QUADRATICALLY HYPONORMAL WEIGHTED SHIFTS WITH TWO EQUAL WEIGHTS

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We present an extensive analysis of *positively* quadratically hyponormal weighted shifts W_α with $\alpha_0 = \alpha_1 = 1$. Our main result states that such weighted shifts abound! Specifically, by focusing on recursively generated weighted shifts of the form $W_{1,(1,\sqrt{x},\sqrt{y})^\wedge}$, we establish that the planar set $\mathcal{R} := \{(x,y) : W_{1,(1,\sqrt{x},\sqrt{y})^\wedge}$ is positively quadratically hyponormal$\}$ is a closed convex set with nonempty interior. In addition, we are able to describe in detail the boundary of $\mathcal{R}$.

1 Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. It is well known that normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal, with converses false. In an attempt to bridge the gap between hyponormality and subnormality, the classes of k-hyponormal and weakly k-hyponormal operators were introduced and studied in [Ath], [Cu1], [Cu2], [CuFi1], [CuFi2] and [CMX].

For $A, B \in \mathcal{L}(\mathcal{H})$, let $[A,B] := AB - BA$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For $k \geq 1$ and $T \in \mathcal{L}(\mathcal{H})$, T is *k-hyponormal* if $(I, T, \cdots, T^k)$ is hyponormal. The n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ is *weakly hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_n T_n$ is hyponormal for every $\lambda_i \in \mathbf{C}$, $i = 1, \cdots, n$, where $\mathbf{C}$ is the set of complex numbers. An operator T is *weakly k-hyponormal* if $(T, T^2, \cdots, T^k)$ is weakly hyponormal. It is well-known that subnormal $\Rightarrow$ k-hyponormal $\Rightarrow$ weakly k-hyponormal, for every $k \geq 1$; to study the converses, unilateral weighted shifts were considered in [Cu1], [CuFi1] and [CuFi2].

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For weighted shifts, the distinction between k-hyponormality and weak k-hyponormality is closely related to the flatness of their weight sequence. In particular, weak 2-hyponormality, often referred to as quadratic hyponormality, was first considered in detail in [Cu1]. It was shown there that for a unilateral weighted shift W_α, with $\alpha_n = \alpha_{n+1}$ for some n, 2-hyponormality immediately forces the weight α to be flat, that is, $\alpha_1 = \alpha_2 = \cdots$ (In [Sta], J. Stampfli had previously established this for subnormal shifts, and [Cu1, Corollary 6] shows that the assumed "rigidity" really pertains to 2-hyponormality.) Concerning weak 2-hyponormality, A. Joshi proved in [Jo1], [Jo2] that the unilateral weighted shift with weights $\alpha_0 = \alpha_1 = a,\ \alpha_2 = \alpha_3 = \cdots = b,\ 0 < a < b$, is *not* quadratically hyponormal, and later P. Fan [Fan] established that for $a = 1,\ b = 2$, and $0 < s < \sqrt{5}/5$, the operator $W_\alpha + sW_\alpha^2$ is *not* hyponormal.

With the aid of symbolic manipulation, it was shown in [Cu1], [Cu2] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat. A natural question then arises: Can a quadratically hyponormal shift have *two* equal weights without being flat? The existence of such shifts was established in [CuFi2, Proposition 4.6], and it led to an essential distinction between 2-hyponormality and quadratic hyponormality, which eventually became the starting point for an inductive procedure to separate subnormality from polynomial hyponormality [CuPu1], [CuPu2]. Very recently, Y.B. Choi [Ch] has shown that a quadratically hyponormal weighted shift with *two* equal weights $\alpha_n = \alpha_{n+1}$ $(n \geq 1)$ must be subnormal. Thus, the weight sequence of a non-subnormal quadratically hyponormal shift with two equal weights is necessarily of the form (up to a scalar) $\alpha\colon 1, 1, \alpha_2, \alpha_3, \ldots$ $(1 < \alpha_2 < \alpha_3 < \ldots)$.

Connected with the above example is the problem of finding adequate descriptions of quadratic hyponormality. A significant starting point in this program is the parameterization of all quadratically hyponormal shifts whose first two weights are equal to 1. Symbolic manipulation shows that there are no such shifts with $1 < \alpha_2 = \alpha_3$ [Cu1, Proposition 11] [Ch, Theorem 1], that α_2 is always less than $\sqrt{2}$, and that $\alpha_3 \geq (2 - \alpha_2^2)^{-1/2}$. In this paper, we give an extensive description of *positive* quadratically hyponormal weighted shifts with $\alpha_0 = \alpha_1 = 1$, with special emphasis on those shifts with a recursively generated tail. Our main result states that such weighted shifts abound! (cf. Theorems 4.1 and 4.11).

The organization of this paper is as follows. In Section 2 we recall some terminology and notation concerning quadratic hyponormality and recursively generated weighted shifts, which will be used frequently throughout the paper. In Section 3 we discuss general facts about quadratically hyponormal weighted shifts W_α with $\alpha_0 = \alpha_1 = 1$. We consider in particular the relationship between a sequence of weights and the determinants of successive principal minors of the canonical infinite matrix induced by the associated shift W_α. This analysis leads to a set of sufficient conditions for positive quadratic hyponormality. In Section 4 we characterize the recursively generated positively quadratically hyponormal weighted shifts with weight sequence $1, (1, \sqrt{x}, \sqrt{y})^\wedge$; the existence of such shifts, with $1 < x < y$, was established in [CuFi2, Proposition 4.6]. Concretely, it was shown that for $x = \frac{11}{10}$ there exists $y \in (\frac{1142}{1000}, \frac{1143}{1000})$ such that the associated shift is positively quadratically hyponormal.

In an effort to understand the relative position of positively quadratically hyponormal operators within the class of subnormal ones, we focus here on the planar set $\mathcal{R}$ of pairs

(x, y) giving rise to positively quadratically hyponormal weighted shifts $W_{1,(1,\sqrt{x},\sqrt{y})^\wedge}$ (see Section 2 for a description of this shift). Our main result, Theorem 4.11, describes $\mathcal{R}$ as a closed convex set, with nonempty interior; in addition, we are able to describe in detail the boundary of $\mathcal{R}$. (In recent work, the second-named author and S.S. Park have shown that for these shifts, quadratic hyponormality and positive quadratic hyponormality are identical notions [JuPa].)

Section 5 is devoted to a discussion of some possible generalizations and directions for future research.

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2 Preliminaries and Notation

We recall some notation which will be used frequently throughout the paper. An operator $T \in \mathcal{L}(\mathcal{H})$ is quadratically hyponormal if $T + sT^2$ is hyponormal for every $s \in \mathbf{C}$. Let $\{e_k\}_{k=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, let P_n denote the orthogonal projection onto the subspace generated by $e_0, ..., e_n$, and let W_α be a hyponormal weighted shift with a weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. For $s \in \mathbf{C}$, we let

$$D(s) := [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2].$$

For $n \geq 0$, let

$$\begin{aligned} D_n(s) &= P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n \\ &= \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} q_k &:= u_k + |s|^2 v_k, \\ r_k &:= s\sqrt{w_k}, \\ u_k &:= \alpha_k^2 - \alpha_{k-1}^2, \\ v_k &:= \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2, \\ w_k &:= \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2 \quad (k \geq 0), \end{aligned}$$

and $\alpha_{-1} = \alpha_{-2} := 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbf{C}$ and every $n \geq 0$. To detect this positivity, we consider $d_n(\cdot) := \det(D_n(\cdot))$. By direct computation we have

$$\begin{aligned} d_0 &= q_0, \\ d_1 &= q_0 q_1 - |r_0|^2, \\ d_{n+2} &= q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n \qquad (n \geq 0); \end{aligned}$$

by inspection, d_n is actually a polynomial in $t := |s|^2$ of degree $n+1$, with Maclaurin expansion

$$d_n(t) := \sum_{i=0}^{n+1} c(n,i) t^i. \tag{2.1}$$

This gives at once

$$\begin{aligned} c(0,0) &= u_0, \qquad c(0,1) = v_0, \qquad c(1,0) = u_1 u_0, \\ c(1,1) &= u_1 v_0 + u_0 v_1 - w_0, \qquad c(1,2) = v_1 v_0, \\ c(n+2,i) &= u_{n+2} c(n+1,i) + v_{n+2} c(n+1,i-1) - w_{n+1} c(n,i-1). \end{aligned}$$

Observe that $c(n,0) \geq 0$ and $c(n,n+1) \geq 0$ for all $n \geq 0$, and that $d_0(t) = \alpha_0^2(1 + t\alpha_1^2) \geq 0$ and

$$d_1(t) = \alpha_0^2[(\alpha_1^2 - \alpha_0^2) + \alpha_1^2(\alpha_2^2 - \alpha_0^2)t + \alpha_1^4\alpha_1^2 t^2)].$$

