Cartesian form of Putnam's inequality for doubly commuting hyponormal n-tuples

Muneo Chō[*], Raúl E. Curto[†], Tadasi Huruya and Wieslaw Żelazko

Abstract

We obtain a Cartesian form of Putnam's inequality for doubly commuting hyponormal n-tuples, and we establish new relations between the Taylor and Xia spectra of such n-tuples.

Respectfully dedicated to Professor Michiaki Watanabe on his sixtieth birthday

1 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$, or equivalently, if $i[H, K] \geq 0$, where $[H, K] := HK - KH$, and $T \equiv H + iK$ is the Cartesian decomposition of T in real and imaginary parts. If T is hyponormal, Putnam's inequality holds ([10, p. 43, (3.2.3)]):

$$\|T^*T - TT^*\| \ \leq \ \frac{2}{\pi}\|K\| \cdot m(\sigma(H)),$$

where $\sigma(H)$ is the spectrum of H and m is Lebesgue measure on the real line. This inequality has been extended in several ways. In the case of a single operator, it was generalized to p-hyponormal operators in [4] and to log-hyponormal operators in [12], while a trace estimate for p-hyponormal operators was obtained in [8]. For several operators, D. Xia introduced and studied hyponormal and semi-hyponormal n-tuples, and obtained preliminary versions of Putnam's inequality (cf. [16] and [17]). M. Chō and T. Huruya extended these results to p-hyponormal tuples ([5]), and recently Duggal did so to the class of doubly commuting n-tuples of p-hyponormal operators

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[9]. Finally, a version of Putnam's inequality for doubly commuting n-tuples of log-hyponormal operators has been obtained in [6].

All of the above mentioned extensions have been formulated in terms of the polar decomposition $T_j = U_j|T_j|$ of the operators involved, and the standard assumption has been that U_j is unitary ($j = 1, ..., n$). In this paper, we obtain a Cartesian form of Putnam's inequality for doubly commuting n-tuples of hyponormal operators (i.e., $T_jT_k = T_kT_j$ and $T_jT_k^* = T_k^*T_j$ for all $j \neq k$).

Let $\mathbf{T} \equiv (T_1, ..., T_n) \equiv (H_1 + iK_1, ..., H_n + iK_n)$ be a doubly commuting n-tuple of hyponormal operators. Then

$$\| \prod_{j=1}^{n}(T_j^*T_j - T_jT_j^*)\| = 2^n \| \prod_{j=1}^{n}[H_j, K_j]\| \leq (\frac{2}{\pi})^n \|K_1\| \cdots \|K_n\| \cdot m(\sigma_{ja}(H_1, ..., H_n)), \quad (1)$$

where $\sigma_{ja}(H_1, ..., H_n)$ is the joint approximate point spectrum of $\mathbf{H} := (H_1, ..., H_n)$, and m is Lebesgue measure on $\mathbf{R}^n$.

The inequality in (1) is sharp. For, let T be a hyponormal operator such that $\|T^*T - TT^*\| = \frac{2}{\pi}\|K\| \cdot m(\sigma(H))$ (cf. [10, p. 137]). Then $T_1 := T \otimes I \otimes ... \otimes I, ..., T_n := I \otimes I \otimes ... \otimes T$ satisfies

$$\begin{aligned} \left\| \prod_{j=1}^{n} T_j^*T_j - T_jT_j^* \right\| &= \|T^*T - TT^*\|^n = (\frac{2}{\pi}\|K\| \cdot m(\sigma(H)))^n \\ &= (\frac{2}{\pi})^n \|K\|^n m(\sigma_{ja}(H \bigotimes I \bigotimes ... \bigotimes I, ..., I \bigotimes I \bigotimes ... \bigotimes H) \\ &= (\frac{2}{\pi})^n \|K\|^n m(\sigma_{ja}(H_1, ..., H_n). \end{aligned}$$

Also, observe that the assumption about doubly commutativity is essential for our purposes, because we shall need that both $\mathbf{H} := (H_1, ..., H_n)$ and $\mathbf{K} := (K_1, ..., K_n)$ be commuting, and that $H_jK_k = K_kH_j$ whenever $j \neq k$. Since the joint approximate point spectrum of a commuting n-tuple of hermitian operators coincides with any other joint spectrum σ^*([18], [19, Theorem 5]), we can replace σ_{ja} in (1) by, e.g., the Taylor spectrum σ_T or the Harte spectrum σ_H.

