PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 11, November 1997, Pages 3299–3301
S 0002-9939(97)03948-8

A JOINT SPECTRAL CHARACTERIZATION OF PRIMENESS FOR C*-ALGEBRAS

RAÚL E. CURTO AND CARLOS HERNÁNDEZ G.

(Communicated by Palle E.T. Jorgensen)

Abstract. We prove that a C*-algebra $\mathcal{A}$ is prime iff $\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b)$ for every $a, b \in \mathcal{A}$, where σ_T denotes Taylor spectrum and L_a, R_b are the left and right multiplication operators acting on $\mathcal{A}$.

Let $\mathcal{A}$ be a unital C*-algebra, let L_a, R_b denote the *left* and *right* multiplication operators induced by $a, b \in \mathcal{A}$ (i.e., $L_a(x) := ax,\ R_b(x) := xb,\ x \in \mathcal{A}$), and set $M_{a,b} := L_a R_b$. We obtain a characterization of primeness for C*-algebras in terms of the spectral theory of (L_a, R_b). Recall that an ideal $\mathcal{I}$ in $\mathcal{A}$ is said to be *prime* if, whenever $\mathcal{I}_1, \mathcal{I}_2$ are ideals in $\mathcal{A}$ such that $\mathcal{I}_1\mathcal{I}_2 \subseteq \mathcal{I}$, it follows that $\mathcal{I}_1 \subseteq \mathcal{I}$ or $\mathcal{I}_2 \subseteq \mathcal{I}$; $\mathcal{A}$ is said to be prime if (0) is a prime ideal. In [Ma1], M. Mathieu obtained the following result.

Theorem 1 ([Ma1])**.** *Let $\mathcal{A}$ be a unital C*-algebra. The following statements are equivalent.*

(i) *$\mathcal{A}$ is prime.*
(ii) *$\|M_{a,b}\| = \|a\| \, \|b\|$ for all $a, b \in \mathcal{A}$.*
(iii) *$\sigma(M_{a,b}) = \sigma(a)\sigma(b)$ for all $a, b \in \mathcal{A}$.*

In this note we prove a joint spectral analogue of Theorem 1. First, we recall the definition of the (joint) Taylor spectrum of (L_a, R_b). The *Koszul complex* associated with (L_a, R_b) is

$$\mathcal{K}_{a,b} : 0 \longrightarrow \mathcal{A} \xrightarrow{D^0_{a,b}} \mathcal{A} \bigoplus \mathcal{A} \xrightarrow{D^1_{a,b}} \mathcal{A} \longrightarrow 0,$$

where

$$D^0_{a,b}(x) := ax \bigoplus xb \ \ (x \in \mathcal{A})$$

and

$$D^1_{a,b}(x \bigoplus y) := -xb + ay \ \ (x, y \in \mathcal{A}).$$

We say that (L_a, R_b) is *Taylor invertible* if $\mathcal{K}_{a,b}$ is exact, i.e., if the following three implications hold:

$$Ker D^0_{a,b} = 0 : \quad x \in \mathcal{A},\ ax = 0 = xb \Rightarrow x = 0;$$

Received by the editors December 6, 1995 and, in revised form, June 12, 1996.
1991 *Mathematics Subject Classification.* Primary 46L05, 47A10, 47A13, 47C15, 47D25; Secondary 47A62, 18G35.
Key words and phrases. Taylor spectrum, multiplication operators, prime C*-algebras.

$$KerD^1_{a,b} = RanD^0_{a,b} : \quad x, y \in \mathcal{A},\ ay = xb \Rightarrow \exists\, z \in \mathcal{A} : x = az \ \text{ and } \ y = zb;$$

and

$$RanD^1_{a,b} = \mathcal{A} : \quad z \in \mathcal{A} \Rightarrow \exists x, y \in \mathcal{A} : -xb + ay = z.$$

The *Taylor* spectrum of (L_a, R_b) is then

$$\sigma_T((L_a, R_b), \mathcal{A}) := \{(\lambda, \mu) \in \mathbf{C}^2 : (L_a - \lambda, R_b - \mu) \text{ is not Taylor invertible}\}$$

([Ta1], [Ta2], [Cu1]).

Theorem 2. *Let $\mathcal{A}$ be a unital C^*-algebra. The following statements are equivalent.*

(i) *$\mathcal{A}$ is prime.*
(ii) *$\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b)$ $(a, b \in \mathcal{A})$.*

Corollary 3. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $a, b \in \mathcal{A}$. The following statements are equivalent.*

(i) *$\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b)$ $(a, b \in \mathcal{A})$.*
(ii) *$\sigma(M_{a,b}) = \sigma(a)\sigma(b)$.*

Remark 4. (i) Since the implication (i)⇒(ii) in Corollary 3 is an easy consequence of the Spectral Mapping Theorem for the Taylor spectrum ([Ta2]), and since (ii) implies the primeness of $\mathcal{A}$ by Theorem 1, the content of Theorem 2 is really the validity of (i)⇒(ii).

(ii) For $\mathcal{A} = \mathcal{L}(\mathcal{H})$, the C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$, Theorem 2 is a special case of [CuF, Theorem 3.1].

(iii) Theorem 2 gives an affirmative answer to a question raised in [Cu2, Problem 5.3; case $n = 1$]; it also sheds new light on the so-called mixed interpolation problem [Har].