We also recall [CuFi1] that a weighted shift W_α is said to be *recursively generated* if there exist $i \geq 1$ and $\Psi = (\Psi_0, \cdots, \Psi_{i-1}) \in \mathbf{C}^i$ such that

$$\gamma_n = \Psi_{i-1}\gamma_{n-1} + \cdots + \Psi_0 \gamma_{n-i} \qquad (n \geq i), \tag{2.2}$$

where $\gamma_n\,(n \geq 0)$ is the moment sequence of W_α, i.e., $\gamma_0 := 1, \gamma_{n+1} := \alpha_n^2 \gamma_n \ (n \geq 0)$. Furthermore, (2.2) is equivalent to

$$\alpha_n^2 = \Psi_{i-1} + \frac{\Psi_{i-2}}{\alpha_{n-1}^2} + \cdots + \frac{\Psi_0}{\alpha_{n-1}^2 \cdots \alpha_{n-i+1}^2} \qquad (n \geq i). \tag{2.3}$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots, \alpha_{2k}$ $(k \geq 0)$, there is a canonical procedure to generate a sequence (denote $\widehat{\alpha}$) in such a way that $W_{\widehat{\alpha}}$ is a recursively generated shift having α as an initial segment of weights (cf. [CuFi1, p. 219]). We now review this procedure in a special case. Given

$$\alpha\ : \alpha_0, \alpha_1, \alpha_2 \quad (0 < \alpha_0 < \alpha_1 < \alpha_2),$$

let

$$\mathbf{v}_0 := \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}, \quad \mathbf{v}_1 := \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad \mathbf{v}_2 := \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix}.$$

The vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ are linearly independent in $\mathbf{R}^2$, so there exists a unique $\Psi = (\Psi_0, \Psi_1) \in \mathbf{R}^2$ such that $\mathbf{v}_2 = \Psi_0 \mathbf{v}_0 + \Psi_1 \mathbf{v}_1$. In fact,

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}. \tag{2.4}$$

Let $\widehat{\gamma}_n := \gamma_n$ $(0 \le n \le 1)$ and let

$$\widehat{\gamma}_n := \Psi_1 \widehat{\gamma}_{n-1} + \Psi_0 \widehat{\gamma}_{n-2} \qquad (n \ge 2). \tag{2.5}$$

Since $\widehat{\gamma}_n > 0\,(n \ge 0)$ (cf. [CuFi1, Proof of Theorem 3.5]), we define

$$\widehat{\alpha}_n := \left(\frac{\widehat{\gamma}_{n+1}}{\widehat{\gamma}_n} \right)^{\frac{1}{2}} \qquad (n \ge 0) \tag{2.6}$$

(so that $\widehat{\alpha}_n = \alpha_n$ for $0 \le n \le 2$). Hence we obtain the coefficients of a recursively generated weighted shift, and

$$\widehat{\alpha}_n^2 = \Psi_1 + \frac{\Psi_0}{\widehat{\alpha}_{n-1}^2} \qquad (n \ge 1). \tag{2.7}$$

By rearranging indices in some of the results of [CuFi1], [CuFi2] (cf. [CuFi2, p. 394]), we obtain the following lemma.

Lemma 2.1 *Let*

$$\alpha : \alpha_0, \alpha_1, \cdots, \alpha_{k-2}, (\alpha_{k-1}, \alpha_k, \alpha_{k+1})^\wedge$$

with $0 < \alpha_{k-1} < \alpha_k < \alpha_{k+1}$ $(k \ge 1)$ *and let* W_α *be the unilateral weighted shift associated with* α*. Then*
(i) $v_{n+1} = \Psi_1 (u_{n+1} + u_n) \quad (n \ge k)$;
(ii) $w_n = u_n v_{n+1} \quad (n \ge k)$ *and*

$$u_n = -\Psi_0 \frac{u_{n-1}}{\alpha_{n-2}^2 \alpha_{n-1}^2} \quad (n \ge k+1), \tag{2.8}$$

where

$$\Psi_0 = -\frac{\alpha_k^2 \alpha_{k-1}^2 (\alpha_{k+1}^2 - \alpha_k^2)}{\alpha_k^2 - \alpha_{k-1}^2} \quad \text{and} \quad \Psi_1 = \frac{\alpha_k^2 (\alpha_{k+1}^2 - \alpha_{k-1}^2)}{\alpha_k^2 - \alpha_{k-1}^2}.$$

3 Quadratically Hyponormal Weighted Shifts with $\alpha_0 = \alpha_1 = 1$

Let W_α be a hyponormal weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$ and assume that $d_n(t) > 0$ for all $t > 0$ and all $n \ge 0$. It follows that W_α is quadratically hyponormal (cf. [CuFi2]). Conversely, if W_α is a quadratically hyponormal weighted shift, then obviously $d_n(t) \ge 0$ for all $t \ge 0$ and all $n \ge 0$. Moreover, if $d_{n_0}(t_0) = 0$ for some $n_0 \ge 0$ and $t_0 > 0$, then $\alpha_0 \le \alpha_1 = \alpha_2 = \cdots$ (cf. [CuFi1]), so W_α is subnormal. Therefore, the positivity of $d_n(t)$ is a fairly good detector of quadratic hyponormality.

Lemma 3.1 *Let* W_α *be a weighted shift with* $\alpha_0 = \alpha_1 = 1$*. If* $d_3(t) \ge 0$*, then* $\alpha_2 = 1$ *or* $\alpha_3^2 \ge 1/(2 - \alpha_2^2)$*.*

Proof. With $\alpha_0 = \alpha_1 = 1$, $d_3(t) = c(3,2)t^2 + c(3,3)t^3 + c(3,4)t^4$, and $c(3,2) = \alpha_2^2(\alpha_2^2 - 1)(2\alpha_3^2 - \alpha_2^2\alpha_3^2 - 1) \ge 0$. The result is now obvious. ■

Proposition 3.2 *If $\alpha_0 = \alpha_1 = \alpha_2 = 1$ and $d_4(t) \geq 0$, then $d_n(t) \geq 0$ for all $n \geq 1$.*

Proof. By the proof of [Cu1, Lemma 9], it is easy to show that $\alpha_3 = 1$, which implies that $d_n(t) = 0$ for all $n \geq 2$. ■

A variation of Proposition 3.2 gives rise to the following result.

Proposition 3.3 *If $\alpha_2 = \alpha_3 = \alpha_4 = 1$ and $d_3(t) \geq 0$, then $d_n(t) \geq 0$ for all $n \geq 0$.*

Proof. Here $d_3(t) = \alpha_0^2 t^2[-\alpha_0^2(1-\alpha_1^2)^3 + ...]$, so the positivity of d_3 forces $\alpha_1 = 1$. A straightforward calculation now shows that $d_n(t) \geq 0$ for every $n \geq 0$, as desired. ■

Proposition 3.4 *Assume that $\alpha_0 = \alpha_1 = 1$ and $\alpha_2 = \alpha_3 \geq 1$. If $d_3(t) \geq 0$, then $d_n(t) \geq 0$ for all $n \geq 0$.*

Proof. Since $d_3(t) \geq 0$, Lemma 3.1 implies that $\alpha_2 = 1$ or $\alpha_3^2 \geq 1/(2-\alpha_2^2)$. If $\alpha_2 \neq 1$, then $\alpha_3^2 \geq 1/(2-\alpha_2^2)$, so

$$-(\alpha_2^2-1)^2 = 2\alpha_2^2 - \alpha_2^4 - 1 = \alpha_2^2(2-\alpha_2^2) - 1 = \alpha_3^2(2-\alpha_2^2) - 1 \geq 0,$$

which forces $\alpha_2 = 1$, a contradiction. It follows that $\alpha_2 = \alpha_3 = 1$. It is now straightforward to verify that $d_0(t) = 1 + t$, $d_1(t) = t^2$, and $d_n \equiv 0$ $(n \geq 2)$. ■

Corollary 3.5 *Let W_α be a hyponormal weighted shift with $\alpha_0 = \alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \sqrt{a}$. The following statements are equivalent.*
(i) $a = 1$;
(ii) $d_3(t) \geq 0$;
(iii) $d_n(t) \geq 0$ *for all* $n \geq 0$.

The next result establishes a link between the proximity of α_3 to α_2 and the proximity of α_2 to α_1, under the assumption $d_3 \geq 0$.

Proposition 3.6 *Assume that $\alpha_0 = \alpha_1 = 1$, $\alpha_2 = \sqrt{b}$ and $\alpha_3 = \sqrt{b+\epsilon}$. Suppose $d_3(t) \geq 0$. If $\epsilon \to 0$, then $b \to 1$.*

Proof. By Lemma 3.1, $b + \epsilon \geq 1/(2-b)$, that is, $b^2 - (\epsilon - 2)b + (1-2\epsilon) \leq 0$. Hence

$$\frac{(2-\epsilon) - \sqrt{\epsilon(\epsilon+4)}}{2} \leq b \leq \frac{(2-\epsilon) + \sqrt{\epsilon(\epsilon+4)}}{2}.$$

It now follows that $\epsilon \to 0$ implies $b \to 1$. ■

Next, we consider hyponormal weighted shifts W_α with $1 = \alpha_0 = \alpha_1 < \alpha_2$.

Proposition 3.7 *Assume that $1 = \alpha_0 = \alpha_1 < \alpha_2$. Then W_α is quadratically hyponormal if and only if $d_n(t) > 0$ $(t > 0)$ for all $n \geq 0$.*

Proof. $\Longleftarrow$) See [CuFi2, Lemma 4.1].