2 Joint Hyponormality

Let $\mathbf{H} \equiv (H_1, ..., H_n)$ be a commuting n-tuple of hermitian operators, and put

$$\mathbf{D}_j R := i[H_j, R], \quad R \in \mathcal{B}(\mathcal{H}), \; j = 1, 2, ..., n.$$

It can be easily verified that $\mathbf{D}_j\mathbf{D}_k = \mathbf{D}_k\mathbf{D}_j,\ 1 \le j,k \le n$. Let Y be a hermitian operator. The $(n+1)$-tuple $(\mathbf{H}, Y) \equiv (H_1, ..., H_n, Y)$ is said to be hyponormal if

$$\mathbf{D}_{j_1} \cdots \mathbf{D}_{j_m} Y \ \ge\ 0$$

for all $1 \le j_1 < \cdots < j_m \le n$. If $(\mathbf{H}, Y)$ is hyponormal, then $H_j + iY$ is also hyponormal, $1 \le j \le n$.

We say that an operator $R \in \mathcal{B}(\mathcal{H})$ has symbols $\mathcal{S}_j^+(R)$ and $\mathcal{S}_j^-(R)$ with respect to H_j if for each x in $\mathcal{H}$ the limits

$$\mathcal{S}_j^+(R)x := \lim_{t\to\infty} e^{itH_j} R e^{-itH_j} x \ \textit{and}\ \mathcal{S}_j^-(R)x := \lim_{t\to\infty} e^{-itH_j} R e^{itH_j} x$$

exist (cf. [16]). Since such limits exist also when we replace the continuous real variable t by any sequence $t_n \to \infty$, and since strong limits of sequences of bounded operators are bounded, the symbols $\mathcal{S}_j^\pm(R)$ belong to $\mathcal{B}(\mathcal{H})$ when they exist. If $(\mathbf{H}, Y)$ is hyponormal, then all symbols $\mathcal{S}_j^\pm(R), j = 1, ..., n$, exist. For $\mathbf{k} \in [0,1]^n$ we now let

$$Y_{\mathbf{k}} := [\prod_{j=1}^{n} (k_j \mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)]Y,\ \mathbf{k} = (k_1, ..., k_n), 0 \le k_j \le 1 \tag{2}$$

([16],[17]); the $Y_{\mathbf{k}}$'s are the general symbols of Y with respect to $\mathbf{H}$.

Let $\mathbf{H} \equiv (H_1, ..., H_n)$ be a commuting n-tuple of hermitian operators, let $\mathbf{k} \equiv (k_1, ..., k_n) \in [0,1]^n$, and let R be an operator in $\mathcal{B}(\mathcal{H})$ for which

$$R_{\mathbf{k}} := [\prod_{j=1}^{n} (k_j \mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)]R$$

exists. Then $H_j R_{\mathbf{k}} = R_{\mathbf{k}} H_j$ for $j = 1, ..., n$.

We can write $R_{\mathbf{k}}$ as a linear combination of operators of the form $\tilde{R}_m$, where $\tilde{R}_0 = R$, $\tilde{R}_m := \mathcal{S}_m^{a_m}\mathcal{S}_{m-1}^{a_{m-1}} \cdots \mathcal{S}_1^{a_1} R$ $(n \ge m \ge 1)$, and $a_m, a_{m-1}, ..., a_1 \in \{+,-\}$. Therefore, it suffices to show that $H_j\tilde{R}_n = \tilde{R}_n H_j$ for $j = 1, 2, ..., n$. This follows immediately from the following three identities:

$$\begin{aligned} \mathcal{S}_m^{a_m}(H_jR) &= H_j\mathcal{S}_m^{a_m}(R), \\ \mathcal{S}_m^{a_m}(RH_j) &= \mathcal{S}_m^{a_m}(R)H_j\ (m \ne j), \textit{and} \\ H_j\mathcal{S}_j^{a_j}(R) &= \mathcal{S}_j^{a_j}(R)H_j. \end{aligned} \tag{3}$$

