SPHERICALLY QUASINORMAL PAIRS OF COMMUTING OPERATORS

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Abstract. We first discuss the spherical Aluthge and spherical Duggal transforms for commuting pairs of operators on Hilbert space. Second, we study the fixed points of these transforms, which are the spherically quasinormal commuting pairs. In the case of commuting 2-variable weighted shifts, we prove that spherically quasinormal pairs are intimately related to spherically isometric pairs. We show that each spherically quasinormal 2-variable weighted shift is completely determined by a subnormal unilateral weighted shift (either the 0-th row or the 0-th column in the weight diagram). We then focus our attention on the case when this unilateral weighted shift is recursively generated (which corresponds to a finitely atomic Berger measure). We show that in this case the 2-variable weighted shift is also recursively generated, with a finitely atomic Berger measure that can be computed from its 0-th row or 0-th column. We do this by invoking the relevant Riesz functionals and the functional calculus for the columns of the associated moment matrix.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$; *quasinormal* if T commutes with T^*T; *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$; and *hyponormal* if $T^*T \geq TT^*$. It is well known that

$$\text{normal} \Longrightarrow \text{quasinormal} \Longrightarrow \text{subnormal} \Longrightarrow \text{hyponormal}.$$

For $T \in \mathcal{B}(\mathcal{H})$, consider now the *canonical polar decomposition* of T, $T \equiv VP$, where V is a partial isometry, $P := (T^*T)^{\frac{1}{2}}$, and $\ker T = \ker V = \ker P$. The *Aluthge transform* $\widehat{T}$ is $\widehat{T} := P^{\frac{1}{2}} V P^{\frac{1}{2}}$; on the other hand, the *Duggal transform* is $\widehat{T}^D := PV$. The Aluthge transform was first introduced in [1] and it has attracted considerable attention over the last two decades (see, for instance, [2], [9], [27], [32], [33], [36] and [41]).

It is well known that $T \in \mathcal{B}(\mathcal{H})$ is quasinormal if and only if T commutes with the positive factor P in the canonical polar decomposition $T \equiv VP$; equivalently, if V commutes with P. It follows easily that T is quasinormal if and only if $T = \widehat{T}$, that is, if and only if T is a fixed point for the Aluthge transform. One can similarly establish that the fixed points of the Duggal transform are also the quasinormal operators.

2000 *Mathematics Subject Classification.* Primary 47B20, 47B37, 47A13, 28A50; Secondary 44A60, 47-04, 47A20.

Key words and phrases. Aluthge transform, Duggal transform, spherically quasinormal, recursively generated 2-variable weighted shifts.

The first named author was partially supported by NSF Grant-1302666.

The second named author was partially supported by a grant from the University of Texas System and the Consejo Nacional de Ciencia y Tecnología de México (CONACYT).

To study the bivariate situation, we need some notation. The class of commuting pairs of operators on Hilbert space will be denoted by $\mathfrak{C}_0$; the subclass of commuting pairs of subnormal operators will be denoted by $\mathfrak{H}_0$; and the subclass of jointly subnormal pairs by $\mathfrak{H}_\infty$.

For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S, T] := ST - TS$. We say that a commuting pair $\mathbf{T} = (T_1, T_2)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}$$

is positive on the direct sum of two copies of $\mathcal{H}$ (cf. [3], [11]). The a commuting pair $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and *subnormal* if $\mathbf{T}$ is the restriction of a normal pair to a common invariant subspace. For $k \geq 1$, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is said to be *k-hyponormal* [21] if

$$\mathbf{T}(k) := \left(T_1, T_2, T_1^2, T_2T_1, T_2^2, \cdots, T_1^k, T_2T_1^{k-1}, \cdots, T_2^k\right)$$

is hyponormal.

For $k \geq 1$, we let $\mathfrak{H}_k$ denote the class of k-hyponormal pairs in $\mathfrak{H}_0$. It is now clear that

$$\mathfrak{H}_\infty \subseteq \mathfrak{H}_k \subseteq \mathfrak{H}_0 \subseteq \mathfrak{C}_0.$$

The multivariable Bram-Halmos Theorem states that $\mathbf{T}$ is subnormal if and only if $\mathbf{T}$ is k-hyponormal for all $k \geq 1$ [21, Theorem 2.3]; that is, $\mathfrak{H}_\infty = \bigcap_{k \geq 1} \mathfrak{H}_k$.

We next consider a suitable polar decomposition (and corresponding Aluthge and Duggal transforms) for $\mathbf{T} = (T_1, T_2) \in \mathfrak{C}_0$. Given a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ of operators acting on $\mathcal{H}$, let

$$Q := (T_1^*T_1 + T_2^*T_2)^{\frac{1}{2}}. \tag{1.1}$$

Clearly, $\ker Q = \ker T_1 \bigcap \ker T_2$. For $x \in \ker Q$, let $V_i x := 0$, and for $y \in \operatorname{Ran} Q$, say $y = Qx$, let $V_i y := T_i x$ $(i = 1, 2)$. It is easy to see that V_1 and V_2 are well defined. We then have

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} Q,$$

as operators from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$. Moreover, this is the unique canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. It follows that $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $(\ker Q)^\perp$ onto $\overline{\operatorname{Ran} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}}$.

Following [5] and [30], we say that $\mathbf{T}$ is *(jointly) quasinormal* if T_i commutes with $T_j^*T_j$ for all $i, j = 1, 2$; and *spherically quasinormal* if T_i commutes with Q for $i = 1, 2$. By [5], for all $k \geq 1$, one has

$$\begin{aligned} \text{normal} &\implies \text{(jointly) quasinormal} \implies \text{spherically quasinormal} \\ &\implies \text{subnormal} \implies k\text{-hyponormal}. \end{aligned} \tag{1.2}$$

On the other hand, the results in [21] and [30] show that the reverse implications in (1.2) do not necessarily hold.

We are now ready to introduce two bivariate operator transforms.

Definition 1.1. *(cf. [25], [26]) With* $\mathbf{T}$*,* V_1*,* V_2 *and* Q *as above, the spherical Aluthge transform of* $\mathbf{T}$ *is*

$$\widehat{\mathbf{T}} \equiv (\widehat{T}_1, \widehat{T}_2),$$

where

$$\widehat{T}_i := Q^{\frac{1}{2}} V_i Q^{\frac{1}{2}} \ (i = 1, 2). \tag{1.3}$$

Lemma 1.2. *([7], [26])* $\widehat{\mathbf{T}}$ *is commutative.*

Definition 1.3. *(cf. [35]) With* $\mathbf{T}$*,* V_1*,* V_2 *and* Q *as above, the spherical Duggal transform of* $\mathbf{T}$ *is*

$$\widehat{\mathbf{T}}^D := (\widehat{T}_1^D, \widehat{T}_2^D),$$

where

$$\widehat{T}_i^D := QV_i \ (i = 1, 2). \tag{1.4}$$

A simple application of Lemma 1.2 together with the fact that $\ker V_1 \bigcap \ker V_2 = \ker Q$ readily implies the following result.

Lemma 1.4. [35] $\widehat{\mathbf{T}}^D$ *is commutative.*

Remark 1.5. Note that, in general, $\widehat{T}_i \equiv (\widehat{\mathbf{T}})_i$ (resp. $\widehat{T}_i^D \equiv (\widehat{\mathbf{T}}^D)_i$) is *not* the Aluthge (resp. Duggal) transform of T_i $(i = 1, 2)$. □

The spherical Aluthge transform was introduced in [25]; its general theory was developed in [26]. In this paper we focus on the spherical quasinormal pairs, which are the fixed points of the spherical Aluthge and Duggal transforms. After characterizing the spherically quasinormal 2-variable weighted shifts, we study the case when a row or column in the weight diagram corresponds to a recursively generated unilateral weighted shift, that is, a weighted shift with finitely atomic Berger measure.

The organization of this paper is as follows. In Section 2 we will characterize the fixed points of the Aluthge and Duggal bivariate operator transforms; these are the spherically quasinormal pairs, that is, those commuting pairs for which T_i commutes with $T_1^*T_1 + T_2^*T_2$ for all $i = 1, 2$. In Section 3 we characterize the spherically quasinormal 2-variable weighted shifts. In Section 4 we provide a concrete construction of spherically quasinormal 2-variable weighted shifts, in terms of the 0-th row or 0-th column in their weight diagram (see Figure 1(i)). In Section 5 we focus our attention on the case when the 0-th row or 0-th column corresponds to a recursively generated subnormal unilateral weighted shift. Finally, we list in the Appendix some known results which are needed somewhere else in the paper.