$\Longrightarrow$) Without loss of generality, we can assume that there does not exist an index n such that $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$, for otherwise $\alpha_1 = \alpha_2 = 1$, by [Cu1, Proposition 11]. Thus, $v_k > 0$ for all $k \geq 2$ and $w_k > 0$ for all $k \geq 0$. If there exists $n_0 \geq 0$ and $t_0 > 0$ such that $d_{n_0}(t_0) = 0$, choose n_0 minimal. Since $d_0(t) = 1 + t > 0$, $d_1(t) = t(\alpha_2^2 - 1 + \alpha_2^2 t) > 0$, and $d_2(t) = \alpha_2^2 t^2[(\alpha_2^2 - 1)\alpha_3^2 + (\alpha_2^2\alpha_3^2 - 1)t] > 0$, it follows that $n_0 \geq 3$. From

$$d_{n_0+1}(t_0) = q_{n_0+1}(t_0)d_{n_0}(t_0) - |r_{n_0}(t_0)|^2 d_{n_0-1}(t_0) = -|r_{n_0}(t_0)|^2 d_{n_0-1}(t_0),$$

and the fact that W_α is quadratically hyponormal, we obtain at once $\sqrt{t_0 w_{n_0}} = r_{n_0}(t_0) = 0$, a contradiction. ■

The following example should be compared with Proposition 3.7.

Example 3.8 *Let W_α be a weighted shift with weight sequence $\alpha : 0 < \alpha_0 \leq \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 < \alpha_5 \leq \cdots$. Then W_α is not quadratically hyponormal. However, $d_n(t) \geq 0$ for all $n \geq 0$. Indeed, $q_3 = r_2 = r_3 = 0$ and $d_3(t) = d_4(t) = 0$. Hence $d_n(t) \geq 0$ for all $n \geq 0$.*

Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence. Recall from [CuFi2] that W_α is positively quadratically hyponormal if $c(n, i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$ and $c(n, n+1) > 0$ for all $n \geq 0$. This notion is stronger than quadratic hyponormality, as shown recently by the second-named author and S.S. Park [JuPa]: they exhibit an example of a quadratically hyponormal shift which is not positively quadratically hyponormal.

We now briefly discuss two examples of unilateral weighted shifts closely related to the Bergman shift. In one instance, we shall establish that the shift is positively quadratically hyponormal, in the other that it is not quadratically hyponormal.

Example 3.9 *Let $\alpha : 1, 1, \sqrt{9/8}, \sqrt{12/10}, \sqrt{15/12}, \sqrt{18/14}, \cdots$ be a weight sequence. Then by [Cu1] W_α is positively quadratically hyponormal. (Note that the sequence*

$$\sqrt{2/3}\alpha : \sqrt{2/3}, \sqrt{2/3}, \sqrt{3/4}, \sqrt{4/5}, \sqrt{5/6}, \sqrt{6/7}, \cdots$$

has a Bergman tail.)

Example 3.10 *Let $\alpha : 1, 1, \sqrt{x}, \sqrt{6/4}, \sqrt{8/5}, \cdots$ be a weight sequence. Then W_α is not quadratically hyponormal for any x with $1 \leq x \leq 6/4$. Indeed, direct computation shows that $d_3 \geq 0 \Rightarrow 1 \leq x \leq 4/3$, and that for any x in $[1, 4/3]$, the polynomial d_{16} takes on a negative value at $t = 3/20$. (Note that the sequence*

$$(1/\sqrt{2})\alpha : \sqrt{1/2}, \sqrt{1/2}, y, \sqrt{3/4}, \sqrt{4/5}, \sqrt{5/6}, \cdots$$

also has a Bergman tail.)

Theorem 3.11 *Let $\alpha : 1, 1, \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \alpha_6, \cdots$ be a weight sequence and assume that W_α is hyponormal. Assume that*
(i) $(cad - ab)(2b - ab - 1) \geq b^2(c - a)^2$,
(ii) $(cd - ab)(abc - bc - 2a^2 + 3a - 1) \geq (ab - 1)(c - a)^2$, *and*
(iii) $(\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_n^2\alpha_{n-1}^2)(\alpha_n^2 - \alpha_{n-1}^2) \geq \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2 \ \ (n \geq 4)$.
Then W_α is positively quadratically hyponormal.

Proof. **Proof.** Observe that for $n \geq 5$ and $0 \leq i \leq n+1$,

$$\begin{aligned} c(n,i) = {} & u_n c(n-1,i) + v_n v_{n-1} \cdots v_5 (v_4 c(3, i-n+3) - w_3 c(2, i-n+3)) \\ & - w_3 c(2, i-n+3)) + \{(v_n u_{n-1} - w_{n-1}) c(n-2, i-1) \\ & + v_n (v_{n-1} u_{n-2} - w_{n-2}) c(n-3, i-2) + \ldots \\ & + v_n v_{n-1} \cdots v_6 (v_5 u_4 - w_4) c(3, i-n+4)\}. \end{aligned} \tag{3.1}$$

We shall establish that

$$v_4 c(3, i-n+3) - w_3 c(2, i-n+3) \geq 0, \ \ (i \geq 2), \tag{3.2}$$

and

$$v_{n+1} u_n - w_n \geq 0, \ \ (n \geq 4). \tag{3.3}$$

Note that (iii) holds if and only if (3.3) holds. To prove (3.2) we observe that

$$v_4 c(3,2) - w_3 c(2,2) \geq 0, \tag{3.4}$$

$$v_4 c(3,3) - w_3 c(2,3) \geq 0, \tag{3.5}$$

and

$$v_4 c(3,4) \geq 0. \tag{3.6}$$

However, (3.4) and (3.5) are equivalent to (i) and (ii), respectively. Since $c(2,i) \geq 0$, $c(1,i) \geq 0$, and $c(0,i) \geq 0$, by (3.4), we have $c(3,2) \geq 0$. By (3.5), we have $c(3,3) \geq 0$. Hence $c(3,i) \geq 0$, $0 \leq i \leq 4$. Consider $n = 4$; we have

$$c(4,i) = u_4 c(3,i) + (v_4 c(3, i-1) - w_3 c(2, i-1)). \tag{3.7}$$

By (3.4), (3.5) and (3.6), we have $c(4,i) \geq 0$, $0 \leq i \leq 5$. Hence, by mathematical induction and (3.1), we have $c(n,i) \geq 0$, for all $n \geq 0$ and all $i \geq 0$. It follows that W_α is positively quadratically hyponormal. ■

The sufficient conditions in Theorem 3.11 really pertain to non-recursively generated weighted shifts, as the following example shows.

Corollary 3.12 *Let* $\alpha : 1, 1, \sqrt{a}, (\sqrt{b}, \sqrt{c}, \sqrt{d})^\wedge$. *Assume that*
(i) $(cd - ab)(2b - ab - 1) \geq b^2 (c-a)^2$, *and*
(ii) $(cd - ab)(abc - bc - 2a^2 + 3a - 1) \geq (ab-1)(c-a)^2$.
Then W_α *is positively quadratically hyponormal.*

Proof. For $n \geq 4$, W_α satisfies $v_{n+1} u_n - w_n \geq 0$. The result now easily follows from Theorem 3.11. ■

Example 3.13 *Let* $1 \leq a \leq b$ *and let* $\alpha : 1, (1, \sqrt{a}, \sqrt{b})^\wedge$. *If* W_α *satisfies property* (i) *in Theorem* 3.11, *then* W_α *is the unweighted unilateral shift. Indeed,* W_α *always satisfies properties* (ii) *and* (iii) *in Theorem* 3.11, *and it satisfies property* (i) *if and only if* $a = b = 1$.

4 Quadratically Hyponormal Shifts with Weights $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^{\wedge}$

In this section we focus on recursively generated weighted shifts of the form W_α, where $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^{\wedge}$ and $1 \leq x \leq y$. We discuss in detail the region

$$\mathcal{R} := \{(x, y) | W_\alpha \text{ is positively quadratically hyponormal}\}.$$

We are now ready to state the main result of this paper.

Theorem 4.1 *The planar set $\mathcal{R}$ is a closed convex set with nonempty interior.*

In what follows, we assume $1 < x < y$ (to avoid the trivial case), and consider the set $\tilde{\mathcal{R}} := \mathcal{R} \setminus \{(1, 1)\}$. We begin with some notation. As in Lemma 2.1, let

$$\Psi_0 := -\frac{x(y-x)}{x-1} \quad \textit{and} \quad \Psi_1 := \frac{x(y-1)}{x-1}, \tag{4.1}$$

and let

$$K := -\frac{\Psi_1^2}{\Psi_0} L, \tag{4.2}$$

where

$$L := \frac{\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0}}{2}. \tag{4.3}$$

By [CuFi2, Theorem 4.3] and its proof, we have that W_α is positively quadratically hyponormal if and only if

$$1 \leq h_2^+ := \Big(\frac{x^2 y + x^2(y-1)K + x(y-x)K^2}{x + x(y-1)K + (1 + xy - 2x)K^2} \Big)^{1/2}. \tag{4.4}$$

By direct computation, we have that

$$\tilde{\mathcal{R}} = \{(x, y) | x(xy-1) + x(x-1)(y-1)K - (x-1)^2 K^2 \geq 0\}, \tag{4.5}$$

where

$$K = \frac{x(y-1)^2 \left(x(y-1) + \sqrt{x^2(y-1)^2 - 4x(x-1)(y-x)} \right)}{2(x-1)^2(y-x)}.$$