For,

$$\begin{aligned} H_j\tilde{R}_n &= H_j\mathcal{S}_n^{a_n} \cdots \mathcal{S}_{j+1}^{a_{j+1}}\mathcal{S}_j^{a_j}(\tilde{R}_{j-1}) \\ &= \mathcal{S}_n^{a_n} \cdots \mathcal{S}_{j+1}^{a_{j+1}} H_j \mathcal{S}_j^{a_j}(\tilde{R}_{j-1}) \\ &= \mathcal{S}_n^{a_n} \cdots \mathcal{S}_{j+1}^{a_{j+1}} [\mathcal{S}_j^{a_j}(\tilde{R}_{j-1})H_j] \\ &= \tilde{R}_n H_j. \end{aligned}$$

Now, the first two identities in (3) follow from the definition of $\mathcal{S}_m^{\pm}$ and from the fact that H_j and H_m commute for $j \neq m$. For the reader's convenience we reproduce here the proof of the third identity, given in [16]. For simplicity, we focus on the case when $a_j = +$ (the case when $a_j = -$ being entirely similar). Fix a real number τ and write

$$\begin{aligned} & \mathcal{S}_j^+(R)x \\ = & \lim_{t\to\infty} e^{i(t+\tau)H_j} R e^{-i(t+\tau)H_j} x \\ = & e^{i\tau H_j}(\lim_{t\to\infty} e^{itH_j} R e^{-itH_j}) e^{-i\tau H_j} x \\ = & e^{i\tau H_j} \mathcal{S}_j^+(R) e^{-i\tau H_j} x. \end{aligned}$$

Thus $\mathcal{S}_j^+(R) = e^{i\tau H_j}\mathcal{S}_j^+(R)e^{-i\tau H_j}$, or

$$\mathcal{S}_j^+(R)e^{i\tau H_j} = e^{i\tau H_j}\mathcal{S}_j^+(R)(all \tau \in \mathbf{R}).$$

If we now pass to Taylor expansions we obtain

$$\sum_{n=0}^{\infty} \mathcal{S}_j^+(R)\frac{H_j^n(i\tau)^n}{n!} = \sum_{n=0}^{\infty} \frac{H_j^n(i\tau)^n}{n!}\mathcal{S}_j^+(R). \tag{4}$$

Comparing the coefficients of τ on both sides of (4), we obtain $\mathcal{S}_j^+(R)H_j = H_j\mathcal{S}_j^+(R)$. The conclusion now follows.

The following concept (without its name) was used by D. Xia in [16].

Let $\mathbf{H} \equiv (H_1, ..., H_n)$ be a commuting n-tuple of hermitian operators, and let Y also be a hermitian operator. The Xia spectrum of $(\mathbf{H}, Y)$ is

$$\sigma_X(\mathbf{H}, Y) := \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(H_1, ..., H_n, Y_{\mathbf{k}}), \tag{5}$$

where $Y_{\mathbf{k}}$ is given by (2).

In Definition 2, Y need not commute with H_j; however, if $(H_1, ..., H_n, Y)$ is hyponormal, then Lemma 2 ensures that $(H_1, ..., H_n, Y_{\mathbf{k}})$ is commuting for all $\mathbf{k} \in [\mathbf{0}, \mathbf{1}]^n$, so in particular $\sigma_X(\mathbf{H}, Y)$ is nonempty.

Our proof of Theorem 1 will make use of the following result due to Xia ([16], Theorem 10).

Theorem A. Let $\mathbf{H} \equiv (H_1, ..., H_n)$ be a commuting n-tuple of hermitian operators, and let Y be hermitian. If $(\mathbf{H}, Y)$ is hyponormal, then

$$\|\mathbf{D}_1 \cdots \mathbf{D}_n Y\| \leq \frac{1}{(2\pi)^n} \cdot m(\sigma_X(\mathbf{H}, Y)),$$

where $m(\cdot)$ denotes Lebesgue measure in $\mathbf{R}^{n+1}$.

3 Proof of Theorem 1

We start with the following result.

Let $\mathbf{T} \equiv \mathbf{H} + i\mathbf{K} \equiv (H_1 + iK_1, ..., H_n + iK_n)$ be a doubly commuting n-tuple of hyponormal operators, and let $Y := K_1 \cdot ... \cdot K_n$. If $K_j \;\geq\; 0$ for all j $(j = 1, ..., n)$, then $(\mathbf{H}, Y)$ is hyponormal.