We devote the rest of this section to establishing some additional basic terminology and notation. For $\omega \equiv \{\omega_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\omega \equiv \text{shift}(\omega_0, \omega_1, \cdots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\omega e_n := \omega_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The moments of $\omega \equiv \{\omega_n\}_{n=0}^{\infty}$ are given as

$$\gamma_k \equiv \gamma_k(W_\omega) := \begin{cases} 1, & \text{if } k = 0 \\ \omega_0^2 \cdots \omega_{k-1}^2, & \text{if } k > 0. \end{cases} \tag{1.5}$$

The (unweighted) unilateral shift is $U_+ := \text{shift}(1, 1, 1, \cdots)$. For $0 < a < 1$ we let $S_a :=$

shift$(a, 1, 1, \cdots)$.

We now recall a well known characterization of subnormality for unilateral weighted shifts, due to C. Berger (cf. [10, III.8.16]) and independently established by Gellar and Wallen [29]: W_ω is subnormal if and only if there exists a probability measure σ supported in $[0, \|W_\omega\|^2]$ (called the *Berger measure* of W_ω) such that $\gamma_k(\omega) = \omega_0^2 \cdot ... \cdot \omega_{k-1}^2 = \int t^k d\sigma(t)$ $(k \geq 1)$.

Observe that U_+ and S_a are subnormal, with Berger measures δ_1 and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support the singleton set $\{p\}$.

Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \text{ and } T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}, \tag{1.6}$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \ \left(\text{all } \mathbf{k} \in \mathbb{Z}_+^2\right). \tag{1.7}$$

Moreover, for $\mathbf{k} \in \mathbb{Z}_+^2$ we have

$$T_1^* e_{0,k_2} = 0 \quad \text{and} \quad T_1^* e_{\mathbf{k}} = \alpha_{\mathbf{k}-\boldsymbol{\varepsilon_1}} e_{\mathbf{k}-\varepsilon_1} \ (k_1 \geq 1); \tag{1.8}$$

$$T_2^* e_{k_1,0} = 0 \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}-\boldsymbol{\varepsilon_2}} e_{\mathbf{k}-\varepsilon_2} \ (k_2 \geq 1). \tag{1.9}$$

In an entirely similar way one can define multivariable weighted shifts (see. [23], [24]).

We now recall the definition of moments for a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)} \in \mathfrak{C}_0$. Given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the moment of $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(W_{(\alpha,\beta)}) := \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \tag{1.10}$$

We remark that, due to the commutativity condition (1.7), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2). Given a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)} \in \mathfrak{C}_0$, and given $k_1, k_2 \geq 0$, we let

$$W_{k_2} := \text{shift}(\alpha_{(0,k_2)}, \alpha_{(1,k_2)}, \cdots) \tag{1.11}$$

be the k_2-th horizontal slice of T_1; similarly we let

$$V_{k_1} := \text{shift}(\beta_{(k_1,0)}, \beta_{(k_1,1)}, \cdots) \tag{1.12}$$

be the k_1-th vertical slice of T_2. (Clearly, W_0 and V_0 are the unilateral weighted shifts associated with the 0-th row and 0-column in the weight diagram for $\mathbf{T}$, resp.) By the commutativity condition (1.7), we note that

$$\gamma_{(k_1,k_2)}(W_{(\alpha,\beta)}) = \frac{\gamma_{k_1}(W_{k_2})}{\beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2}, \tag{1.13}$$

where $\gamma_{k_1}(W_{k_2})$ is given by (1.5). A similar identity holds for V_{k_1}.

A straightforward generalization of the above mentioned Berger-Gellar-Wallen result was proved in [31]. That is, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ admits a commuting normal extension if and only if there is a probability measure μ (which we call the Berger measure of $\mathbf{T}$) defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ (where $a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \int_R s^{k_1} t^{k_2} d\mu(s,t)$, for all $\mathbf{k} \in \mathbb{Z}_+^2$.

In the single variable case, if W_ω is subnormal with Berger measure σ_ω and $h \geq 1$, and if we let $\mathcal{L}_h := \bigvee\{e_n : n \geq h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\omega|_{\mathcal{L}_h}$ is $\frac{s^h}{\gamma_h} d\sigma_\omega(s)$; alternatively, if $S : \ell^\infty(\mathbb{Z}_+) \to \ell^\infty(\mathbb{Z}_+)$ is defined by

$$S(\omega)(n) := \omega(n+1) \ (\omega \in \ell^\infty(\mathbb{Z}_+), n \geq 0), \tag{1.14}$$

then

$$d\sigma_{S(\omega)}(s) = \frac{s}{\omega_0^2} d\sigma_\omega(s). \tag{1.15}$$

2. The Spherical Aluthge and Duggal Transforms

In [32], I.B. Jung, E. Ko and C. Pearcy proved that an operator $T \in B(\mathcal{H})$ with dense range has a nontrivial invariant subspace if and only if $\widehat{T}$ does. On the other hand, one can show that T has a nontrivial invariant subspace if and only if $\widehat{T}^D$ does, where $\widehat{T}^D$ is the Duggal transform for T. In (cf. [34], [35]), the authors studied the common invariant subspaces between the spherical Aluthge (resp. Duggal) transform and its original pair. By Lemmas 1.2 and 1.4, we know that $\widehat{\mathbf{T}}$, $\widehat{\mathbf{T}}^D \in \mathfrak{C}_0$ whenever $\mathbf{T} \in \mathfrak{C}_0$ (cf. [25], [35]). In (cf. [34], [35]), the authors showed that for $\mathbf{T} \in \mathfrak{C}_0$ with dense ranges, $\mathbf{T}$ has a common nontrivial invariant subspace if and only if $\widehat{\mathbf{T}}$ does if and only if $\widehat{\mathbf{T}}^D$ does.

In [33], I.B. Jung, E. Ko and C. Pearcy also proved that T and $\widehat{T}$ have the same spectrum. This result can be extended to pairs $\mathbf{T} \in \mathfrak{C}_0$ (cf. [7], [22]). That is, one can show that for a commuting pair $\mathbf{T} \equiv (T_1, T_2)$

$$\sigma_T(\widehat{\mathbf{T}}) = \sigma_T(\mathbf{T}), \tag{2.1}$$

where $\sigma_T(\mathbf{T})$ is the Taylor spectrum of $\mathbf{T}$. (For more information on the notion of Taylor spectrum and related results, the reader is referred to [12], [13], [37], [38]).

Related to the above mentioned results, it is well known, and easy to prove, that if $T \in \mathcal{B}(\mathcal{H})$ is invertible, then $\widehat{T}$ is also invertible. In this case, $\widehat{T} = |T|^{\frac{1}{2}} T |T|^{-\frac{1}{2}}$. Similarly, for $\mathbf{T} \in \mathfrak{C}_0$, one can use a bit of homological algebra applied to the appropriate Koszul complexes to prove directly that $\widehat{\mathbf{T}}$ is Taylor invertible when $\mathbf{T}$ is Taylor invertible [7]. If $\mathbf{T} \equiv (T_1, T_2)$ is Taylor invertible and we represent it as a column matrix, then one can see that Q is also invertible, and in this case,

$$\widehat{\mathbf{T}} = Q^{\frac{1}{2}} \mathbf{T} \left(Q^{-\frac{1}{2}} \oplus Q^{-\frac{1}{2}}\right).$$

We next consider the structure of commuting pairs which are fixed points of the spherical Aluthge and Duggal transform. It is known that T is quasinormal if and only if $T = \widehat{T}$ if and only if $T = \widehat{T}^D$. We will extend this result to the case of commuting pairs $\mathbf{T} \equiv (T_1, T_2)$. First, we need an auxiliary result.