4.1 Description of $\tilde{\mathcal{R}}$

Theorem 4.2

$$\tilde{\mathcal{R}} = \{(x, y) \,|\, 1 < x < y \textit{ and } f(x, y) \geq 0\}, \tag{4.6}$$

where $f(x, y) = \sum_{i=0}^{7} \phi_i y^i$ *with*

$$\begin{aligned}
\phi_0 &:= x^2 - 4x^3 + 5x^4 - x^5, \\
\phi_1 &:= -2x + 7x^2 - 2x^3 - 11x^4 + x^5, \\
\phi_2 &:= 1 - 2x - 7x^2 + 4x^3 + 29x^4 - 5x^5 + x^6, \\
\phi_3 &:= -2x + 22x^2 - 33x^3 - 22x^4,
\end{aligned}$$

$$\begin{aligned}
\phi_4 &:= -11x^2 + 24x^3 + 25x^4 - 3x^5,\\
\phi_5 &:= 3x^2 - 12x^3 - 15x^4 + 3x^5,\\
\phi_6 &:= 4x^3 + 4x^4 - x^5,\\
\phi_7 &:= -x^3.
\end{aligned}$$

Lemma 4.3 *Let $h := x - 1$ and $k := y - x$. Then $(x, y) \in \tilde{\mathcal{R}}$ if and only if $(h, k) \in \mathcal{U}_1 \cap \mathcal{U}_2$, where*

$$\mathcal{U}_1 := \{(h, k) \mid f_1(h, k) \geq 0\}, \tag{4.7}$$

$$\mathcal{U}_2 := \{(h, k) \mid f_2(h, k) \geq 0\},$$

and

$$f_1(h, k) := \sum_{i=0}^{7} \eta_i k^i = \sum_{j=0}^{11} \lambda_j h^j,$$

with

$$\begin{aligned}
\eta_0 &:= -2h^7 - 7h^8 - 9h^9 - 5h^{10} - h^{11},\\
\eta_1 &:= -7h^6 - 30h^7 - 45h^8 - 28h^9 - 6h^{10},\\
\eta_2 &:= 4h^4 - 5h^5 - 53h^6 - 96h^7 - 66h^8 - 15h^9,\\
\eta_3 &:= 4h^3 - h^4 - 57h^5 - 117h^6 - 85h^7 - 20h^8,\\
\eta_4 &:= h^2 - 6h^3 - 50h^4 - 93h^5 - 65h^6 - 15h^7,\\
\eta_5 &:= -9h^2 - 36h^3 - 51h^4 - 30h^5 - 6h^6,\\
\eta_6 &:= -5h - 16h^2 - 18h^3 - 8h^4 - h^5,\\
\eta_7 &:= -1 - 3h - 3h^2 - h^3,\\
\lambda_0 &:= -k^7,\\
\lambda_1 &:= -5k^6 - 3k^7,\\
\lambda_2 &:= k^4 - 9k^5 - 16k^6 - 3k^7,\\
\lambda_3 &:= 4k^3 - 6k^4 - 36k^5 - 18k^6 - k^7,\\
\lambda_4 &:= 4k^2 - k^3 - 50k^4 - 51k^5 - 8k^6,\\
\lambda_5 &:= -5k^2 - 57k^3 - 93k^4 - 30k^5 - k^6,\\
\lambda_6 &:= -7k - 53k^2 - 117k^3 - 65k^4 - 6k^5,\\
\lambda_7 &:= -2 - 30k - 96k^2 - 85k^3 - 15k^4,\\
\lambda_8 &:= -7 - 45k - 66k^2 - 20k^3,\\
\lambda_9 &:= -9 - 28k - 15k^2,\\
\lambda_{10} &:= -5 - 6k,\\
\lambda_{11} &:= -1,
\end{aligned}$$

and

$$f_2(h, k) := \sum_{i=0}^{6} \delta_i k^i = \sum_{j=0}^{9} \varepsilon_j h^j \tag{4.8}$$

with

$$\begin{aligned}
\delta_0 &:= -h^6 - 3h^7 - 3h^8 - h^9,\\
\delta_1 &:= -3h^5 - 12h^6 - 15h^7 - 6h^8,\\
\delta_2 &:= 4h^3 - h^4 - 21h^5 - 33h^6 - 15h^7,\\
\delta_3 &:= 2h^2 - 24h^4 - 42h^5 - 20h^6,
\end{aligned}$$

$$
\begin{aligned}
\delta_4 &:= -3h^2 - 21h^3 - 33h^4 - 15h^5, \\
\delta_5 &:= -3h - 12h^2 - 15h^3 - 6h^4, \\
\delta_6 &:= -1 - 3h - 3h^2 - h^3,
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_0 &:= -k^6, \\
\epsilon_1 &:= -3k^5 - 3k^6, \\
\epsilon_2 &:= 2k^3 - 3k^4 - 12k^5 - 3k^6, \\
\epsilon_3 &:= 4k^2 - 21k^4 - 15k^5 - k^6, \\
\epsilon_4 &:= -k^2 - 24k^3 - 33k^4 - 6k^5,
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_5 &:= -3k - 21k^2 - 42k^3 - 15k^4, \\
\epsilon_6 &:= -1 - 12k - 33k^2 - 20k^3, \\
\epsilon_7 &:= -3 - 15k - 15k^2, \\
\epsilon_8 &:= -3 - 6k, \\
\epsilon_9 &:= -1.
\end{aligned}
$$

Proof. By expanding the inequality in (4.5) we have

$$
x(xy-1) + x(x-1)(y-1)K - (x-1)^2K^2 \geq 0 \quad \Longleftrightarrow A \geq B\sqrt{C}, \tag{4.9}
$$

where

$$
\begin{aligned}
A &:= 2x^6y - 2xy^2 + x^2(4y + 4y^2 + 2y^3) + x^3(-2 - 11y + 6y^2 - 22y^3 + 12y^4 - 3y^5), \\
&\quad + x^4(6 + 3y - y^2 + 28y^3 - 24y^4 + 9y^5 - y^6) \\
&\quad + x^5(-5 + 8y - 22y^2 + 12y^3 - 3y^4), \\
B &:= x^2(-y + 3y^2 - 3y^3 + y^4) + x^3(3y - 10y^2 + 12y^3 - 6y^4 + y^5) \\
&\quad + x^4(-1 + 3y - 3y^2 + y^3), \\
C &:= x^2(-1 + y)^2 - 4x(-1 + x)(-x + y).
\end{aligned}
$$

In terms of h and k,

$$
\begin{aligned}
A &= f_2(h,k)(1+h), \\
B &= k^5 + h(4k^4 + 3k^5) + h^2(7k^3 + 13k^4 + 3k^5) + h^3(7k^2 + 24k^3 + 14k^4 + k^5) \\
&\quad + h^4(4k + 24k^2 + 27k^3 + 5k^4) + h^5(1 + 13k + 27k^2 + 10k^3) \\
&\quad + h^6(3 + 14k + 10k^2) + h^7(3 + 5k) + h^8, \\
C &= (1+h)((h-k)^2 + h^3 + 2h^2k + hk^2).
\end{aligned}
$$

Clearly, $B \geq 0$; therefore, the right-hand inequality in (4.9) is equivalent to

$$
\left\{ \begin{array}{l} A^2 \geq B^2C \\ A \geq 0 \end{array} \right. , \tag{4.10}
$$

which induce easily the sets $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. ■

The next lemma will be proved in Subsection 4.3.

Lemma 4.4 *$\mathcal{U}_1 \subseteq \mathcal{U}_2$.*

Combining all of the above lemmas, we obtain the following theorem, which is an obvious modification of Theorem 4.2.

Theorem 4.5 *Let $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ with $1 < x < y$ and let $h := x - 1$, $k := y - x$. Then W_α is positively quadratically hyponormal if and only if $(h, k) \in \mathcal{U}_1$, i.e., if and only if $f_1(h, k) \geq 0$.*

4.2 The Shape of $\mathcal{U}_1$

In this subsection we discuss the region $\mathcal{U}_1$; as we have observed, this is equivalent to the study of $\tilde{\mathcal{R}}$. To this end, we replace k by th, where t is a nonnegative real number. Then we have

$$f_1(h, th) = (\phi_1(t) - G_1(h, t))h^5, \tag{4.11}$$

where

$$\begin{aligned}
\phi_1(t) &:= 4t^2 + 4t^3 + t^4, \\
G_1(h, t) &:= \varphi_1(t)h + \varphi_2(t)h^2 + \varphi_3(t)h^3 + \varphi_4(t)h^4 + \varphi_5(t)h^5, \\
\varphi_1(t) &:= 2 + 7t + 5t^2 + t^3 + 6t^4 + 9t^5 + 5t^6 + t^7, \\
\varphi_2(t) &:= 7 + 30t + 53t^2 + 57t^3 + 50t^4 + 36t^5 + 16t^6 + 3t^7, \\
\varphi_3(t) &:= 9 + 45t + 96t^2 + 117t^3 + 93t^4 + 51t^5 + 18t^6 + 3t^7, \\
\varphi_4(t) &:= 5 + 28t + 66t^2 + 85t^3 + 65t^4 + 30t^5 + 8t^6 + t^7, \\
\varphi_5(t) &:= 1 + 6t + 15t^2 + 20t^3 + 15t^4 + 6t^5 + t^6.
\end{aligned}$$

Let

$$\tilde{r}_1(h, t) := \phi_1(t) - G_1(h, t)$$

and let

$$C := \{(h, t) \in R^+ \times R^+ : \tilde{r}_1(h, t) = 0\}.$$

Since it is easy to exhibit pairs (h, t) and (h', t') such that $\tilde{r}_1(h, t) > 0$ and $\tilde{r}_1(h', t') < 0$, the Intermediate Value Theorem readily implies that $C \neq \emptyset$.

