)(\widetilde{\mathbf{Z}}, \widetilde{\bar{\mathbf{Z}}})$.

In view of the preceding remarks we are now able to establish the generalization of Theorem 5.4.

THEOREM 7.8. *If γ is flat and $M_{n,r} \geq 0$, then $M_{n,r}$ admits a unique flat extension of the form $M_{n+1,r}$.*

COROLLARY 7.9. *If γ is flat and $M_{n,r} \geq 0$, then $M_{n,r}$ admits a unique positive extension of the form $M_{\infty,r}$, and this is a flat extension of $M_{n,r}$.*

THEOREM 7.10. *The truncated moment sequence γ has a* $\operatorname{rank} M_{n,r}$*-atomic representing measure if and only if $M_{n,r} \geq 0$ and $M_{n,r}$ admits a flat extension $M_{n+1,r}$.*

COROLLARY 7.11. *If γ is flat and $M_{n,r} \geq 0$, then γ has a* $\operatorname{rank} M_{n,r}$*-atomic representing measure.*

We conclude this chapter by establishing a correspondence between the multivariable truncated complex moment problem and the subnormal completion problem for multivariable weighted shifts. In analogy with the material in Chapter 6.2, given $r \geq 1$ and $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \ldots, \alpha_{2r-1}, \alpha_{2r}) \in (\ell^\infty(\mathbb{Z}_+^{2r}))^{2r}$, we let $W_{\boldsymbol{\alpha}} \equiv (W_{\alpha_1}, W_{\alpha_2}, \ldots, W_{\alpha_{2r-1}}, W_{\alpha_{2r}})$ be the associated $2r-$variable weighted shift, defined by $W_{\alpha_i} e_{\mathbf{k}} := \alpha_i(\mathbf{k}) e_{\mathbf{k}+\eta_i}$, ($\mathbf{k} \in \mathbb{Z}_+^{2r}$, $i = 1, \ldots, 2r$). As usual, one assumes that $W_{\boldsymbol{\alpha}}$ is commuting, i.e., $\alpha_i(\mathbf{k}+\eta_j)\alpha_j(\mathbf{k}) = \alpha_j(\mathbf{k}+\eta_i)\alpha_i(\mathbf{k})$ ($\mathbf{k} \in \mathbb{Z}_+^{2r}$, $i, j = 1, \ldots, 2r$); then $W_{\boldsymbol{\alpha}}$ is subnormal if and only if there exists a compactly supported positive Borel measure μ on $\mathbb{R}_+^{2r}$ such that

$$\int t_1^{k_1} \ldots t_{2r}^{k_{2r}} d\mu = \widetilde{\gamma}_{(k_1,\ldots,k_{2r})}$$

where $\widetilde{\gamma}_{(0,\ldots,0)} := 1$ and

$$\widetilde{\gamma}_{(k_1,\ldots,k_{2r})} := \alpha_i^2(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_{2r})\widetilde{\gamma}_{(k_1,\ldots,k_{2r})-\eta_i}$$

whenever $k_i \geq 1$ [**JL**].

As in Chapter 6, for a given finite collection $C \equiv \{\alpha_1(\mathbf{k}), \ldots, \alpha_{2r}(\mathbf{k})\}_{|\mathbf{k}| \leq m}$, we define a complex linear functional $\widetilde{\varphi}$ on $\mathbb{C}[t_1, \ldots, t_{2r}]_{m+1}$ by $\widetilde{\varphi}(t_1^{k_1} \ldots t_{2r}^{k_{2r}}) :=$

$\widetilde{\gamma}_{(k_1,\dots,k_{2r})}$ and we set $\gamma_{\mathbf{ij}} := \widetilde{\varphi}(\prod_{s=1}^{r}(t_{2s-1} - it_{2s})^{i_s}(t_{2s-1} + it_{2s})^{j_s})$ $(\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^r, |\mathbf{i}| + |\mathbf{j}| \leq m + 1)$. Assuming an interpolating measure ν has been found for the data $\{\gamma_{\mathbf{ij}}\}_{|\mathbf{i}|+|\mathbf{j}|\leq m+1}$, one then defines μ by

$$d\mu(t_1, \dots, t_{2r}) := d\nu(t_1 - it_2, \dots, t_{2r-1} - t_{2r}, t_1 + it_2, \dots, t_{2r-1} + it_{2r});$$