Lemma 2.1. *For $i = 1, 2$, T_i commutes with Q if and only if V_i commutes with Q.*

Proof. Recall that, for $i = 1, 2$, $T_i = V_i Q$. If T_i commutes with Q, then $V_i Q^2 = (V_i Q)Q = T_i Q = QT_i = Q(V_i Q)$, and as a consequence $(V_I Q - QV_i)Q = 0$; that is, V_i commutes with Q on $\operatorname{Ran} Q$. On the other hand, $V_i Q - QV_i$ vanishes on $\ker Q$. It now easily follows that V_i commutes with Q. The converse is trivial. $\square$

We next consider spherical quasinormality for commuting pairs. Suppose a commuting pair $\mathbf{T}$ is spherically quasinormal. Since for $i = 1, 2$, T_i commutes with $T_1^* T_1 + T_2^* T_2$, then for $i = 1, 2$ T_i commutes with Q (by the continuous functional calculus for Q). Observe now that

$$\widehat{(T_1, T_2)}\sqrt{Q} = \left(\sqrt{Q}V_1\sqrt{Q}, \sqrt{Q}V_2\sqrt{Q}\right)\sqrt{Q} = \left(\sqrt{Q}T_1, \sqrt{Q}T_2\right) = (T_1, T_2)\sqrt{Q},$$

so that

$$\widehat{(T_1, T_2)} = (T_1, T_2) \text{ on } \overline{\operatorname{Ran} \sqrt{Q}} \; (= \overline{\operatorname{Ran} Q}). \tag{2.2}$$

On the other hand, since $\ker Q = \ker T_1 \bigcap \ker T_2$, it follows easily that

$$\widehat{(T_1, T_2)} = (T_1, T_2) \text{ on } \ker Q. \tag{2.3}$$

Since $\mathcal{H} = \left(\overline{\operatorname{Ran} Q}\right) \oplus \ker Q$, we can combine (2.2) and (2.3) to prove that $\widehat{(T_1, T_2)} = (T_1, T_2)$.

We are now ready to state the main result of this section.

Theorem 2.2. *Let $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{C}_0$. The following statements are equivalent.*
(i) $\mathbf{T}$ *is spherically quasinormal.*
(ii) $\widehat{(T_1, T_2)} = (T_1, T_2)$.
(iii) $\widehat{(T_1, T_2)}^D = (T_1, T_2)$.

Proof. (i) $\Rightarrow$ (ii): This follows from the discussion preceding the statement of Theorem 2.2.
(ii) $\Rightarrow$ (iii):

$$\begin{aligned}
\widehat{\mathbf{T}} = \mathbf{T} &\implies \left(\sqrt{Q}V_1\sqrt{Q}, \sqrt{Q}V_2\sqrt{Q}\right) = (V_1 Q, V_2 Q) \\
&\implies \left(\sqrt{Q}T_1, \sqrt{Q}T_2\right) = \left(T_1\sqrt{Q}, T_2\sqrt{Q}\right) \\
&\implies T_i \text{ commutes with } \sqrt{Q} \; (i = 1, 2) \\
&\implies T_i \text{ commutes with } Q \; (i = 1, 2) \\
&\implies V_i \text{ commutes with } Q \; (i = 1, 2) \\
&\implies \widehat{\mathbf{T}}^D = \mathbf{T}.
\end{aligned}$$

(iii) $\Rightarrow$ (i): Assume that $\widehat{\mathbf{T}}^D = \mathbf{T}$. It follows that V_i commutes with Q $(i = 1, 2)$. As a consequence, T_i commutes with Q, which implies that T_i commutes with Q^2 $(i = 1, 2)$, as desired. $\square$

3. A Characterization of Spherically Quasinormal 2-variable Weighted Shifts

In this section we present a characterization of spherical quasinormality for 2-variable weighted shifts. The following theorem was announced in [25]. Before we state it, we list a simple fact about quasinormality for 2-variable weighted shifts.

Remark 3.1. A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ is (jointly) quasinormal if and only if $\alpha_{(k_1,k_2)} = \alpha_{(0,0)}$ and $\beta_{(k_1,k_2)} = \beta_{(0,0)}$ for all $k_1, k_2 \geq 0$. This can be seen via a simple application of (1.7) and (1.8). As a result, up to a scalar multiple in each component, a quasinormal 2-variable weighted shift is identical to the so-called Helton-Howe shift; that is, the shift that corresponds to the pair of multiplications by the coordinate functions in the Hardy space $H^2(\mathbb{T} \times \mathbb{T})$ of the 2-torus, with respect to arclength measure on each circle $\mathbb{T}$ (cf. [30]). This fact is entirely consistent with the one-variable result: a unilateral weighted shift W_ω is quasinormal if and only if $W_\omega = cU_+$ for some $c > 0$. $\square$

Theorem 3.2. *Let $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)} \in \mathfrak{C}_0$ be a 2-variable weighted shift. Then the following statements are equivalent.*
(i) $\mathbf{T} \equiv (T_1, T_2)$ is spherically quasinormal.
(ii) There exists a constant $c > 0$ such that for all $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$,

$$\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} = c.$$

(iii) $T_1^* T_1 + T_2^* T_2 = c\, I$.

Proof. (i) $\Longrightarrow$ (ii): Assume that $\mathbf{T} \equiv (T_1, T_2)$ is spherically quasinormal. Then, $\widehat{(T_1, T_2)} = (T_1, T_2)$, where $\widehat{(T_1, T_2)}$ is the spherical Aluthge transform of $\mathbf{T}$. Thus, we have

$$\left(\sqrt{Q}V_1\sqrt{Q}, \sqrt{Q}V_2\sqrt{Q}\right) = (V_1 Q, V_2 Q) \Longrightarrow \left(\sqrt{Q}T_1, \sqrt{Q}T_2\right) = \left(T_1\sqrt{Q}, T_2\sqrt{Q}\right),$$

that is, for all $i = 1, 2$, T_i is commutes with $\sqrt{Q}$. Hence, the continuous functional calculus imposes that T_1 and T_2 commute with Q. We now consider the following: for all $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, $\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} = c$. If we fix an orthonormal basis vector $e_{\mathbf{k}}$, then by (1.6) and (1.1) we have

$$T_1 e_{\mathbf{k}} = \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}, \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

and

$$Q e_{\mathbf{k}} = \sqrt{\alpha^2_{(k_1,k_2)} + \beta_{(k_1,k_2)}}.$$

We thus obtain

$$QT_1 e_{\mathbf{k}} = \alpha_{(k_1,k_2)}\sqrt{\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)}} \tag{3.1}$$

$$T_1 Q e_{\mathbf{k}} = \sqrt{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}}\,\alpha_{(k_1,k_2)}. \tag{3.2}$$

It follows that

$$\sqrt{\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)}} = \sqrt{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}}. \tag{3.3}$$

Similarly, we have

$$QT_2e_{\mathbf{k}} = \beta_{(k_1,k_2)}\sqrt{\alpha^2_{(k_1,k_2+1)}+\beta^2_{(k_1,k_2+1)}} \tag{3.4}$$

$$T_2Qe_{\mathbf{k}} = \sqrt{\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)}}\beta_{(k_1,k_2)}. \tag{3.5}$$

Hence,

$$\sqrt{\alpha^2_{(k_1,k_2+1)}+\beta^2_{(k_1,k_2+1)}} = \sqrt{\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)}}. \tag{3.6}$$

Therefore, by (3.3) and (3.6), for all $\mathbf{k}\equiv(k_1,k_2)\in\mathbb{Z}^2_+$ we obtain

$$\sqrt{\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)}} = \sqrt{\alpha^2_{(k_1+1,k_2)}+\beta^2_{(k_1+1,k_2)}} = \sqrt{\alpha^2_{(k_1,k_2+1)}+\beta^2_{(k_1,k_2+1)}};$$

that is, for all $\mathbf{k}\equiv(k_1,k_2)\in\mathbb{Z}^2_+$ we have

$$\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)} = c := \alpha^2_{(0,0)}+\beta^2_{(0,0)} > 0,$$

as desired.
(ii) $\Longrightarrow$ (iii): We assume that $\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)} = c > 0$ for all $\mathbf{k}\equiv(k_1,k_2)\in\mathbb{Z}^2_+$. Then, by (3.2) and (3.5), we clearly get that

$$T_1^*T_1+T_2^*T_2 = c\cdot I.$$

(iii) $\Longrightarrow$ (i): We assume that $T_1^*T_1+T_2^*T_2 = c\cdot I$. Then, for all $i=1,2$, we have

$$T_i\left(T_1^*T_1+T_2^*T_2\right) = c\cdot T_i = \left(T_1^*T_1+T_2^*T_2\right)T_i,$$

so that we get that T_1 and T_2 commute with Q. Thus, by the same argument in the proof of Theorem 2.2, we have that $\widehat{(T_1,T_2)} = (T_1,T_2)$. Therefore, by Theorem 2.2, $\mathbf{T}$ is spherically quasinormal. □

Remark 3.3. If $\mathbf{T}\equiv(T_1,T_2) = W_{(\alpha,\beta)}\in\mathfrak{C}_0$ is a spherically quasinormal 2-variable weighted shift, then Q is injective, so that by the continuous functional calculus, we have that $(T_1,T_2)\in\mathfrak{C}_0$ is spherically quasinormal if and only if each T_i is commute with $Q^2 = T_1^*T_1+T_2^*T_2$ for all $i=1,2$. Observe also that when Q is injective, we always have $V_1^*V_1+V_2^*V_2 = I$. □

We now investigate the weight diagrams of $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{T}}^D$ for a 2-variable weighted shift $\mathbf{T}\equiv(T_1,T_2) = W_{(\alpha,\beta)}$.