Lemma 4.6 *C is a loop with polar form $r = f(\theta)$, $0 \leq \theta \leq \pi/2$.*

Proof. Observe first that $\tilde{r}_1(0, 0) = \phi_1(0) - G_1(0, 0) = 0$. Now fix $t_0 \in R^+$. Since $\varphi_i(t_0) > 0$, $i = 1, 2, 3, 4, 5$, $G_1(h, t_0)$ is strictly increasing in h, so there exists a unique h_0 such that $\phi_1(t_0) = G_1(h_0, t_0)$. This proves the lemma. ∎

We now consider the tangent line to the curve C near $(0, 0)$.

Lemma 4.7 *The tangent line to C converges to the x-axis as $(h, t) \to (0, 0)$, and it converges to the y-axis as $(k, t) \to (0, \infty)$.*

Proof. Since $k \equiv th$, we have

$$\frac{dk}{dh} = \frac{dt}{dh}h + t, \tag{4.12}$$

and since $\tilde{r}_1(h,t) = 0$ on C, we must have

$$\frac{\partial \tilde{r}_1}{\partial t}\frac{dt}{dh} + \frac{\partial \tilde{r}_1}{\partial h} = 0. \tag{4.13}$$

That is,

$$\left(\frac{d\phi_1(t)}{dt} + (-\varphi_1'(t)h - \varphi_2'(t)h^2 - \cdots - \varphi_5'(t)h^5)\right)\frac{dt}{dh} \\ + (-\varphi_1(t) - \varphi_2(t)2h - ... - \varphi_5(t)5h^4) = 0. \tag{4.14}$$

Hence we have

$$\frac{dt}{dh} = \frac{\varphi_1(t) + \varphi_2(t)2h + \cdots + \varphi_5(t)5h^4}{\frac{d\phi_1(t)}{dt} - \varphi_1'(t)h - \varphi_2'(t)h^2 - \cdots - \varphi_5'(t)h^5}. \tag{4.15}$$

Furthermore, since $t \to 0^+$as $h \to 0^+$on C, by (4.12) we have

$$\lim_{h\to 0^+}\frac{dk}{dh} = \lim_{h\to 0^+}\frac{dt}{dh}\cdot h. \tag{4.16}$$

By (4.15) and (4.11), $\lim_{h\to 0^+}\frac{dt}{dh} = \infty$, so the limit in (4.16) is an indeterminate form, and we may apply L'Hôpital's rule to calculate it:

$$\lim_{h\to 0^+}\frac{dt}{dh}\cdot h = \lim_{h\to 0^+}\frac{h}{\frac{dh}{dt}} = \lim_{h\to 0^+}\frac{1}{\frac{d}{dh}\left(\frac{dh}{dt}\right)}. \tag{4.17}$$

Notice that

$$\frac{d}{dh}\left(\frac{dh}{dt}\right) = \frac{\partial}{\partial t}\left(\frac{dh}{dt}\right)\cdot\frac{dt}{dh} + \frac{\partial}{\partial h}\left(\frac{dh}{dt}\right). \tag{4.18}$$

By direct computation, we have

$$\frac{d}{dh}\left(\frac{dh}{dt}\right) = \frac{F_1(h,t)}{F_2(h,t)}, \tag{4.19}$$

for some expressions $F_1(h,t)$ and $F_1(h,t)$ such that

$$\lim_{h\to 0^+} F_1(h,t) = -32 \text{ and } \lim_{h\to 0^+} F_2(h,t) = 0, \tag{4.20}$$

and $F_2(h,t) < 0$ for small $(h,t) \in R_+ \times R_+$. This implies that

$$\lim_{h\to 0^+}\frac{d}{dh}\left(\frac{dh}{dt}\right) = +\infty, \tag{4.21}$$

and then

$$\lim_{h\to 0^+}\frac{dk}{dh} = \lim_{h\to 0^+}\frac{dt}{dh}\cdot h = 0. \tag{4.22}$$

Now we consider the tangent line to C as $(k,t) \to (0,\infty)$. Since $k \to 0^+$ is equivalent to $h \to 0^+$ on C under the equation $k = th$, we have

$$\lim_{k\to 0^+} \frac{dk}{dh} = \lim_{k\to 0^+} \left[\frac{dt}{dh}h + t\right] = +\infty. \tag{4.23}$$

The proof is now complete. ■

By combining Lemmas 4.6 and 4.7 we obtain the following result.

Proposition 4.8 *The boundary of the region $\mathcal{U}_1$ is a loop with polar form $r = \tilde{r}_1(\theta)$ whose tangent lines near the origin $(0,0)$ converge to the x- and y-axes.*

We shall draw the graph of $r = \tilde{r}_1(\theta)$ in the next subsection.

Recall Descartes's Rule of Signs that if $p(x)$ is a polynomial with real coefficients, then the number of positive roots either is equal to the number of variations in sign of $p(x)$ or is less than that number by an even number; and the number of negative roots either is equal to the number of variations in sign of $p(-x)$ or is less than that number by an even number.

Lemma 4.9 *Given $h > 0$, there exist at most two roots (possibly a double root) $k_0 > 0$ such that $f_1(h, k_0) = 0$.*

Proof. Recall that

$$f_1(h,k) = \sum_{i=0}^{7} \phi_i(h)k^i,$$

where all of the coefficients of $\phi_i(h)$, $i = 0, 1, 5, 6, 7$ are negative (see Lemma 4.3). Moreover, $\phi_i(h)$, $i = 2, 3, 4$, has one variation in sign, so it has exactly one positive root α_i, $i = 2, 3, 4$, respectively. Hence we have
(i) $\phi_2(h) \geq 0$ on $0 < h \leq \alpha_2 \cong 0.20004$ and $\phi_2(h) < 0$ on $h > \alpha_2$,
(ii) $\phi_3(h) \geq 0$ on $0 < h \leq \alpha_3 \cong 0.21044$ and $\phi_3(h) < 0$ on $h > \alpha_3$,
(iii) $\phi_4(h) \geq 0$ on $0 < h \leq \alpha_4 \cong 0.08901$ and $\phi_4(h) < 0$ on $h > \alpha_4$.
Comparing the zeros of ϕ_i, $i = 2, 3, 4$, it follows from the Descartes's Rule of Signs that for fixed $h > 0$, the sign of $f_1(h,k)$ changes twice as a polynomial in k. Hence the number of roots of $f_1(h,k) = 0$ is 0 or 2. ■

Using a similar technique in Lemma 4.9 with functions in Lemma 4.3, we obtain the following lemma.

Lemma 4.10 *Given $k > 0$, there exist at most two roots (possibly a double root) $h_0 > 0$ such that $f_1(h_0, k) = 0$.*

The following is one of our main results, describing the shape of $\mathcal{U}_1$.

Theorem 4.11 *The region $\mathcal{U}_1$ is convex.*

Before proving Theorem 4.11 we need some calculations to obtain the behavior of $\frac{d^2k}{dh^2}(h,t)$ for a given parameter $t \in (0,\infty)$. Let $\widetilde{r}_1(h,t) := \phi_1(t) + G_1(h,t)$ as in the remarks preceding Lemma 4.6, and consider the curve $\widetilde{r}_1(h,t) = 0$. By (4.13), we have

$$\frac{dt}{dh} = -\frac{\frac{\partial \tilde{r}_1}{\partial h}}{\frac{\partial \tilde{r}_1}{\partial t}}. \tag{4.24}$$

Also, $k = th$ leads to the equation

$$\frac{d^2k}{dh^2} = \frac{d^2t}{dh^2} \cdot h + 2 \cdot \frac{dt}{dh}. \tag{4.25}$$

Now let us differentiate the equation (4.13) with respect to h and obtain the equation

$$\begin{aligned}\left[\frac{\partial}{\partial t}\left(\frac{\partial \widetilde{r}_1}{\partial t}\right) \cdot \frac{dt}{dh} + \frac{\partial}{\partial h}\left(\frac{\partial \widetilde{r}_1}{\partial t}\right)\right] \cdot \frac{dt}{dh} + \frac{\partial \widetilde{r}_1}{\partial t} \cdot \left[\frac{\partial}{\partial t}\left(\frac{dt}{dh}\right) \cdot \frac{dt}{dh} + \frac{d^2t}{dh^2}\right] \\ + \frac{\partial}{\partial t}\left(\frac{\partial \widetilde{r}_1}{\partial h}\right) \cdot \frac{dt}{dh} + \frac{\partial^2 \widetilde{r}_1}{\partial h^2} = 0.\end{aligned} \tag{4.26}$$