For the proof of Theorem 2 we shall need the following lemma. Recall that the (joint) *Harte* spectrum of (L_a, R_b) is the union of the *left* and *right* spectra; alternatively, $(\lambda, \mu) \notin \sigma_H((L_a, R_b), \mathcal{A})$ if and only if $D^0_{a-\lambda, b-\mu}$ is bounded below and $D^1_{a-\lambda, b-\mu}$ is onto. Clearly $\sigma_H \subseteq \sigma_T$.

Lemma 5 ([Ma2, Theorem 3.12])**.** *Let $\mathcal{A}$ be a unital prime C^*-algebra, and let $a, b \in \mathcal{A}$. Then*

$$\sigma_H((L_a, R_b), \mathcal{A}) = [\sigma_\ell(a) \times \sigma_r(b)] \cup [\sigma_r(a) \times \sigma_\ell(b)].$$

Proof of Theorem 2. Let $\lambda \in \sigma(a)$, $\mu \in \sigma(b)$. We wish to prove that $(\lambda, \mu) \in \sigma_T((L_a, R_b), \mathcal{A})$. Assume not. Without loss of generality, $\lambda = \mu = 0$, and by Lemma 5, we can also assume that either (i) $0 \in [\sigma_\ell(a) \setminus \sigma_r(a)]$, $0 \in [\sigma_\ell(b) \setminus \sigma_r(b)]$ or (ii) $0 \in [\sigma_r(a) \setminus \sigma_\ell(a)]$, $0 \in [\sigma_r(b) \setminus \sigma_\ell(b)]$.

Case (i). Let c be a right inverse for a, i.e., $ac = 1$. Then $x := 1 - ca$ is such that $ax = 0$ and $x \neq 0$. Similarly, $y := 1 - db$ is such that $by = 0$, $y \neq 0$, where $bd = 1$. Since $\mathcal{A}$ is prime, $\|M_{x,y}\| = \|x\|\,\|y\| > 0$, which implies that

$$xzy \neq 0 \tag{1}$$

for some $z \in \mathcal{A}$. Then $D^1_{a,b}(0 \bigoplus xz) = axz = 0$. Since $\mathcal{K}_{a,b}$ is exact, the kernel of $D^1_{a,b}$ must equal the range of $D^0_{a,b}$, that is, $0 \bigoplus xz = au \bigoplus ub$ for some $u \in \mathcal{A}$. Then $au = 0$ and $xz = ub$. It follows that $xzy = uby = 0$, contradicting (1).

Case (ii). Let c', d' be left inverses for a, b, respectively. Then $x' := 1 - ac'$ and $y' := 1 - bd'$ are such that $x'a = 0$, $y'b = 0$, $x' \neq 0$ and $y' \neq 0$. As in the previous case, we can find $z' \in \mathcal{A}$ such that $x'z'y' \neq 0$. Since $D^1_{a,b}(z'y' \bigoplus 0) = z'y'b = 0$, there exists $u' \in \mathcal{A}$ such that $z'y' \bigoplus 0 = au' \bigoplus u'b$. Then $z'y' = au'$, which implies $x'z'y' = x'au' = 0$, a contradiction. □

Remark 6. Although a detailed analysis of the case $n = 1$ in [Cu2, Problem 5.3] was sufficient for our purposes here, one wonders if a similar technique might also work for $n > 1$. We have attempted this for $a = (a_1, a_2)$ and $b = (b_1)$, without success: the argument related to exactness at the middle stage of $\mathcal{K}_{a,b}$ in the proof of Theorem 2 does not easily extend to either of the middle stages of $\mathcal{K}_{(a_1,a_2),b_1}$. At present, we are searching for alternative ways to deal with the intermediate stages of $\mathcal{K}_{(a_1,\dots,a_n),(b_1,\dots,b_n)}$.

Acknowledgment

We are indebted to the referee for several useful comments.

References

[Cu1] R. Curto, Applications of several complex variables to multiparameter spectral theory, Surveys of Some Recent Results in Operator Theory, J.B. Conway and B. Morrel, eds., vol. II, Pitman Res. Notes in Math. Ser. 192, Longman Publ. Co., London, 1988; pp. 25-90. MR **90d:**47007

[Cu2] R. Curto, Spectral theory of elementary operators, Proceedings of the International Workshop on Elementary Operators and Applications, M. Mathieu, ed., World Scientific, 1992; pp. 3-52. MR **93i:**47041

[CuF] R. Curto and L. Fialkow, The spectral picture of (L_A, R_B), J. Funct. Anal. 71(1987), 371-392. MR **88c:**47006

[Har] R. Harte, A problem of mixed interpolation, Math. Z. 143(1975), 149-153. MR **53:**8933

[Ma1] M. Mathieu, Applications of ultraprime Banach algebras in the theory of elementary operators, Thesis, Univ. Tübingen, 1986.

[Ma2] M. Mathieu, Elementary operators on prime C*-algebras, I, Math. Ann. 284(1989), 223-244. MR **90h:**46092

[Ta1] J.L. Taylor, A joint spectrum for several commuting operators, J. Funct. Anal. 6(1970), 172-191. MR **42:**3603

[Ta2] J.L. Taylor, The analytic functional calculus for several commuting operators, Acta Math. 125(1970), 1-38. MR **42:**6622

Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242
E-mail address: curto@math.uiowa.edu

Instituto de Matemáticas, UNAM, Ciudad Universitaria, 04510 Mexico, D.F., Mexico
E-mail address: carlosh@servidor.unam.mx