Proposition 3.4. *Let $\mathbf{T}\equiv(T_1,T_2) = W_{(\alpha,\beta)}$ be a 2-variable weighted shift. Then*

$$\widehat{T}_1e_{\mathbf{k}} = \alpha_{\mathbf{k}}\frac{(\alpha^2_{\mathbf{k}+\epsilon_1}+\beta^2_{\mathbf{k}+\epsilon_1})^{1/4}}{(\alpha^2_{\mathbf{k}}+\beta^2_{\mathbf{k}})^{1/4}}e_{\mathbf{k}+\epsilon_1};\ \widehat{T}_2e_{\mathbf{k}} = \beta_{\mathbf{k}}\frac{(\alpha^2_{\mathbf{k}+\epsilon_2}+\beta^2_{\mathbf{k}+\epsilon_2})^{1/4}}{(\alpha^2_{\mathbf{k}}+\beta^2_{\mathbf{k}})^{1/4}}e_{\mathbf{k}+\epsilon_2} \tag{3.7}$$

and

$$\widehat{T}_1^De_{\mathbf{k}} = \alpha_{\mathbf{k}}\frac{(\alpha^2_{\mathbf{k}+\epsilon_1}+\beta^2_{\mathbf{k}+\epsilon_1})^{1/2}}{(\alpha^2_{\mathbf{k}}+\beta^2_{\mathbf{k}})^{1/2}}e_{\mathbf{k}+\epsilon_1};\ \widehat{T}_2^De_{\mathbf{k}} = \beta_{\mathbf{k}}\frac{(\alpha^2_{\mathbf{k}+\epsilon_2}+\beta^2_{\mathbf{k}+\epsilon_2})^{1/2}}{(\alpha^2_{\mathbf{k}}+\beta^2_{\mathbf{k}})^{1/2}}e_{\mathbf{k}+\epsilon_2} \tag{3.8}$$

for all $\mathbf{k}\in\mathbb{Z}^2_+$.

Proof. Straightforward from (1.1), (1.3), (1.4) and (1.6). □

Remark 3.5. By (3.7) and (3.8) in Proposition 3.4, if $\widehat{W_{(\alpha,\beta)}} = \widehat{W_{(\alpha,\beta)}}^D$, then for all $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, $\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} = c > 0$, so that $W_{(\alpha,\beta)}$ is a spherically quasinormal. Thus, consistent with Theorem 3.2, we see that $W_{(\alpha,\beta)}$ is spherically quasinormal if and only if $\widehat{W_{(\alpha,\beta)}} = \widehat{W_{(\alpha,\beta)}}^D$. □

We now recall the class of spherically isometric commuting pairs of operators (cf. [4], [5], [6], [28], [30]).

Definition 3.6. *A commuting pair* $\mathbf{T} \equiv (T_1, T_2)$ *is a spherical isometry if* $T_1^* T_1 + T_2^* T_2 = I$.

The following result is a straightforward application of Definition 3.6.

Lemma 3.7. *A* 2*-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ *is a spherical isometry if and only if*

$$\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}} = 1$$

for all $\mathbf{k} \in \mathbb{Z}_+^{\mathbf{2}}$.

By Theorem 3.2, we have:

Corollary 3.8. *A 2-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ *is spherically quasinormal if and only if there exists* $c > 0$ *such that* $\frac{1}{\sqrt{c}}\mathbf{T}$ *is a spherical isometry, that is,* $T_1^* T_1 + T_2^* T_2 = I$.

We pause to recall an important result about spherical isometries.

Lemma 3.9. [28] *Any spherical isometry is subnormal.*

Combining Corollary 3.8 and Lemma 3.9, we easily obtain the following result.

Theorem 3.10. *Any quasinormal* 2*-variable weighted shift is subnormal.*

Remark 3.11. (cf. [25, Remark 2.14])
(i) A. Athavale and S. Poddar have recently proved that a commuting spherically quasinormal pair is always subnormal [5, Proposition 2.1]; this provides a different proof of Theorem 3.10.
(ii) In a different direction, let $Q_{\mathbf{T}}(X) := T_1^* X T_1 + T_2^* X T_2$. By induction, it is easy to prove that if $\mathbf{T}$ is spherically quasinormal, then $Q^n_{\mathbf{T}}(I) = (Q_{\mathbf{T}}(I))^n$ $(n \geq 0)$; by [8, Remark 4.6], $\mathbf{T}$ is subnormal. □

4. Construction of Spherically Quasinormal 2-variable Weighted Shifts

As observed in [26], within the class of 2-variable weighted shifts there is a simple description of spherical isometries, in terms of the weight sequences $\alpha \equiv \{\alpha_{(k_1,k_2)}\}$ and $\beta \equiv \{\beta_{(k_1,k_2)}\}$. Indeed, since spherical isometries are (jointly) subnormal, we know that the unilateral weighted shift associated with the 0-th row in the weight diagram must be subnormal. Thus, without loss of generality, we can always assume that the 0-th row corresponds to a subnormal unilateral weighted shift, and denote its weights by $\{\alpha_{(k,0)}\}_{k=0,1,2,\cdots}$. Also, in view of Corollary 3.8 we can assume that $c = 1$. Using the identity

$$\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}} = 1 \quad (\mathbf{k} \in \mathbb{Z}_+^2) \tag{4.1}$$

and the above mentioned 0-th row, we can compute $\beta_{(k,0)} := \sqrt{1-\alpha_{k,0}^2}$ for $k = 0, 1, 2, \cdots$. With these new values at our disposal, we can use the commutativity property (1.7) to generate the values of α in the first row (see Figure 1(i)); that is,

$$\alpha_{(k,1)} := \alpha_{(k,0)}\beta_{(k+1,0)}/\beta_{(k,0)}.$$

We can now repeat the algorithm, and calculate the weights $\beta_{(k,1)}$ for $k = 0, 1, 2, \cdots$, again using the identity (4.1). This in turn leads to the α weights for the second row, and so on.

This simple construction of spherically isometric 2-variable weighted shifts will allow us to study properties like recursiveness (tied to the existence of finitely atomic Berger measures) and propagation of recursive relations. We pursue this in Section 5 below.

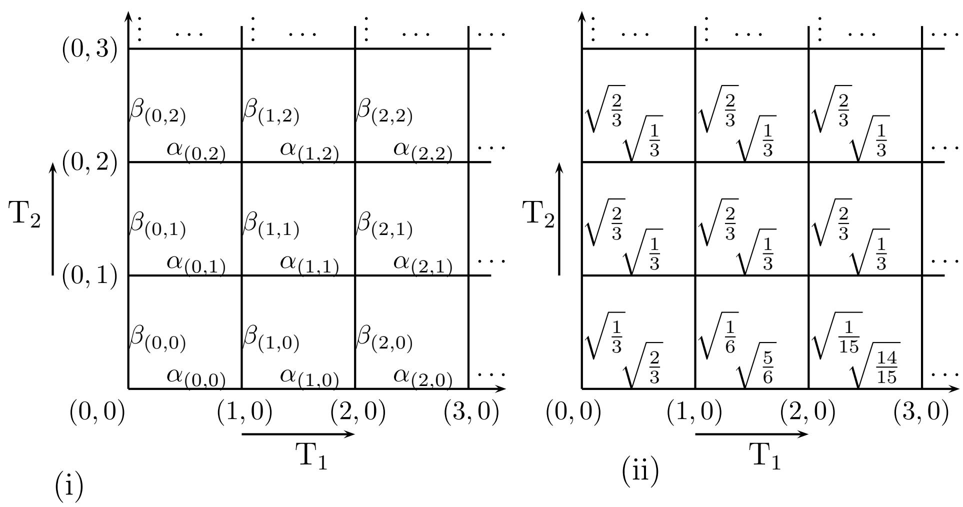


Figure 1. Weight diagram of a generic 2-variable weighted shift and weight diagram of the 2-variable weighted shift in Example 5.10, respectively.