By (4.24) and the definition of $\widetilde{r}_1(h,t)$, we may solve for $\frac{d^2t}{dh^2}$ in (4.26). Now substituting $\frac{d^2t}{dh^2}$ and $\frac{dt}{dh}$ in (4.25), we have $\frac{d^2k}{dh^2}$. For convenience, we abbreviate this as follows:

$$\frac{d^2k}{dh^2} = \frac{P(h,t)}{Q(h,t)^2}, \tag{4.27}$$

for some function $Q(h,t)$ and for

$$P(h,t) \equiv \gamma(t) + \rho_1(t)h + \rho_2(t)h^2 + \cdots + \rho_9(t)h^9$$

satisfying
(i) all coefficients of γ are positive,
(ii) all coefficients of ρ_i, $i = 3, 4, \cdots, 9$ are negative,
(iii)

$$\begin{aligned}\rho_1(t) = &-14 + 267t + 1833t^2 + 4746t^3 + 6747t^4 + 7155t^5 \\ &+ 6080t^6 + 4088t^7 + 1584t^8 - 115t^9 - 489t^{10} \\ &- 258t^{11} - 65t^{12} - 7t^{13},\end{aligned}$$

(iv)

$$\begin{aligned}\rho_2(t) = &-196 - 256t + 2848t^2 + 11380t^3 + 18760t^4 \\ &+ 15364t^5 + 2608t^6 - 9880t^7 - 15764t^8 - 14352t^9 - 8744t^{10} \\ &- 3468t^{11} - 808t^{12} - 84t^{13}.\end{aligned}$$

In particular, let us consider the variation in sign of ρ_i, $i = 1, 2$. It is easy to show that ρ_i, $i = 1, 2$, has at most two positive roots. However, by direct calculation, we obtain exactly two positive roots $\beta_j^{(1)}$ of $\rho_1(t)$, $j = 1, 2 : \beta_1^{(1)} \cong 0.0401248$, $\beta_2^{(1)} \cong 2.08778$, $\beta_1^{(2)} \cong 0.202432$, and $\beta_2^{(2)} \cong 0.989119$, respectively. Note that for each $t > 0$, we know that $\gamma(t) > 0$ and $\rho_i(t) < 0$, $i = 3, \cdots, 9$. Hence we can easily deduce that in all cases, the variation in sign of $P(h,t)$ is 1. We have thus proved the following result.

Lemma 4.12 *Given $t > 0$, the equation*

$$\frac{d^2k}{dh^2}(h,t) = 0 \tag{4.28}$$

has exactly one positive root.

Now recall that the curvature κ of a curve C is the rate of change of the direction angle with respect to arclength; that is, if C is of the form $k = f(h)$, then

$$\kappa(h) := \frac{f''(h)}{\{1 + [f'(h)]^2\}^{3/2}}. \tag{4.29}$$

Similarly, for a curve of the form $h = g(k)$,

$$\kappa(k) := \frac{f''(k)}{\{1 + [g'(k)]^2\}^{3/2}}. \tag{4.30}$$

Now let us return to our curve $C \equiv \{(h,t) : \widetilde{r}_1(h,t) = 0\}$. By Lemma 4.9, $\widetilde{r}_1(h,t) = 0$ consists of two arcs of the form $k = f(h)$ by intersecting the curve with a ray of the form $t = t_0$. We thus have the following result.

Lemma 4.13 *For a sufficiently small value of t with $0 < t < t_0$ and $h > 0$ such that $\widetilde{r}_1(h,t) = 0$, we have $\kappa(h,t) > 0$.*

Proof. Since the function $P(h,t)$ in (4.27) is positive for small values of t and h on C, an application of (4.29) completes the proof. ■

By Lemma 4.10, the curve $\widetilde{r}_1(h,t) = 0$ also consists of two arcs of the form $h = f(k)$ by intersection with a ray $t = t_1$. Using the above method we obtain similarly the following result.

Lemma 4.14 *For a sufficiently large value of t with $t > t_1$ and $h > 0$ such that $\widetilde{r}_1(h,t) = 0$, we have $\kappa(h,t) < 0$.*

Now we are ready to prove Theorem 4.11.

Proof of Theorem 4.11. (It follows from Lemma 4.12 that the number of zeros of the curvature $\kappa(h,t)$ of C is one with respect to h, and by Lemma 4.13, $\kappa(h,t)$ is positive for sufficiently small h such that $\widetilde{r}_1(h,t) = 0$. Let $\mathcal{U}_1 = \mathcal{V} \cup \mathcal{W}$, where

$$\mathcal{V} := \{(h,th) | \widetilde{r}_1(h,t) \geq 0, 0 < t \leq t_0\} \tag{4.31}$$

and

$$\mathcal{W} := \{(h,th) | \widetilde{r}_1(h,t) \geq 0, t_0 < t\}. \tag{4.32}$$

Suppose that $\mathcal{V}$ is not convex. By the definition of convexity, it is easy to show that $\kappa(h,t) < 0$ for some $t < t_0$. Since the curve is a closed curve, geometric considerations readily imply that the equation $\kappa(h,t) = 0$ must have at least two positive roots with respect to h, which gives a contradiction. Thus, $\mathcal{V}$ must be convex. Using Lemmas 4.10 and 4.14, we can prove similarly that $\mathcal{W}$ is convex. The definition of t_0 and an analysis of Figure 1 below now establish that $\mathcal{U}_1$ is convex. ■

4.3 The Shape of $\mathcal{U}_2$

In this section, we discuss the shape of $\mathcal{U}_2$ and prove Lemma 4.4. A simple computation establishes the following result.

Lemma 4.15 *For $t > 0$, let $k = th$. Then*

$$f_2(h, th) = \tilde{r}_2(h, t)h^5, \tag{4.33}$$

where

$$\tilde{r}_2(h, t) := \phi_{22}(t) - \sum_{i=1}^{4} \tau_i(t)h^i$$

with

$$\begin{aligned}
&\phi_{22}(t) := t^2(4 + 2t),\\
&\tau_1(t) := 1 + 3t + t^2 + 3t^4 + 3t^5 + t^6,\\
&\tau_2(t) := 3 + 12t + 21t^2 + 24t^3 + 21t^4 + 12t^5 + 3t^6,\\
&\tau_3(t) := 3 + 15t + 33t^2 + 42t^3 + 33t^4 + 15t^5 + 3t^6,\\
&\tau_4(t) := 1 + 6t + 15t^2 + 20t^3 + 15t^4 + 6t^5 + t^6.
\end{aligned}$$

As a matter of fact, if we follow the method in the proofs of Lemmas 4.6 and 4.7, we show easily that the boundary of $\mathcal{U}_2$ is a loop with polar form $r = g(\theta)$ whose tangent lines near $(0, 0)$ converge to the x- and y-axes. Also we may draw the graph of $f_2(h, k) = 0$ as in Figure 1 by using a table like Table 1.

From calculus, let us recall Taylor's formula with derivative remainder. Suppose that $f(x)$ has continuous derivatives of orders $1, 2, ..., n$ when $\alpha \le x \le \beta$, and a derivative of order $n + 1$ when $\alpha < x < \beta$. Then, if a and x are any two numbers such that either $\alpha \le a < x \le \beta$ or $\alpha \le x < a \le \beta$, then Taylor's formula is given by

$$f(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1}, \tag{4.34}$$

where ξ is some number between a and x. In particular, if we put $x = a + h$, (4.34) can be expressed as

$$f(a + h) = f(a) + hf'(a) + \cdots + h^n\frac{f^{(n)}(a)}{n!} + \frac{h^{n+1}}{(n + 1)!}f^{(n+1)}(a + \theta h),\ 0 < \theta < 1. \tag{4.35}$$

We now prove Lemma 4.4.

Proof of Lemma 4.4. To compare $\tilde{r}_1(h, t)$ and $\tilde{r}_2(h, t)$, we define

$$R(h, t) := \tilde{r}_2(h, t) - \tilde{r}_1(h, t), \tag{4.36}$$

so that we have

$$R(h, t) = -\omega(t) + \sum_{i=1}^{5} \omega_i(t)h^i, \tag{4.37}$$

where

$$\begin{aligned}
\omega(t) &:= t^4 + 2t^3, \\
\omega_1(t) &:= 1 + 4t + 4t^2 + t^3 + 3t^4 + 6t^5 + 4t^6 + t^7, \\
\omega_2(t) &:= 4 + 18t + 32t^2 + 33t^3 + 29t^4 + 24t^5 + 13t^6 + 3t^7, \\
\omega_3(t) &:= 6 + 30t + 63t^2 + 75t^3 + 60t^4 + 36t^5 + 15t^6 + 3t^7, \\
\omega_4(t) &:= 4 + 22t + 51t^2 + 65t^3 + 50t^4 + 24t^5 + 7t^6 + t^7, \\
\omega_5(t) &:= 1 + 6t + 15t^2 + 20t^3 + 15t^4 + 6t^5 + t^6.
\end{aligned}$$

If we follow the method used to prove Lemmas 4.6 and 4.7, we see that $R(h,t) = 0$ is a loop with polar form $r = r(\theta),\ 0 \le \theta \le \frac{\pi}{2}$. For each fixed t_0, there exists $h_0 > 0$ such that

$$\begin{aligned}
R(h_0, t_0) &= 0, \\
R(h, t_0) &< 0 \qquad (0 < h < h_0), \\
R(h, t_0) &> 0 \qquad (h > h_0).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\tilde{r}_1(h_0, t_0) &= \tilde{r}_2(h_0, t_0), \\
\tilde{r}_1(h, t_0) &> \tilde{r}_2(h, t_0) \qquad (0 < h < h_0), \\
\tilde{r}_1(h, t_0) &< \tilde{r}_2(h, t_0) \qquad (h > h_0).
\end{aligned}$$

For a given t, let h_1 and h_2 denote the unique values such that $\tilde{r}_1(h_1, t) = \tilde{r}_2(h_2, t) = 0$; we must show that $h_1 \le h_2$ (see Figure 1 below).