Since

$$\mathbf{D}_j Y = i(H_j Y - Y H_j) = i(H_j K_j - K_j H_j)(\prod_{i \neq j} K_i) \; (j = 1, ..., n),$$

it follows that

$$\mathbf{D}_{j_1} \cdots \mathbf{D}_{j_m} Y = i[H_{j_1}, K_{j_1}] \cdots i[H_{j_m}, K_{j_m}](\prod_{i \neq j_1, ..., j_m} K_i) \;\geq\; 0$$

for every $1 \leq j_1 < \cdots < j_m \leq n$, because $(i[H_{j_1}, K_{j_1}], ..., i[H_{j_m}, K_{j_m}], \prod_{i \neq j_1,...,j_m} K_i)$ is a commuting $(m+1)$-tuple of positive operators. Hence $(\mathbf{H}, Y)$ is hyponormal.

Proof of Theorem 1. Let $a_j := \|K_j\|$ $(j = 1, ..., n)$; then $K_j + a_j \geq 0$ (all $j = 1, ..., n$). Also, let $Y(a) := (K_1 + a_1) \cdots (K_n + a_n)$. By Lemma 3, $(H_1, ..., H_n, Y(a))$ is hyponormal. Since

$$\mathbf{D}_1 \cdots \mathbf{D}_n Y(a) = \prod_{j=1}^{n} i[H_j, K_j] = \frac{1}{2^n} \prod_{j=1}^{n} (T_j^* T_j - T_j T_j^*),$$

Theorem A implies that

$$\| \prod_{j=1}^{n} (T_j^* T_j - T_j T_j^*) \| = 2^n \, \|\mathbf{D}_1 \cdots \mathbf{D}_n Y(a)\| \leq \frac{1}{\pi^n} m(\sigma_X(\mathbf{H}, Y(a))). \tag{6}$$

On the other hand, the projection property for σ_{ja} ([2], [11]) implies that

$$\sigma_{ja}(\mathbf{H}, Y(a)_{\mathbf{k}}) \;\subseteq\; \sigma_{ja}(\mathbf{H}) \;\times\; \sigma(Y(a)_{\mathbf{k}}), \tag{7}$$

for every $\mathbf{k} \in [0, 1]^n$. Let $\mathbf{k} = (k_1, ..., k_n) \in [0, 1]^n$ and put

$$B_j := k_j \mathcal{S}_j^+(K_j) + (1 - k_j)\mathcal{S}_j^-(K_j) \;\; (j = 1, ..., n).$$

Then

$$\|B_j\| \leq k_j \|\mathcal{S}_j^+(K_j)\| + (1 - k_j)\|\mathcal{S}_j^-(K_j)\| \leq k_j \|K_j\| + (1 - k_j)\|K_j\| = \|K_j\|.$$

Since $(T_1, ..., T_n) \equiv (H_1 + iK_1, ..., H_n + iK_n)$ is a doubly commuting n-tuple of hyponormal operators, we have

$$
\begin{aligned}
Y(a)_{\mathbf{k}} &= [\prod_{j=1}^{n}(k_k \mathcal{S}_j^{+} + (1-k_j)\mathcal{S}_j^{-})]Y(a) \\
&= [\prod_{j\neq i}(k_j \mathcal{S}_j^{+} + (1-k_j)\mathcal{S}_j^{-})]((K_1 + a_1)\cdot ... \cdot \{k_i \mathcal{S}_i^{+}(K_i) \\
&\quad +(1-k_i)\mathcal{S}_i^{-}(K_i) + a_i\} \cdot ... \cdot (K_n + a_n)) \\
&= (B_1 + a_1)\cdots(B_n + a_n).
\end{aligned}
$$

Hence

$$\|Y(a)_{\mathbf{k}}\| \leq \|(B_1 + a_1)\| \cdots \|(B_n + a_n)\| \leq 2^n \|K_1\| \cdots \|K_n\|.$$