5. Recursively Generated Spherically Quasinormal 2-variable Weighted Shifts

We begin by recalling some terminology and basic results from [15] and [16]. A subnormal unilateral weighted shift W_ω is said to be *recursively generated* if the sequence of moments γ_n admits a finite-step recursive relation; that is, if there exists an integer $k \geq 1$ and real coefficients $\varphi_0, \varphi_1, \cdots, \varphi_{k-1}$ such that

$$\gamma_{n+k} = \varphi_0\gamma_n + \varphi_1\gamma_{n+1} + \cdots + \varphi_{k-1}\gamma_{n+k-1} \ \ (\text{all } n \geq 0). \tag{5.1}$$

In conjunction with (5.1) we consider the generating function

$$g_\omega(s) := s^k - (\varphi_0 + \varphi_1 s + \cdots + \varphi_{k-1}s^{k-1}). \tag{5.2}$$

The following result characterizes recursively generated subnormal unilateral weighted shifts.

Lemma 5.1. [17] *Let W_ω be a subnormal unilateral weighted shift. The following statements are equivalent.*
(i) W_ω is recursively generated.
(ii) The Berger measure μ of W_ω is finitely atomic, and $\operatorname{supp}\mu \subseteq \mathcal{Z}(g_\omega)$, where $\mathcal{Z}(g_\omega)$ denotes the zero set of g_ω, that is, the set of roots of the equation $g_\omega = 0$.

Our first result in this section establishes the propagation of a recursive relation from the 0-th row of a spherically quasinormal 2-variable weighted shift to the first row. Given a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$, recall from (1.11) the notation W_0 and W_1.

Theorem 5.2. *Let $\mathbf{T}$ be a spherically quasinormal 2-variable weighted shift, and assume that W_0 is recursively generated, with coefficients $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$; that is,*

$$\gamma_{n+k}(W_0) = \varphi_0\gamma_k(W_0) + \varphi_1\gamma_{k+1}(W_0) + \cdots + \varphi_{n-1}\gamma_{n+k-1}(W_0) \quad (\text{all } k \geq 0)\,. \tag{5.3}$$

Then W_1 is recursively generated, with the same recursion coefficients.

Proof. Since $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{C}_0$ is spherically quasinormal, by Theorem 3.2, for all $k \geq 0$ observe that

$$\begin{aligned} \beta^2_{(0,0)}\gamma_{k_1}(W_1) &= \beta^2_{(k_1,0)}\gamma_{k_1}(W_0) \\ &= \left(c - \alpha^2_{(k_1,0)}\right)\gamma_{k_1}(W_0) \\ &= c\gamma_{k_1}(W_0) + \gamma_{k_1+1}(W_0)\,, \end{aligned}$$

Thus, we have

$$\beta^2_{(0,0)}\gamma_{n+k_1}(W_1) = c\gamma_{n+k_1}(W_0) - \gamma_{n+k_1+1}(W_0) \tag{5.4}$$

$$= c\left(\sum_{i=0}^{n-1}\varphi_i\gamma_{k_1+i}(W_0)\right) - \sum_{i=0}^{n-1}\varphi_i\gamma_{k_1+i+1}(W_0) \tag{5.5}$$

$$= \sum_{i=0}^{n-1}\varphi_i\gamma_{k_1+i}(W_0)\left(c - \alpha^2_{(k_1+i,0)}\right) \tag{5.6}$$

$$= \sum_{i=0}^{n-1}\varphi_i\gamma_{k_1+i}(W_0)\,\beta^2_{(k_1+i,0)} \tag{5.7}$$

$$= \beta^2_{(0,0)}\sum_{i=0}^{n-1}\varphi_i\gamma_{k_1+i}(W_1)\,. \tag{5.8}$$

It follows from (5.8) that

$$\gamma_{n+k_1}(W_1) = \varphi_0\gamma_{k_1}(W_1) + \varphi_1\gamma_{k_1+1}(W_1) + \cdots + \varphi_{n-1}\gamma_{n+k_1-1}(W_1)\,. \tag{5.9}$$

Thus, we see that W_1 is a recursively generated weighted shift with the same recursion coefficients; that is, (5.3) holds for W_1. □

A straightforward induction argument yields the following result.

Corollary 5.3. *Let $\mathbf{T}$ be a spherically quasinormal 2-variable weighted shift, and assume that W_0 is recursively generated, with coefficients $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$, and let $k_2 > 1$. Then W_{k_2} is recursively generated, with the same recursion coefficients.*

In view of Theorem 5.2, one is naturally led to the following question. If W_0 is recursively generated, is it also the case that V_0 is recursively generated? To study this question, we will take advantage of the theory of truncated moment problems in two real variables. (The reader is referred to [18], [19] and [20] for terminology and basic results.) Here we will only make use of the moment matrix associated with $W_{(\alpha,\beta)}$; that is, the infinite matrix $M(\alpha,\beta)$ whose rows and columns are indexed by $\mathbf{k} \in \mathbb{Z}_+^2$ and whose $(\mathbf{i},\mathbf{j})$-entry is given by $\gamma_{\mathbf{i}+\mathbf{j}}$. As typically done in the theory of truncated real moment problems, it is natural to label the rows and columns of $M(\alpha,\beta)$ using the homogenous monomials of ascending degree $1, S, T, S^2, ST, T^2, S^2, S^2T, ST^2, T^3, \cdots$. For instance, when we refer to the entry in the position $((1,2),(0,1))$, we mean the entry corresponding to row $(1,2)$ and column $(0,1)$, that is, the row labeled by the monomial ST^2 and the column labeled by the monomial T.

The proof of the following result is straightforward.

Lemma 5.4. *Let $W_{(\alpha,\beta)}$ be a 2-variable weighted shift, let $c > 0$ and fix $\mathbf{k} \in \mathbb{Z}_+^2$. The following statements are equivalent.*
(i) $\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = c$.

(ii) $\gamma_{\mathbf{k}+\varepsilon_1} + \gamma_{\mathbf{k}+\varepsilon_2} = c\gamma_{\mathbf{k}}$.

Corollary 5.5. *Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c > 0$. Then the columns of the moment matrix $M(\alpha,\beta)$ satisfy the linear relation $S + T = c\,1$.*

Corollary 5.6. *Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c > 0$, and let σ and τ be the Berger measures of W_0 and V_0, respectively. Then* $\operatorname{supp}\tau = c - \operatorname{supp}\sigma := \{c - s : s \in \operatorname{supp}\sigma\}$.

Proof. Since the columns of the moment matrix $M(\alpha,\beta)$ satisfy the linear relation $S + T = c\,1$, the Riesz functionals Λ_α and Λ_β for σ and τ (resp.) satisfy the condition

$$\Lambda_\beta(p(t)) = \Lambda_\alpha(p(c-t)),$$

for every polynomial p in one real variable. This immediately leads to the desired result about the supports of the Berger measures. □

We are now ready to prove that for spherically quasinormal 2-variable weighted shifts the property of being recursively generated transfers from the 0-th row in the weight diagram to the 0-th column.