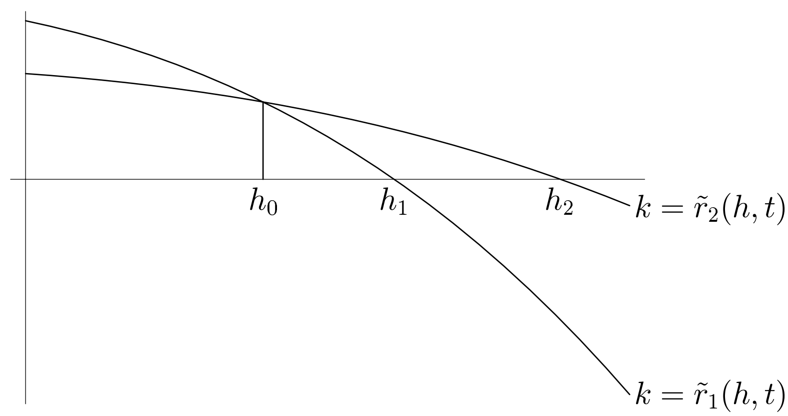

Figure 1: h_1 and h_2 such that $\tilde{r}_1(h_1, t) = \tilde{r}_2(h_2, t) = 0$.

First, observe that $\mathcal{U}_1 \cup \mathcal{U}_2 \subseteq [0, \frac{3}{10}) \times R^+$. Moreover,

$$R(\frac{1}{10}, t) = \frac{1}{100000}(14641 + 61226t + ... + 13310t^7) > 0, \text{ for any } t > 0,$$

so we have $R(h,t) > 0$ for $\frac{1}{10} \le h < \frac{3}{10}$. Thus if we let t_{11} and t_{12} denote the values satisfying $\tilde{r}_2(\frac{1}{10}, t_{1i}) = 0$ $(i = 1, 2)$, with $t_{11} < t_{12}$, then for any $t_{11} \le t \le t_{12}$, we have $h_1 < h_2$; notice that $t_{11} \cong 0.243479$ and $t_{12} \cong 1.889552$ (see Figure 2 below).

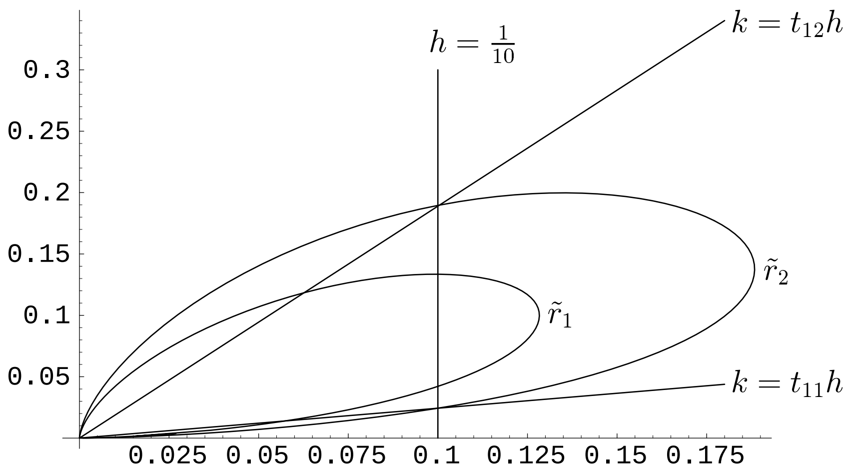

Figure 2: Geometric interpretation of the second part of the proof of Lemma 4.4.

If we now consider a smaller value of h, say $h = \frac{1}{20}$, we have

$$R\left(\frac{1}{20}, t\right) = \frac{194481 + 796446t + ... + 185220t^7}{3200000} > 0 \qquad (0 < t < 0.49286, t > 1.75845).$$

We thus have $R(h,t) > 0$ for $\frac{1}{20} \le h < \frac{1}{10}$ and $0 < t < 0.492$ (or $t > 1.758$). If we denote by t_{21} and t_{22} the values satisfying $\tilde{r}_2(\frac{1}{20}, t_{2i}) = 0$ $(i = 1, 2)$, with $t_{21} < t_{22}$, then for any $t_{11} \le t \le t_{21}$, $t_{12} \le t \le t_{22}$ we have $h_1 < h_2$ (observe that $t_{21} \cong 0.14173$ and $t_{22} \cong 2.8077$). Continuing this process, we can obtain a collection $\{t_{mi}\}_{\substack{m=1,\dots,M\\ i=1,2}}$ such that $t_{M1} < \cdots < t_{11}$, $t_{12} < \cdots < t_{M2}$, t_{M1} smaller than any prescribed positive number (provided M is large enough), and t_{M2} bigger than any prescribed positive number (again provided that M is large enough). It follows that $h_1 < h_2$ for $t \in [t_{M1}, t_{M2}]$.

It remains to prove that the inequality $h_1 \le h_2$ also holds near the origin and near infinity. Since the variation in sign of $\widetilde{r}_2(h,t)$ with respect to h is 1 (see Lemma 4.15), $\widetilde{r}_2(\cdot, t)$ has exactly one positive root for each t. Moreover, $\widetilde{r}_2(h,t)$ is a polynomial of degree 4 with respect to h, so by direct computation we may obtain the positive root function $p(t)$ for $t \ge 0$. (p has a rather complicated expression, so we omit it here.) We claim that $P(t) := \widetilde{r}_1(p(t), t) < 0$ for all values of t in a small neighborhood Ω of the origin. (The size of the neighborhood will then determine the number M of needed iterations of the previous argument, as t_{M1} must belong to Ω.) To do so, we consider Taylor's formula for $P(t)$. Notice that

$$\frac{d\widetilde{r}_1}{dt} = \frac{\partial \widetilde{r}_1}{\partial t} + \frac{\partial \widetilde{r}_1}{\partial h} \cdot \frac{dh}{dt} \tag{4.38}$$

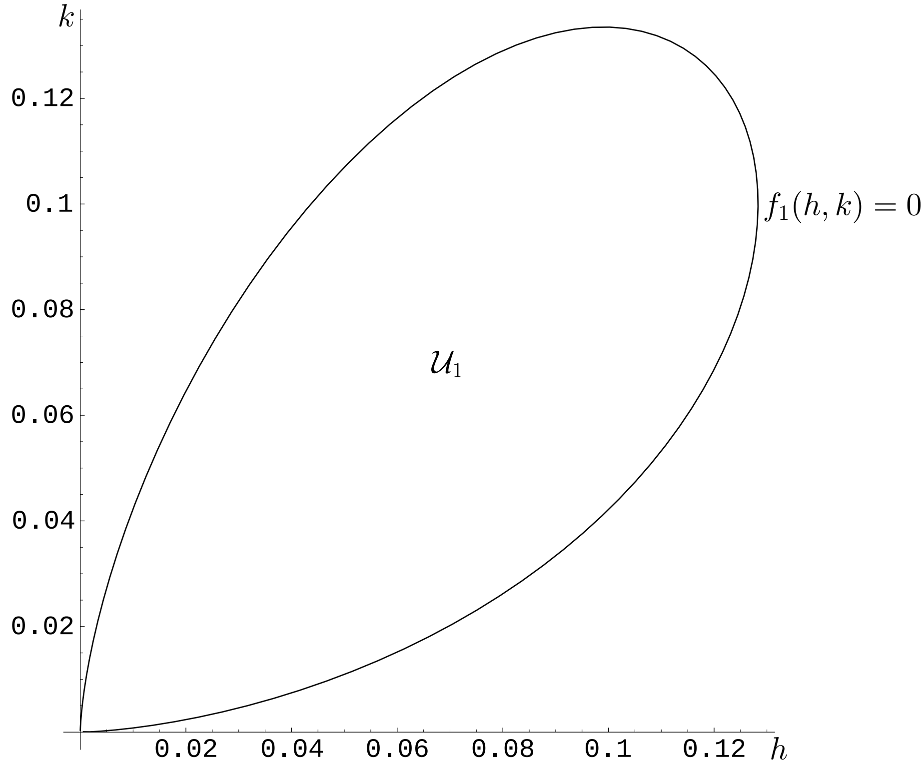

Figure 3: For $\alpha : 1, (1, \sqrt{1+h}, \sqrt{1+h+k})^\wedge$, W_α is positively quadratically hyponormal $\Longleftrightarrow$ $(h,k) \in \mathcal{U}_1$.