Since $Y(a)_{\mathbf{k}}$ is positive, we know that $\sigma(Y(a)_{\mathbf{k}}) \subseteq [0, 2^n \|K_1\| \cdots \|K_n\|]$. It follows that

$$m(\bigcup_{\mathbf{k}\in[0,1]^n} \sigma(Y(a)_{\mathbf{k}})) \leq 2^n \|K_1\| \cdots \|K_n\|. \tag{8}$$

By combining (6), (5), (7) and (8) we see that

$$
\begin{aligned}
\|\prod_{j=1}^{n}(T_j^* T_j - T_j T_j^*)\| &\leq \frac{1}{\pi^n} m(\sigma_X(\mathbf{H}, Y(a))) \leq \frac{1}{\pi^n} m(\bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{H}, Y_{\mathbf{k}})) \\
&\leq \frac{1}{\pi^n} m(\sigma_{ja}(\mathbf{H})) m(\bigcup_{\mathbf{k}\in[0,1]^n} \sigma(Y(a)_{\mathbf{k}})) \\
&\leq (\frac{2}{\pi})^n \cdot m(\sigma_{ja}(\mathbf{H})) \|K_1\| \cdots \|K_n\|,
\end{aligned}
$$

as desired. The proof is now complete.

4 Relations between the Taylor Spectrum and the Xia Spectrum

Let $\mathbf{T} \equiv \mathbf{H} + i\mathbf{K}$ be a doubly commuting n-tuple of hyponormal operators. The Xia spectrum of $\mathbf{T}$ is

$$\sigma_X(\mathbf{T}) := \sigma_X(\mathbf{H}, Y),$$

where $Y := K_1 \cdots K_n$.

Observe that unlike the Taylor spectrum $\sigma_T(\mathbf{T})$ ([7], [13], [14]), the Xia spectrum of $\mathbf{T}$ is a compact nonempty subset of $\mathbf{R}^{n+1}$. In this section we prove the following result.

Let $\mathbf{T} \equiv (T_1, ..., T_n)$ be a doubly commuting n-tuple of hyponormal operators, and consider the map $P : \mathbf{C}^n \to \mathbf{R}^{n+1}$ given by $P(a_1+ib_1, ..., a_n+ib_n) := (a_1, ..., a_n, b_1 \cdot ... \cdot b_n)$. Then

$$P(\sigma_T(\mathbf{T})) = \sigma_X(\mathbf{T}).$$

We have already discussed the role of doubly commutativity in Theorems 1 and 4 (see last paragraph of the Introduction). Moreover, a careful analysis of the proof of Lemma 5 below exhibits the significance of hyponormality in the statement of Theorem 4.

We first consider the single operator case. Let $T \equiv H + iK$ be hyponormal. In this case $\sigma_X(T) = \bigcup_{k \in [0,1]} \sigma_{ja}(H, K_k)$, where $K_k := k\mathcal{S}_H^+(K) + (1-k)\mathcal{S}_H^-(K)$ $(k \in [0,1])$. As shown in [17, Chapter 2, Theorem 2.6], K_k is self-adjoint and $HK_k = K_kH$ for all $k \in [0,1]$.

Let $T \equiv H + iK$ be a hyponormal operator. Then

$$a + ib \in \sigma(T) \Leftrightarrow (a, b) \in \sigma_X(T).$$

For $k \in [0,1]$ let $T_k := k\mathcal{S}_H^+(T) + (1-k)\mathcal{S}_H^-(T)$. Clearly $T_k = H + iK_k$, and by [17, Chapter 4, Theorem 3.1] we have $\sigma(T) = \bigcup_{k \in [0,1]} \sigma(T_k)$. Let $f(x, y) := x + iy$ and let $k \in [0,1]$. Since (H, K_k) is a commuting pair of hermitian operators and $T_k = H + iK_k = f(H, K_k)$ (observe that T_k is indeed normal), the spectral mapping theorem for σ_T, when applied to f, yields

$$f(\sigma_{ja}(H, K_k)) = f(\sigma_T(H, K_k)) = \sigma(f(H, K_k)) = \sigma(T_k).$$

Then

$$\begin{aligned} \sigma(T) &= \bigcup_{k \in [0,1]} \sigma(T_k) = \bigcup_{k \in [0,1]} f(\sigma_{ja}(H, K_k)) \\ &= f[\bigcup_{k \in [0,1]} \sigma_{ja}(H, K_k)] = f[\sigma_X(H, K)] = f[\sigma_X(T)]. \end{aligned}$$

We shall also need the Berberian Extension Theorem ([1], Theorem 1).