μ then interpolates the original data $\{\widetilde{\gamma}_{(k_1,\dots,k_{2r})}\}_{0\leq k_1+\cdots+k_{2r}\leq m+1}$. Thus, a solution to the multivariable complex truncated moment problem gives at once a solution to the subnormal completion problem for multivariable weighted shifts. Conversely, if there exists a subnormal completion for C, then the associated truncated moment problem admits a solution.

Finally, we observe that the study of completion problems for $(2r-1)$-variable weighted shifts can be reduced to that of $(2r)$-variable ones, via the following device: If $\alpha_i(\mathbf{k})$ are the given weights of a $(2r-1)$-variable shift $(1 \leq i \leq 2r-1,\ \mathbf{k} \in \mathbb{Z}_+^{2r-1})$, we let

$$\widetilde{\alpha}_i(\mathbf{k}, k_{2r}) := \alpha_i(\mathbf{k}) \ \ (1 \leq i \leq 2r-1,\ \mathbf{k} \in \mathbb{Z}_+^{2r-1}),$$

and

$$\widetilde{\alpha}_{2r}(\mathbf{k}, k_{2r}) := 1 \ \ (\mathbf{k} \in \mathbb{Z}_+^{2r-1}).$$

Added in Proof. We have recently shown that (vii) $\Rightarrow$ (vi) in Conjecture 1.1 whenever $\bar{Z} = \alpha 1 + \beta Z$ $(\alpha,\ \beta \in \mathbb{C})$ or $Z^k \in \langle \bar{Z}^i Z^j \rangle_{0\leq i+j\leq k-1}$ $(k \leq [\frac{n}{2}] + 1)$. On the other hand, using [**Sch1**] we can exhibit an example of $M(3)(\gamma)$ that is positive and invertible (and hence satisfies property (RG)), but such that γ admits *no* representing measure; in particular, via Theorem 5.13, $M(3)(\gamma)$ admits no flat extension $M(4)$. Thus, (vii) $\Rightarrow$ (vi) in Conjecture 1.1 is false for arbitrary moment matrices. We are currently searching for the proper condition which must be added to (RG) to guarantee the existence of a representing measure for γ. For additional recent related results see [**P3**], [**P4**] and [**StSz**].

References

[AK] N.I. Ahiezer and M.G. Krein, *Some Questions in the Theory of Moments*, Transl. Math. Monographs, vol. 2, Amer. Math. Soc., Providence, RI, 1962.

[Akh] N.I. Akhiezer, *The Classical Moment Problem*, Hafner Publ. Co., New York, 1965.

[Atz] A. Atzmon, *A moment problem for positive measures on the unit disc*, Pacific J. Math. **59** (1975), 317–325.

[Ber] C. Berg, *The multidimensional moment problem and semigroups, Moments in Mathematics*, Proc. Sympos. Appl. Math. **37** (1987), 110–124.

[BCJ] C. Berg, J.P.R. Christensen and C.U. Jensen, *A remark on the multidimensional moment problem*, Math. Ann. **223** (1979), 163–169.

[BM] C. Berg and P.H. Maserick, *Polynomially positive definite sequences*, Math. Ann. **259** (1982), 487–495.

[Cas] G. Cassier, *Problème des moments sur un compact de* $\mathbf{R}^n$ *et décomposition des polynômes à plusieurs variables*, J. Funct. Anal. **58** (1984), 254–266.

[Cu] R. Curto, *Applications of several complex variables to multiparameter spectral theory*, Surveys of Some Recent Results in Operator Theory, vol. II (J. Conway and B. Morrel, eds.), Pitman Res. Notes Math. Ser. 192, Longman Sci. and Techn., London, 1988, pp. 25–90.