and

$$\begin{aligned}
\frac{d}{dt}\left(\frac{d\widetilde{r}_1}{dt}\right) &= \frac{d}{dt}\left(\frac{\partial \widetilde{r}_1}{\partial t}\right) + \frac{d}{dt}\left(\frac{\partial \widetilde{r}_1}{\partial h}\cdot\frac{dh}{dt}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial \widetilde{r}_1}{\partial t}\right) + \frac{\partial}{\partial h}\left(\frac{\partial \widetilde{r}_1}{\partial t}\right)\cdot\frac{dh}{dt} + \frac{d}{dt}\left(\frac{\partial \widetilde{r}_1}{\partial h}\right)\cdot\frac{dh}{dt} + \frac{\partial \widetilde{r}_1}{\partial h}\cdot\frac{d}{dt}\left(\frac{dh}{dt}\right) \\
&= \frac{\partial^2 \widetilde{r}_1}{\partial t^2} + \frac{\partial^2 \widetilde{r}_1}{\partial h \partial t}\cdot\frac{dh}{dt} + \left[\frac{\partial}{\partial t}\left(\frac{\partial \widetilde{r}_1}{\partial h}\right) + \frac{\partial}{\partial h}\left(\frac{\partial \widetilde{r}_1}{\partial h}\right)\cdot\frac{dh}{dt}\right]\cdot\left(\frac{dh}{dt}\right) \\
&\quad + \frac{\partial \widetilde{r}_1}{\partial h}\cdot\frac{d^2 h}{dt^2} \\
&= \frac{\partial^2 \widetilde{r}_1}{\partial t^2} + 2\frac{\partial^2 \widetilde{r}_1}{\partial h \partial t}\cdot\frac{dh}{dt} + \frac{\partial^2 \widetilde{r}_1}{\partial h^2}\cdot\left(\frac{dh}{dt}\right)^2 + \frac{\partial \widetilde{r}_1}{\partial h}\cdot\frac{d^2 h}{dt^2}.
\end{aligned} \tag{4.39}$$

By direct computation, $\lim_{t\to 0} p'(t) = 0$, so by (4.38) we have $\lim_{t\to 0^+} P'(t) = 0$. Also by (4.39) it is easy to show that

$$\begin{aligned}
\lim_{t\to 0^+} P''(t) &= \lim_{t\to 0^+}\left(\frac{\partial^2 \widetilde{r}_1}{\partial t^2} + \frac{\partial \widetilde{r}_1}{\partial h}\cdot p''(t)\right) \\
&= 8 - 2\cdot \lim_{t\to 0^+} p''(t).
\end{aligned} \tag{4.40}$$

A calculation shows that

$$\xi_0 := \lim_{t\to 0^+} p''(t) \approx 8 > 5, \tag{4.41}$$

N	h	k	N	h	k	N	h	k
1	0.000474	0.000074	34	0.121582	0.071903	67	0.070793	0.124114
2	0.001819	0.000057	35	0.123134	0.075456	68	0.066764	0.121444
3	0.003924	0.000185	36	0.124484	0.079000	69	0.062715	0.118449
4	0.006679	0.000420	37	0.125631	0.082524	70	0.058666	0.115138
5	0.009982	0.000785	38	0.126576	0.086021	71	0.054635	0.111525
6	0.013740	0.001298	39	0.127317	0.089480	72	0.050645	0.107626
7	0.017866	0.001972	40	0.127855	0.092892	73	0.046714	0.103461
8	0.022281	0.002814	41	0.128189	0.096247	74	0.042864	0.099054
9	0.026916	0.003830	42	0.128320	0.099535	75	0.039115	0.094431
10	0.031710	0.005022	43	0.128247	0.102745	76	0.035483	0.089621
11	0.036606	0.006388	44	0.127970	0.105866	77	0.031988	0.084654
12	0.041560	0.007928	45	0.127490	0.108886	78	0.028644	0.079562
13	0.046531	0.009636	46	0.126806	0.111795	79	0.025466	0.074380
14	0.051483	0.011507	47	0.125921	0.114579	80	0.022465	0.069142
15	0.056386	0.013537	48	0.124834	0.117227	81	0.019653	0.063883
16	0.061215	0.015717	49	0.123547	0.119725	82	0.017036	0.058638
17	0.065948	0.018041	50	0.122060	0.122060	83	0.014620	0.053441
18	0.070567	0.020501	51	0.120377	0.124220	84	0.012408	0.048326
19	0.075057	0.023090	52	0.118500	0.126190	85	0.010401	0.043325
20	0.079404	0.025800	53	0.116431	0.127956	86	0.008598	0.038468
21	0.083598	0.028622	54	0.114173	0.129504	87	0.006996	0.033786
22	0.087630	0.031549	55	0.111731	0.130820	88	0.005590	0.029307
23	0.091492	0.034572	56	0.109110	0.131891	89	0.004373	0.025057
24	0.095179	0.037684	57	0.106315	0.132703	90	0.003335	0.021061
25	0.098684	0.040876	58	0.103353	0.133242	91	0.002468	0.017344
26	0.102003	0.044141	59	0.100231	0.133496	92	0.001759	0.013928
27	0.105134	0.047470	60	0.096959	0.133453	93	0.001196	0.010836
28	0.108074	0.050855	61	0.093545	0.133101	94	0.000764	0.008088
29	0.110819	0.054290	62	0.090001	0.132433	95	0.000449	0.005705
30	0.113369	0.057764	63	0.086339	0.131438	96	0.000233	0.003708
31	0.115721	0.061271	64	0.082571	0.130112	97	0.000099	0.002119
32	0.117875	0.064802	65	0.078714	0.128450	98	0.000030	0.000956
33	0.119829	0.068349	66	0.074782	0.126450	99	0.000003	0.000242

Table 1: Numerical data for Figure 3

and then

$$\lim_{t\to 0^+} P''(t) = 8 - 2\cdot\xi_0 < -2. \tag{4.42}$$

Thus, for t sufficiently small, $P''(s) \le -1$ provided $0 < s \le 2t$. Moreover, since $\lim_{t\to 0^+} P(t) = 0$, we may redefine P so that $P(0) = 0$. By (4.35) we have

$$P(\eta + t) = P(\eta) + P'(\eta)t + \frac{P''(\eta+\theta(\eta)t)}{2!}t^2 \quad (0 < t < \infty) \tag{4.43}$$

for some number $\theta(\eta)$ with $0 < \theta(\eta) < 1$. Taking $\eta \to 0^+$, we see that

$$P(t) = \limsup_{\eta\to 0^+} \frac{P''(\eta+\theta(\eta)t)}{2!}t^2 \le -\frac{t^2}{2}, \tag{4.44}$$

because $s := \eta + \theta(\eta)t \le \eta + t \le 2t$ as $\eta \to 0^+$. Therefore, $P(t)$ is negative for small values of t, which shows that $h_1 < h_2$ for t in a small neighborhood of the origin. Furthermore, using $t = \frac{1}{z}$, we can prove similarly that $h_1 < h_2$ for values of t in a neighborhood of infinity. Hence we have $h_1 \le h_2$ in all cases, thus establishing that $\mathcal{U}_1 \subseteq \mathcal{U}_2$. ■

4.4 Graph of $f_1(h, k) = 0$ and $f_2(h, k) = 0$

We conclude this section by presenting a drawing of the region $\mathcal{U}_1$ using numerical approximation. First we consider a scale $\theta = \frac{\pi}{200}$ and put

$$T(N) := \tan(\frac{\pi}{200}N). \tag{4.45}$$

Finding the roots h such that

$$\tilde{r}_1(h, T(N)h) = 0 \tag{4.46}$$

for $N = 1, 2, \cdots, 99$, we obtain Table 1 above. The graph of $f_1(h, k) = 0$ obtained from Table 1 is shown in Figure 3. In a similar way we can draw the graph of $f_2(h, k) = 0$.

5 Concluding Remarks and Open Questions

Let $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ with $1 \leq x \leq y$. Since $\mathcal{R} := \{(x, y) | W_\alpha$ is positively quadratically hyponormal$\}$ is a closed set, there exist maximum values x_M and y_M of x and y such that $\mathcal{R} \cap (\{x_M\} \times \mathbf{R})$ and $\mathcal{R} \cap (\mathbf{R} \times \{y_M\})$ are singletons.

Problem 5.1 *Find a concrete expression for x_M and y_M.*

Consider now the case of $\alpha : 1, 1, (\sqrt{x}, \sqrt{y}, \sqrt{z})^\wedge$ with $1 \leq x \leq y \leq z$. From the results in this paper, it is clear that $\mathcal{V} := \{(x, y, z) | W_\alpha$ is positively quadratically hyponormal$\}$ is a solid in 3-dimensional space.

Problem 5.2 *Describe $\mathcal{V}$ and its boundary.*

In a different direction, we have

Problem 5.3 *Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with $1 \leq x \leq a \leq b \leq c$. Find $\mathcal{W} := \{x | W_\alpha$ is quadratically hyponormal$\}$.*

In Section 3 we discussed general facts about quadratically hyponormal weighted shifts W_α with $\alpha_0 = \alpha_1 = 1$. We considered in particular the relationship between a sequence of weights and the determinants of successive principal minors of the canonical infinite matrix induced by the associated shift W_α. We later investigated in detail quadratic hyponormality for shifts of recursive type, with the aid of recursive coefficients associated with the tail of the weight sequence. For shifts whose tail is subnormal of nonrecursive type, however, the Nested Determinants approach is not very effective, and we may need to find another useful machine to study quadratic hyponormality.

Problem 5.4 *Let $\tilde{\alpha} : 1, 1, \sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, ...$, where W_α is a nonrecursive subnormal weighted shift. Characterize quadratic hyponormality for $W_{\tilde{\alpha}}$.*

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