Theorem B. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. There exist an extension space $\mathcal{K}$ of $\mathcal{H}$ and a faithful *-representation $\pi : B(\mathcal{H}) \to B(\mathcal{K})$, $S \mapsto S^\circ$ such that

$$\sigma_{ja}(S_1, ..., S_n) = \sigma_{ja}(S_1^\circ, ..., S_n^\circ) = \sigma_p(S_1^\circ, ..., S_n^\circ), \tag{9}$$

where σ_p denotes (joint) point spectrum. Moreover, if T is hyponormal, then T° is also hyponormal.

Proof of Theorem 4. *(i) Proof of the inclusion* $\sigma_X(T) \subseteq P(\sigma_T(T))$. Assume that $(a_1, ..., a_n, b) \in \sigma_X(T_1, ..., T_n)$. Using mathematical induction, we shall show that there exist real numbers $b_1, ..., b_n$ such that $(a_1 + ib_1, ..., a_n + ib_n) \in \sigma_T(T_1, ..., T_n)$ and $b = b_1 \cdot ... \cdot b_n$. For $n = 1$ we use Lemma 4. Suppose Theorem 4 holds for $(n-1)$-tuples. Since $(a_1, ..., a_n, b) \in \sigma_X(T_1, ..., T_n)$, there exist $\mathbf{k} = (k_1, ..., k_n) \in [0, 1]^n$ and a sequence $\{x_\ell\}$ of unit vectors such that

$$(H_j - a_j)x_\ell \to 0 \ (j = 1, ..., n) \ \textit{and} \ (Y_{\mathbf{k}} - b)x_\ell \to 0 \ (\ell \to \infty), \tag{10}$$

where as usual $Y := K_1 \cdot ... \cdot K_n$ and $Y_{\mathbf{k}} := [\prod_{j=1}^n (k_j \mathcal{S}_j^+ + (1 - k_j)\mathcal{S}_j^-)]Y$. Let $B_j := k_j \mathcal{S}_j^+(K_j) + (1 - k_j)\mathcal{S}_j^-(K_j)$; B_j is self-adjoint (all $j = 1, ..., n$), $B_j H_m = H_m B_j$ and $B_j B_m = B_m B_j$ (all $j, m = 1, ..., n$). Moreover, $Y_{\mathbf{k}} = B_1 \cdot ... \cdot B_n$. Using Theorem B, let $\mathcal{K}$ be the Berberian extension space of $\mathcal{H}$. Define

$$\mathcal{M} := Ker(H_n^\circ - a_n).$$

By (10) and (9), $\mathcal{M} \neq (0)$. Since $(H_1^\circ, ..., H_n^\circ, B_1^\circ, ..., B_n^\circ)$ is a commuting $2n$-tuple, $\mathcal{M}$ is invariant under $B_1^\circ, ..., B_n^\circ$. Also, since $b \in \sigma(Y_{\mathbf{k}}^\circ|_{\mathcal{M}})$, there exist b_0, b_n and a non-zero vector $x^\circ \in \mathcal{M}$ such that

$$(B_1^\circ \cdot ... \cdot B_{n-1}^\circ - b_0)x^\circ = 0, \quad (B_n^\circ - b_n)x^\circ = 0 \ \ and b = b_0 b_n$$

(by the spectral mapping theorem for σ_{ja} and Theorem B). Put $Y_0 := K_1 \cdot ... \cdot K_{n-1}$ and $\mathbf{k}_0 := (k_1, ..., k_{n-1})$. Then

$$[\prod_{j=1}^{n-1} (k_j \mathcal{S}_j^+ + (1 - k_j)\mathcal{S}_j^-)]Y_0 = B_1 \cdot ... \cdot B_{n-1}.$$

It follows that

$$(a_1, ..., a_{n-1}, b_0) \in \sigma_X(H_1^\circ|_{\mathcal{M}}, ..., H_{n-1}^\circ|_{\mathcal{M}}, Y_0^\circ|_{\mathcal{M}}).$$