Theorem 5.7. *Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c > 0$, and assume that the unilateral weighted shift W_0 (which corresponds to the 0-th row in the weight diagram of $W_{(\alpha,\beta)}$) is recursively generated. Then the unilateral weighted shift V_0 ((which corresponds to the 0-th column) is also recursively generated.*

Proof. The proof is based on a simple observation at the level of the Riesz functional associated with the moment matrix $M \equiv M(\alpha,\beta)$. Since $S + T = c\,1$ in the column space of M, it follows that, at the level of polynomials in the indeterminates s and t, one can replace any occurrence of s by $c-t$. As a consequence, the same holds for the columns of M, by the functional calculus introduced and studied in [17], [18] and [19]. Thus, the linear relation

$$S^k = \varphi_0 1 + \varphi_1 S + \cdots + \varphi_{k-1} S^{k-1}$$

can be rewritten (in terms of T) as

$$(c\ 1 - T)^k = \varphi_0 1 + \varphi_1 (c\ 1 - T) + \cdots + \varphi_{k-1}(c\ 1 - T)^{k-1}. \tag{5.10}$$

Inspection of (5.10) already shows that T^k can be expressed in terms of columns labeled by monomials of degree up to $k-1$. In what follows we make the recursive relation explicit. Recall that, by the Binomial Theorem,

$$(c\ 1 - T)^p = \sum_{j=0}^{p} (-1)^j \binom{p}{j} c^{p-j} T^j.$$

As a result, (5.10) becomes

$$\begin{aligned} \sum_{j=0}^{k} (-1)^j \binom{k}{j} c^{k-j} T^j &= \sum_{i=0}^{k-1} \varphi_i \sum_{j=0}^{i} (-1)^j \binom{i}{j} c^{i-j} T^j && (5.11) \\ &= \sum_{j=0}^{k-1} \left[\sum_{i=j}^{k-1} (-1)^j \binom{i}{j} \varphi_i c^{i-j} \right] T^j && (5.12) \end{aligned}$$

It follows that

$$(-1)^k T^k + \sum_{j=0}^{k-1} \left[(-1)^j \binom{k}{j} c^{k-j} \right] T^j = \sum_{j=0}^{k-1} (-1)^j \left[\sum_{i=j}^{k-1} \binom{i}{j} \varphi_i c^{i-j} \right] T^j, \tag{5.13}$$

and therefore

$$(-1)^k T^k = \sum_{j=0}^{k-1} (-1)^j \left[\sum_{i=j}^{k-1} \binom{i}{j} \varphi_i c^{i-j} - \binom{k}{j} c^{k-j} \right] T^j, \tag{5.14}$$

so that

$$T^k = \sum_{j=0}^{k-1} (-1)^{k-j} \psi_j T^j, \tag{5.15}$$

where

$$\psi_j := \sum_{i=j}^{k-1} \binom{i}{j} \varphi_i c^{i-j} - \binom{k}{j} c^{k-j}.$$

We have thus found explicitly the recursive coefficients for the moments associated with V_0. This completes the proof. □

Corollary 5.8. *Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c>0$, and assume that the unilateral weighted shift W_0 (which corresponds to the 0-th row in the weight diagram of $W_{(\alpha,\beta)}$) is recursively generated. Let σ be the Berger measure of W_0, and let μ be the Berger measure of $W_{(\alpha,\beta)}$. Then*

(i) $\operatorname{supp}\mu \subseteq \operatorname{supp}\sigma \times (c - \operatorname{supp}\sigma)$; *and*

(ii) μ is finitely atomic.

Proof. Recall that σ and τ are the marginal measures of μ (cf. Definition 6.1). By Lemma 6.2, we know that

$$\operatorname{supp}\mu \subseteq \operatorname{supp}\sigma \times \operatorname{supp}\tau.$$

By Corollary 5.6, we obtain (i). Since σ is finitely atomic, (ii) is now immediate. $\square$

Remark 5.9. Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c > 0$, and assume that the unilateral weighted shift W_0 (which corresponds to the 0-th row in the weight diagram of $W_{(\alpha,\beta)}$) is recursively generated. By Theorem 5.2, W_1 is also recursively generated, and let $\sigma^{(1)}$ be its (finitely atomic) Berger measure. Although the recursive coefficients transfer from W_0 to W_1, it is not necessarily true that σ and $\sigma^{(1)}$ have the same support. By Lemma 6.4 in the Appendix, we know that $\sigma^{(1)} \ll \sigma$, so $\operatorname{supp}\sigma^{(1)} \subseteq \operatorname{supp}\sigma$. Example 5.10 below shows that this inclusion may be proper. $\square$

First, we need some terminology.

Given three positive numbers a, b, c such that $0 < a < b < c$, we recall Stampfli's result on the existence of a subnormal unilateral weighted shift, denoted by $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, whose first three weights are a, b and c [39]. Here we will briefly recall the approach to Stampfli's result presented in [15], [16] and [17]. As proved in those papers, the Berger measure σ of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is finitely atomic, and the coefficients of recursion are given by

$$\varphi_0 = -\frac{ab(c-b)}{b-a} \text{ and } \varphi_1 = \frac{b(c-a)}{b-a}; \tag{5.16}$$

cf. [14, Section 1, p. 81], [15, Example 3.12], [16, Section 3]. Moreover, the atoms t_0 and t_1 are the roots of the equation

$$s^2 - (\varphi_0 + \varphi_1 s) = 0, \tag{5.17}$$

and the densities ρ_0 and ρ_1 uniquely solve the system of equations

$$\begin{cases} \rho_0 + \rho_1 &= 1 \\ \rho_0 s_0 + \rho_1 s_1 &= \alpha_0^2, \end{cases} \tag{5.18}$$

where

$$s_0 := \frac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_0}}{2},\ s_1 := \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_0}}{2},\ \rho_0 := \frac{s_1 - a}{s_1 - s_0},\text{ and } \rho_1 := \frac{a - s_0}{s_1 - s_0}. \tag{5.19}$$

We can now easily see that $\varphi_0 < 0$, $\varphi_1 > 0$, and $s_0 < s_1 < \varphi_1$. We thus obtain $\sigma = \rho_0\delta_{s_0} + \rho_1\delta_{s_1}$, which is the Berger measure of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$. The recursive relation, at the level of the weights, is

$$\alpha_{k+1}^2 = \varphi_1 + \frac{\varphi_0}{\alpha_k^2} \ \ (k \geq 0). \tag{5.20}$$

In view of the preservation of the recursive relation from W_0 to W_1, one might be tempted to claim that all unilateral weighted shifts W_{k_2} corresponding to horizontal rows have Berger measures $\sigma^{(k_2)}$ with the same support. This is not true. What is actually true is that $\operatorname{supp}\sigma^{(k_2)} = \operatorname{supp}\sigma^{(1)}$ for all $k_2 > 1$. The support of $\sigma^{(1)}$, however, might be strictly smaller than the support of σ. We will exhibit this behavior in the following concrete example. Notice that in this example, the 2-variable weighted shift is actually a spherical isometry.

Example 5.10. Consider the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)} \in \mathfrak{C}_0$ whose weight diagram is given in Figure 1(ii). That is, W_0 is the Stampfli subnormal completion of the initial segment of weights $\{\sqrt{\frac{2}{3}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{14}{15}}\}$. Using (5.16) one gets at once

$$\varphi_0 = -\frac{\alpha_{(0,0)}^2 \alpha_{(1,0)}^2 \left(\alpha_{(2,0)}^2 - \alpha_{(1,0)}^2\right)}{\alpha_{(1,0)}^2 - \alpha_{(0,0)}^2} = -\frac{1}{3} \quad \text{and} \quad \varphi_1 = \frac{\alpha_{(1,0)}^2 \left(\alpha_{(2,0)}^2 - \alpha_{(0,0)}^2\right)}{\alpha_{(1,0)}^2 - \alpha_{(0,0)}^2} = \frac{4}{3}. \tag{5.21}$$

It follows that W_0 is subnormal with Berger measure

$$\sigma = \frac{1}{2}\delta_{\frac{1}{3}} + \frac{1}{2}\delta_1.$$

Since

$$\beta_{(k_1,0)} := \sqrt{1 - \alpha_{(k_1,0)}^2} \quad (k_1 \geq 0)\,,$$

direct calculation yields

$$\beta_{(k_1,0)} = \sqrt{\frac{2}{3\,(3^{k_1}+1)}} \quad k_1 \geq 0.$$

Theorem 5.2 says that $W_1 = \text{shift}(\alpha_{(0,1)}, \alpha_{(1,1)}, \cdots)$ is also a recursively generated weighted shift with the same recursion coefficients φ_0 and φ_1. Moreover, the generating function

$$g\,(t) := t^2 - (\varphi_1 t + \varphi_0)$$

has 2 distinct real roots

$$0 < s_0 \equiv \frac{1}{3} < s_1 \equiv 1.$$

Let

$$V := \begin{pmatrix} 1 & 1 \\ s_0 & s_1 \end{pmatrix}$$

and let

$$\begin{pmatrix} \rho_0\,(W_1) \\ \rho_1\,(W_1) \end{pmatrix} = V^{-1} \begin{pmatrix} \gamma_0\,(W_1) \\ \gamma_1\,(W_1) \end{pmatrix}.$$