[CF1] R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations Operator Theory **17** (1993), 202–246.

[CF2] R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem,* II, Integral Equations Operator Theory (to appear).

[CF3] R. Curto and L. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17** (1991), 603–635.

[CP1] R. Curto and M. Putinar, *Existence of non-subnormal polynomially hyponormal operators*, Bull. Amer. Math. Soc. **25** (1991), 373–378.

[CP2] R. Curto and M. Putinar, *Nearly subnormal operators and moment problems*, J. Funct. Anal. **115** (1993), 480–497.

[Fia] L. Fialkow, *Positivity, extensions and the truncated complex moment problem*, Proc. Sympos. Pure Math. (to appear).

[Fug] B. Fuglede, *The multidimensional moment problem*, Exposition. Math. **1** (1983), 47–65.

[Hau] F. Hausdorff, *Momentprobleme für ein endliches Intervall*, Math. Z. **16** (1923), 220–248.

[Hav1] E.K. Haviland, *On the momentum problem for distributions in more than one dimension,* I, Amer. J. Math. **57** (1935), 562–568.

[Hav2] E.K. Haviland, *On the momentum problem for distributions in more than one dimension,* II, Amer. J. Math. **58** (1936), 164–168.

[Ioh] I.S. Iohvidov, *Hankel and Toeplitz Matrices and Forms: Algebraic Theory*, Birkhäuser-Verlag, Boston, 1982.

[JL] N. Jewell and A. Lubin, *Commuting weighted shifts and analytic function theory in several variables*, J. Operator Theory **1** (1979), 207–223.

[KrN] M.G. Krein and A. Nudel'man, *The Markov Moment Problem and Extremal Problems*, Transl. Math. Monographs, vol. 50, Amer. Math. Soc., Providence, RI, 1977.

[Lan] H. Landau, *Classical background of the moment problem, Moments in Mathematics*, Proc. Sympos. Appl. Math. **37** (1987), 1–15.

[McG] J.L. McGregor, *Solvability criteria for certain N-dimensional moment problems*, J. Approx. Theory **30** (1980), 315–333.

[P1] M. Putinar, *A two-dimensional moment problem*, J. Funct. Anal. **80** (1988), 1–8.

[P2] M. Putinar, *The L problem of moments in two dimensions*, J. Funct. Anal. **94** (1990), 288–307.

[P3] M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Indiana Univ. Math. J. **42** (1993), 969–984.

[P4] M. Putinar, *Extremal solutions of the two-dimensional L-problem of moments*, preprint 1994.

[Sar] D. Sarason, *Moment problems and operators on Hilbert space, Moments in Mathematics*, Proc. Sympos. Appl. Math. **37** (1987), 54–70.

[Sch1] K Schmüdgen, *An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional*, Math. Nachr. **88** (1979), 385–390.

[Sch2] K. Schmüdgen, *The K-moment problem for semi-algebraic sets*, Math. Ann. **289** (1991), 203–206.

[ShT] J. Shohat and J. Tamarkin, *The Problem of Moments*, Math. Surveys I, Amer. Math. Soc., Providence, RI, 1943.

[Smu] J.L. Smul'jan, *An operator Hellinger integral (Russian)*, Mat. Sb. **91** (1959), 381–430.

[St] J. Stochel, private correspondence, 1994.

[StSz] J. Stochel and F. Szafraniec, *Algebraic operators and moments on algebraic sets*, Portugal. Math. **51** (1994), 1–21.

[Sto] M.H. Stone, *Linear Transformation in Hilbert Space*, Amer. Math. Soc., New York, 1932.

[Sza] F. Szafraniec, *Moments on compact sets*, Prediction Theory and Harmonic Analysis (V. Mandrekar and H. Salehi, eds.), North-Holland, Amsterdam, 1983, pp. 379–385.

[Wol] Wolfram Research, Inc., *Mathematica, Version 2.1*, Wolfram Research, Inc., Champaign, IL, 1992.

List of Symbols