By induction, there exist $b_1, ..., b_{n-1}$ such that $b_0 = b_1 \cdots b_{n-1}$ and

$$(a_1 + ib_1, ..., a_{n-1} + ib_{n-1}) \in \sigma_T(T_1^\circ|_{\mathcal{M}}, ..., T_{n-1}^\circ|_{\mathcal{M}}).$$

Since $(T_1^\circ|_{\mathcal{M}}, ..., T_{n-1}^\circ|_{\mathcal{M}})$ is a doubly commuting $(n-1)$-tuple of hyponormal operators on $\mathcal{M}$ and $(a_1 + ib_1, ..., a_{n-1} + ib_{n-1}) \in \sigma_T(T_1^\circ|_{\mathcal{M}}, ..., T_{n-1}^\circ|_{\mathcal{M}})$, the operator

$$\sum_{j=1}^{n-1} (T_j^\circ - (a_j + ib_j))(T_j^\circ - (a_j + ib_j))^* + (H_n^\circ - a_n)^2 + (B_n^\circ - b_n)^2$$

is not invertible on $\mathcal{M}$ and hence not invertible on $\mathcal{K}$ (recall that $\sigma_T = \sigma_r$, the right spectrum, for doubly commuting n-tuples of hyponormal operators [7]). By Theorem B we have

$$(a_1 - ib_1, ..., a_{n-1} - ib_{n-1}, a_n, b_n) \in \sigma_p(T_1^{\circ *}, ..., T_{n-1}^{\circ *}, H_n^{\circ}, B_n^{\circ}).$$

Let $\mathcal{N} := \mathrm{Ker}(H_n^{\circ} - a_n) \bigcap \mathrm{Ker}(B_n^{\circ} - b_n)$. Then there exists a non-zero vector y° in $\mathcal{N}$ such that

$$(T_j^{\circ} - (a_j + ib_j))^* y^{\circ} \ = \ 0 \ (for all j = 1, ..., n-1).$$

Let

$$\mathcal{L} := \bigcap_{j=1}^{n-1} Ker((T_j^{\circ} - (a_j + ib_j))^*).$$

Then $\mathcal{N} \bigcap \mathcal{L}$ is a non-zero subspace of $\mathcal{K}$. Hence we have $(a_n, b_n) \in \sigma_{ja}(H_n^{\circ}|_{\mathcal{L}}, B_n^{\circ}|_{\mathcal{L}})$ and $(a_n, b_n) \in \sigma_X(H_n^{\circ}|_{\mathcal{L}}, K_n^{\circ}|_{\mathcal{L}})$. By induction we have $a_n + ib_n \in \sigma(T_n^{\circ}|_{\mathcal{L}})$. Since $T_n^{\circ}|_{\mathcal{L}}$ is hyponormal on $\mathcal{L}$, the operator

$$\sum_{j=1}^{n} (T_j^{\circ} - (a_j + ib_j))(T_j^{\circ} - (a_j + ib_j))^*$$

is not invertible. Hence, by Theorem B, there exists a non-zero vector $w^{\circ} \in \mathcal{L}$ such that

$$(T_n^{\circ} - (a_n + ib_n))^* w^{\circ} = 0.$$

Therefore, there exists a sequence $\{x_p\}$ of unit vectors in $\mathcal{H}$ such that

$$(T_j - (a_j + ib_j))^* x_p \to 0 \ (p \to \infty) \ (for all j = 1, ..., n).$$

Hence we have $(a_1 + ib_1, ..., a_n + ib_n) \in \sigma_T(T_1, ..., T_n)$ and $b = b_1 \cdot ... \cdot b_n$.

(ii) Proof of the inclusion $P(\sigma_T(T)) \subseteq \sigma_X(T)$. Assume that $(c_1 + id_1, ..., c_n + id_n) \in \sigma_T(T_1, ..., T_n)$. Hence there exists a sequence $\{x_q\}$ of unit vectors such that

$$(T_j - (c_j + id_j))^* x_q \to 0 \ (q \to \infty) \ (for all j = 1, ..., n).$$

Consider again the Berberian extension space $\mathcal{K}$ of $\mathcal{H}$ and let