We then have

$$\sigma^{(1)} = \rho_0\,(W_1)\,\delta_{s_0} + \rho_1\,(W_1)\,\delta_{s_1},$$

where $\sigma^{(1)}$ is the Berger measure of $W_1 \equiv \text{shift}(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \cdots)$. A straightforward calculation yields $\rho_0(W_1) = 1$ and $\rho_1(W_1) = 0$. It follows that $\sigma^{(1)} = \delta_{\frac{1}{3}}$, as desired. Moreover, for $k_1 \geq 0$ we have $\beta_{(k_1,1)} = \sqrt{\frac{2}{3}}$. Now, $W_2 = \text{shift}(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \cdots) = W_1$ and, more generally, $W_{k_2} = W_1$ for all $k_2 \geq 1$. we have thus shown that even within the class of spherically isometric 2-variable weighted shifts it is indeed possible to shrink the support of σ as we move from the 0-th row to the remaining rows in the weight diagram. □

We will now state and prove an improved version of Lemma 6.4. We have known this fact for many years, as it was implicit in the proof of [24, Theorem 3.1]. We have also referred to it in research presentations, but somehow we had never have the occasion to give a formal proof. Let us first recall that for $h \geq 1$ we let $\mathcal{L}_h := \bigvee\{e_n : n \geq h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$. Thus, if W_ω is subnormal, then the Berger measure of $W_\omega|_{\mathcal{L}_h}$ is $\frac{1}{\gamma_h} s^h d\sigma_\omega(s)$, where $W_\omega|_{\mathcal{L}_h}$ means the restriction of W_ω to the invariant subspace $\mathcal{L}_h$. We can extend this result to the case of 2-variable weighted shifts. We first recall that for an arbitrary 2-variable weighted shift $W_{(\alpha,\beta)}$, we let $\mathcal{M}_j$ (resp. $\mathcal{N}_i$) be the subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geq 0$ and $k_2 \geq j$ (resp. $k_1 \geq i$ and $k_2 \geq 0$). If $W_{(\alpha,\beta)}$ is subnormal with Berger measure μ, then the Berger measure of $W_\omega|_{\mathcal{M}_j}$ is $\frac{t^j}{\gamma_{0j}\left(W_{(\alpha,\beta)}\right)} d\mu(s,t)$. We then have:

Theorem 5.11. *Consider the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ given by Figure 1(i). Let $\sigma^{(j)}$ and $\tau^{(i)}$ be as in Lemma 6.4. If $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ is subnormal, then $\sigma^{(j+1)} \simeq \sigma^{(j)}$ and $\tau^{(i+1)} \simeq \tau^{(i)}$ for $(i, j \geq 1)$, where $\simeq$ indicates that the two relevant measures are mutually absolutely continuous.*

Proof. Let $R := X \times Y \equiv [0, a_1] \times [0, a_2]$, where $a_k := \|T_k\|$ $(k = 1, 2)$. By Lemma 6.4, we only need to show the following implication:

$$\text{for } j \geq 1,\ \sigma^{(j)} \ll \sigma^{(j+1)};\ \text{that is, } \sigma^{(j+1)}(E) = 0 \Longrightarrow \sigma^{(j)}(E) = 0 \ \text{ (for all } E \subseteq X).$$

Since $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ is subnormal, we let μ be the Berger measure of $W_{(\alpha,\beta)}$. Then, by Lemma 6.4, for $j \geq 1$ $d\mu_j(s,t) := \frac{1}{\gamma_{0j}\left(W_{(\alpha,\beta)}\right)} t^j d\mu(s,t)$, and as a result

$$\begin{aligned} d\mu_{j+1}(s,t) &= \frac{1}{\gamma_{0j}\left(W_{(\alpha,\beta)}\right)} t^{j+1} d\mu(s,t) \\ &= \frac{\gamma_{0j}\left(W_{(\alpha,\beta)}\right)}{\gamma_{0j+1}\left(W_{(\alpha,\beta)}\right)} t d\mu_j(s,t) \\ &= \frac{\gamma_{0j-1}\left(W_{(\alpha,\beta)}\right)}{\gamma_{0j+1}\left(W_{(\alpha,\beta)}\right)} t^2 d\mu_{j-1}(s,t). \end{aligned}$$

Suppose now that for $j \geq 1$ and for $E \subseteq X$, $\sigma^{(j+1)}(E) = 0$. Then, by Lemma 6.6, we have that

$$\begin{aligned} \sigma^{(j+1)}(E) &= \mu_{j+1}^X(E) = \mu_{j+1}(E \times Y) = \int_{E \times Y} d\mu_{j+1}(s,t) = 0 \\ &\Longrightarrow \int_{E \times Y} t^2 d\mu_{j-1}(s,t) = 0. \end{aligned}$$

Since $t^2 \geq 0$, we know that $t^2 = 0$ a.e. $[\mu_{j-1}]$ on $E \times Y$; it follows that $t = 0$ a.e. $[\mu_{j-1}]$ on $E \times Y$. Then

$$\int_{E \times Y} t d\mu_{j-1}(s,t) = 0 \Longrightarrow \int_{E \times Y} d\mu_j(s,t) = 0 \Longrightarrow \sigma^{(j)}(E) = 0.$$

This completes the proof. □

Remark 5.12. (i) We now refer back to the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ constructed in Example 5.10. Since $W_{(\alpha,\beta)}$ is quasinormal, it is also subnormal by Theorem 3.10; let μ be its Berger measure. It is easy to see that $\mathbf{T}|_{\mathcal{M}} \in \mathfrak{H}_\infty$ and that the Berger measure of $\mathbf{T}|_{\mathcal{M}}$ is $\mu_{\mathcal{M}} = \delta_{\left(\frac{1}{3},\frac{2}{3}\right)}$. By Lemma 6.3 and Figure 1(ii), we can see that $(\mu_{\mathcal{M}})_{ext} = \delta_{\left(\frac{1}{3},\frac{2}{3}\right)}$ and hence it follows that

$$(\mu_{\mathcal{M}})^X_{ext} = \delta_{\frac{1}{3}}.$$

Since $\beta^2_{00} = \frac{1}{3}$, Lemma 6.3 shows that

$$\mu = \frac{1}{2}\left(\delta_{\left(\frac{1}{3},\frac{2}{3}\right)} + \delta_{(1,0)}\right). \quad \square$$

The following problem arises naturally.

Problem 5.13. *Consider a spherically quasinormal* 2*-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ *and let* σ *be the Berger measure of* W_0*. Since* $W_{(\alpha,\beta)}$ *is subnormal by Theorem 3.10, let* μ *be the Berger measure of* $W_{(\alpha,\beta)}$*.*

(i) Describe μ *in terms of* σ*.*

(ii) Assume that W_0 *is recursively generated. By Corollary 5.8, we know that* μ *is finitely atomic, and that* $\operatorname{supp}\mu \subseteq \operatorname{supp}\sigma \times (c - \operatorname{supp}\sigma)$*. What else can we say? Can we give a concrete formula for the atoms and densities of* μ*?*

In Problem 5.13(ii) we know that W_0 carries all the information about $W_{(\alpha,\beta)}$; therefore we know that the atoms and densities of μ must algorithmically be obtained from those of σ. Thus, the question refers to finding such algorithm. In Example 5.14 below, we show how one might go about finding a concrete formula for μ.

Example 5.14. In Problem 5.13, assume that σ is 2-atomic, and write $\sigma \equiv \lambda_0\delta_{s_0} + \lambda_1\delta_{s_1}$, with $0 \le s_0 < s_1 \le 1$ and $\lambda_0, \lambda_1 > 0$. From Corollary 5.8 we know that

$$\operatorname{supp}\mu \subseteq \{(s_0, c - s_0), (s_0, c - s_1), (s_1, c - s_0), (s_1, c - s_1)\}.$$

Moreover, $\operatorname{supp}\mu$ must have at least two atoms, because σ (and τ) are 2-atomic. Thus, we can postulate that $\mu = \rho_{00}\delta_{(s_0,c-s_0)} + \rho_{01}\delta_{(s_0,c-s_1)} + \rho_{10}\delta_{(s_1,c-s_0)} + \rho_{11}\delta_{(s_1,c-s_1)}$, with $\rho_{ij} \ge 0$ $(i, j = 1, 2)$. We now write the moment equations as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ s_0 & s_0 & s_1 & s_1 \\ c - s_0 & c - s_1 & c - s_0 & c - s_1 \\ s_0(c - s_0) & s_0(c - s_1) & s_1(c - s_0) & s_1(c - s_1) \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{pmatrix} = \begin{pmatrix} \gamma_{(0,0)} \\ \gamma_{(0,1)} \\ \gamma_{(1,0)} \\ \gamma_{(1,1)} \end{pmatrix}. \tag{5.22}$$

Denote the 4×4 matrix in (5.22) by V. A calculation using *Mathematica* [40] shows that $\det V = -(s_1 - s_0)^4 < 0$. It follows that we can always find real numbers $\rho_{00}, \rho_{01}, \rho_{10}, \rho_{11}$ satisfying the moment equations. However, that is not sufficient, since we need to guarantee that these four numbers are nonnegative. We do know that $\gamma_{(0,0)} = 1$, $\gamma_{(0,1)} = \lambda_0 s_0 + \lambda_1 s_1$, $\gamma_{(1,0)} = \lambda_0(c - s_0) + \lambda_1(c - s_1)$ and $\gamma_{(1,1)} = \gamma_{(1,0)}\beta^2_{(1,0)}$. We also know that $\beta^2_{(1,0)} = c - \alpha^2_{(1,0)} = c - \frac{\gamma_{(2,0)}}{\gamma_{(1,0)}}$. Using this information, a calculation with *Mathematica* reveals that $\rho_{01} = \rho_{10} = 0$, and that $\rho_{00} = \lambda_0$ and $\rho_{11} = 1 - \lambda_0$. It follows that

$$\mu = \lambda_0\delta_{(s_0,c-s_0)} + (1 - \lambda_0)\delta_{(s_1,c-s_1)}.$$

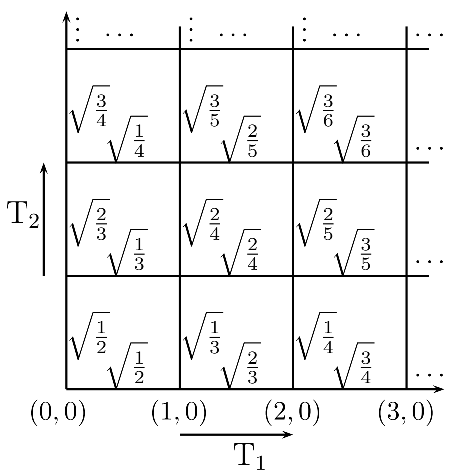
In particular, μ is always 2-atomic. For instance, the Berger measure of the spherical isometry built in Example 5.10 is

$$\mu = \frac{1}{2}(\delta_{(\frac{1}{3},\frac{2}{3})} + \delta_{(1,0)}).$$

This formula for μ is entirely consistent with Remark 5.12. $\square$

We conclude this section with an intriguing question.

Question 5.15. *Let W_0 be the Bergman shift* $\text{shift}(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots)$*, and use Section 4 to build a spherically quasinormal 2-variable weighted shift W (cf. Figure 2). For this shift the j-th row is identical to the j-column, for every $j \geq 0$. Note also that W is a close relative of the Drury-Arveson 2-variable weighted shift, in that the j-row of W is the Agler A_{j+2} shift. What is the Berger measure of W?*



The 2-variable weighted shift $W_{(\alpha,\beta)}$ whose weight diagram is shown on the left has the following properties:

(i) $\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} = 1$ for all $(k_1, k_2) \in \mathbb{Z}^2_+$.

(ii) $\widehat{W_{(\alpha,\beta)}} = \widehat{W_{(\alpha,\beta)}}^D = W_{(\alpha,\beta)}$.

(iii) $W_{(\alpha,\beta)}$ is a spherical isometry.

Figure 2. The weight diagram on the left corresponds to the 2-variable weighted shift in Question 5.15.

6. Appendix

For the reader's convenience, in this section, we gather several well known auxiliary results which are needed for the proofs of the main results in this article. To check subnormality of 2-variable weighted shifts, we introduce some definitions [23, Proposition 3.10].

Definition 6.1. *(i) Let μ and ν be two positive Borel measures on a set X. We say that $\mu \leq \nu$ on X, if $\mu(E) \leq \nu(E)$ for each Borel subset $E \subseteq X$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on X.*
(ii) Let μ be a probability Borel measure on $X \times Y$, and assume that $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} (which is also a probability Borel measure) on $X \times Y$ is given by

$d\mu_{ext}(s,t) := (1-\delta_0(t))\frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}}d\mu(s,t)$.

(iii) Given a Borel measure μ on $X\times Y$, the marginal measure μ^X is given by $\mu^X := \mu\circ\pi_X^{-1}$, where $\pi_X : X\times Y \to X$ is the canonical projection onto X. Thus $\mu^X(E) = \mu(E\times Y)$, for every $E\subseteq X$.

Lemma 6.2. *Let μ be a probability Borel measure on $X\times Y$, and let μ^X and μ^Y be the two marginal measures. Then*

$$\operatorname{supp}\mu \subseteq \operatorname{supp}\sigma\times\operatorname{supp}\tau \subseteq X\times Y.$$

Lemma 6.3. [23, Proposition 3.9] *(Subnormal backward extension) Assume that $W_{(\alpha,\beta)}\in\mathfrak{H}_0$ (see Figure 1(i)) and that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with associated measure $\mu_{\mathcal{M}}$. Then $W_{(\alpha,\beta)}$ is subnormal if and only if the following conditions hold:*
(i) $\frac{1}{t}\in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \le (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}(\mu_{\mathcal{M}})_{ext}^X \le \sigma$.
Moreover, if $\beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$, then $(\mu_{\mathcal{M}})_{ext}^X = \sigma$. In the case when $W_{(\alpha,\beta)}$ is subnormal, the Berger measure μ of $W_{(\alpha,\beta)}$ is given by

$$\mu = \beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}(\mu_{\mathcal{M}})_{ext} + \left(\sigma - \beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}(\mu_{\mathcal{M}})_{ext}^X\right)\times\delta_0. \tag{6.1}$$

Recall that given two positive regular Borel measures μ and ω, μ is said to be absolutely continuous with respect to ω (in symbols, $\mu\ll\omega$) if for every Borel set E, $\omega(E)=0\Rightarrow\mu(E)=0$.

Lemma 6.4. [24, Theorem 3.3] *Consider the 2-variable weighted shift $\mathbf{T}\equiv(T_1,T_2)=W_{(\alpha,\beta)}$ given by Figure 1(i)). If $\mathbf{T}\equiv(T_1,T_2)=W_{(\alpha,\beta)}$ is subnormal, then $\sigma^{(j+1)}\ll\sigma^{(j)}$ and $\tau^{(i+1)}\ll\tau^{(i)}$ $(i,j\ge 0)$, where $\sigma^{(j)}$ (resp. $\tau^{(i)}$) is the Berger measure of the j-th horizontal slice of T_1 (resp. the i-th vertical slice of T_2).*

Lemma 6.5. [24] *Let μ and ν be two regular Borel measures on R, and assume that $\mu\ll\nu$. Then $\mu^X\ll\nu^X$ and $\mu^Y\ll\nu^Y$.*

Lemma 6.6. [24] *Let μ be the Berger measure of a subnormal 2-variable weighted shift, and for $j\ge 0$ let $\sigma^{(j)}$ be as in Lemma 6.4. Then $\sigma^{(j)}=\mu_j^X$, where $d\mu_j(s,t) := \frac{1}{\gamma_{0j}\left(W_{(\alpha,\beta)}\right)}t^j d\mu(s,t)$.*

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