$$\mathcal{U} := Ker((T_n^{\circ} - (c_n + id_n))^*).$$

By Theorem B there exists $z^{\circ} \in \mathcal{U}$ such that

$$(T_j^{\circ} - (c_j + id_j))^* z^{\circ} = 0 \ (for all j = 1, ..., n-1).$$

Since $(T_1^\circ, ..., T_{n-1}^\circ)$ is a doubly commuting $(n-1)$-tuple of hyponormal operators on $\mathcal{U}$, we know that $(c_1 + id_1, ..., c_{n-1} + id_{n-1}) \in \sigma_T(T_1^\circ|_{\mathcal{U}}, ..., T_{n-1}^\circ|_{\mathcal{U}})$. By induction we have

$$(c_1, ..., c_{n-1}, d_1 \cdots d_{n-1}) \in \sigma_X(T_1^\circ|_{\mathcal{U}}, ..., T_{n-1}^\circ|_{\mathcal{U}}).$$

Hence there exist $(m_1, ..., m_{n-1}) \in [0,1]^{n-1}$ and a non-zero vector $u^\circ \in \mathcal{U}$ such that

$$(H_j^\circ - c_j)u^\circ = (E_1^\circ \cdot ... \cdot E_{n-1}^\circ - (d_1 \cdot ... \cdot d_{n-1}))u^\circ = 0,$$

where $E_j := m_j \mathcal{S}_j^+(K_j) + (1 - m_j)\mathcal{S}_j^-(K_j)$ $(j = 1, ..., n-1)$. We next let

$$\mathcal{V} := \bigcap_{j=1}^{n-1} Ker(H_j^\circ - c_j) \bigcap Ker(E_1^\circ \cdot ... \cdot E_{n-1}^\circ - (d_1 \cdot ... \cdot d_{n-1})).$$

Since $\mathcal{U} \bigcap \mathcal{V} \neq (0)$, we have

$$c_n + id_n \in \sigma(T_n^\circ|_{\mathcal{V}}).$$

Thus, by Lemma 4 we have $(c_n, d_n) \in \sigma_X(T_n)$ and hence there exists $0 \leq m_n \leq 1$ such that $(c_n, d_n) \in \sigma_{ja}(H_n, E_n)$, where $E_n := m_n \mathcal{S}_n^+(K_n) + (1 - m_n)\mathcal{S}_n^-(K_n)$. Since $H_n + iE_n$ is a normal operator, there exists $v^\circ \in \mathcal{V}$ such that

$$(H_n^\circ - c_n)v^\circ = (E_n^\circ - d_n)v^\circ = 0.$$

Put $\mathbf{m} = (m_1, ..., m_n)$. Since

$$E_1 \cdot ... \cdot E_n = \{m_1 \mathcal{S}_1^+(K_1) + (1 - m_1)\mathcal{S}_1^-(K_1)\} \cdot ... \cdot \{m_n \mathcal{S}_n^+(K_n) + (1 - m_n)\mathcal{S}_n^-(K_n)\}$$

$$= [\prod_{j=1}^{n} (m_j \mathcal{S}_j^+ + (1 - m_j)\mathcal{S}_j^-)](K_1 \cdot ... \cdot K_n) = Y_{\mathbf{m}},$$

we have

$$(c_1, ..., c_n, d_1 \cdot ... \cdot d_n) \in \sigma_p(H_1^\circ, ..., H_n^\circ, Y_{\mathbf{m}}^\circ).$$

Therefore, since $(c_1, ..., c_n, d_1 \cdot ... \cdot d_n) \in \sigma_{ja}(H_1, ..., H_n, Y_{\mathbf{m}})$, we see at once that $(c_1, ..., c_n, d_1 \cdot ... \cdot d_n) \in \sigma_X(T_1, ..., T_n)$. The proof is now complete.

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Muneo Chō
Department of Mathematics, Kanagawa University, Yokohama 221-8686 JAPAN
e-mail: m-cho@cc.kanagawa-u.ac.jp
Raúl E. Curto
Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242 USA
e-mail: curto@math.uiowa.edu
Tadasi Huruya
Faculty of Education, Niigata University, Niigata 950-2181 JAPAN
e-mail: huruya@ed.niigata-u.ac.jp
Wieslaw Żelazko
Mathematical Institute, Polish Academy of Sciences, Warszawa 00-950 POLAND
e-mail: zelazko@impan.impan.gov.pl