A JOINT SPECTRAL CHARACTERIZATION OF PRIMENESS FOR C*-ALGEBRAS

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Abstract. We prove that a C*-algebra $A$ is prime iff $\sigma_T((L_a, R_b), A) = \sigma(a) \times \sigma(b)$ for every $a, b \in A$, where $\sigma_T$ denotes Taylor spectrum and $L_a, R_b$ are the left and right multiplication operators acting on $A$.

Let $A$ be a unital C*-algebra, let $L_a, R_b$ denote the left and right multiplication operators induced by $a, b \in A$ (i.e., $L_a(x) := ax$, $R_b(x) := xb$, $x \in A$), and set $M_{a,b} := L_a R_b$. We obtain a characterization of primeness for C*-algebras in terms of the spectral theory of $(L_a, R_b)$. Recall that an ideal $\mathcal{I}$ in $A$ is said to be prime if, whenever $\mathcal{I}_1, \mathcal{I}_2$ are ideals in $A$ such that $\mathcal{I}_1 \mathcal{I}_2 \subseteq \mathcal{I}$, it follows that $\mathcal{I}_1 \subseteq \mathcal{I}$ or $\mathcal{I}_2 \subseteq \mathcal{I}$; $A$ is said to be prime if $(0)$ is a prime ideal. In [Ma1], M. Mathieu obtained the following result.

**Theorem 1 ([Ma1]).** Let $A$ be a unital C*-algebra. The following statements are equivalent.

(i) $A$ is prime.

(ii) $\|M_{a,b}\| = \|a\| \|b\|$ for all $a, b \in A$.

(iii) $\sigma(M_{a,b}) = \sigma(a) \sigma(b)$ for all $a, b \in A$.

In this note we prove a joint spectral analogue of Theorem 1. First, we recall the definition of the (joint) Taylor spectrum of $(L_a, R_b)$. The Koszul complex associated with $(L_a, R_b)$ is

$$\mathcal{K}_{a,b} : 0 \longrightarrow A \overset{D^0_{a,b}}{\longrightarrow} A \overset{D^1_{a,b}}{\longrightarrow} A \longrightarrow 0,$$

where

$$D^0_{a,b}(x) := ax \bigoplus xb \quad (x \in A)$$

and

$$D^1_{a,b}(x \bigoplus y) := -xb + ay \quad (x, y \in A).$$

We say that $(L_a, R_b)$ is Taylor invertible if $\mathcal{K}_{a,b}$ is exact, i.e., if the following three implications hold:

$$\text{Ker} D^0_{a,b} = 0 : \quad x \in A, \ ax = 0 = xb \Rightarrow x = 0;$$
Corollary 3. Let \( \sigma \) be a unital prime \( \sigma \)-algebra. The following statements are equivalent.

(i) \( \sigma \) is prime.
(ii) \( \sigma((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b) \quad (a, b \in \mathcal{A}). \)

Corollary 3. Let \( \mathcal{A} \) be a unital \( \sigma \)-algebra, and let \( a, b \in \mathcal{A} \). The following statements are equivalent.

(i) \( \sigma((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b) \quad (a, b \in \mathcal{A}). \)
(ii) \( \sigma(M_{a,b}) = \sigma(a)\sigma(b). \)

Remark 4. (i) Since the implication (i)\( \Rightarrow \) (ii) in Corollary 3 is an easy consequence of the Spectral Mapping Theorem for the Taylor spectrum ([Ta2]), and since (ii) implies the primeness of \( \mathcal{A} \) by Theorem 1, the content of Theorem 2 is really the validity of (i)\( \Rightarrow \) (ii).

(ii) For \( \mathcal{A} = \mathcal{L}(\mathcal{H}) \), the \( \sigma \)-algebra of bounded operators on a Hilbert space \( \mathcal{H} \), Theorem 2 is a special case of [CuF, Theorem 3.1].

(iii) Theorem 2 gives an affirmative answer to a question raised in [Cu2, Problem 5.3; case \( n = 1 \)]; it also sheds new light on the so-called mixed interpolation problem [Har].

For the proof of Theorem 2 we shall need the following lemma. Recall that the (joint) Harte spectrum of \( (L_a, R_b) \) is the union of the left and right spectra; alternatively, \( (\lambda, \mu) \notin \sigma_H((L_a, R_b), \mathcal{A}) \) if and only if \( D_{a-\lambda, b-\mu}^{1} \) is bounded below and \( D_{a-\lambda, b-\mu}^{0} \) is onto. Clearly \( \sigma_H \subseteq \sigma_T \).

Lemma 5 ([Ma2, Theorem 3.12]). Let \( \mathcal{A} \) be a unital prime \( \sigma \)-algebra, and let \( a, b \in \mathcal{A} \). Then

\[
\sigma_H((L_a, R_b), \mathcal{A}) = [\sigma_L(a) \times \sigma_R(b)] \cup [\sigma_L(a) \times \sigma_R(b)].
\]

Proof of Theorem 2. Let \( \lambda \in \sigma(a), \quad \mu \in \sigma(b) \). We wish to prove that \( (\lambda, \mu) \in \sigma_T((L_a, R_b), \mathcal{A}) \). Assume not. Without loss of generality, \( \lambda = \mu = 0 \), and by Lemma 5, we can also assume that either (i) \( 0 \in [\sigma_L(a) \setminus \sigma_R(a)], \quad 0 \in [\sigma_L(b) \setminus \sigma_R(b)] \) or (ii) \( 0 \in [\sigma_L(a) \setminus \sigma_L(a)], \quad 0 \in [\sigma_R(b) \setminus \sigma_R(b)]. \)

Case (i). Let \( c \) be a right inverse for \( a \), i.e., \( ac = 1 \). Then \( x := 1 - ca \) is such that \( ax = 0 \) and \( x \neq 0 \). Similarly, \( y := 1 - db \) is such that \( by = 0, \ y \neq 0 \), where \( bd = 1 \). Since \( \mathcal{A} \) is prime, \( \|M_{x,y}\| = \|x\|\|y\| > 0 \), which implies that

\[
1 \\
\] for some \( z \in \mathcal{A} \). Then \( D_{a,b}^{1}(0 \bigoplus xz) = axz = 0 \). Since \( K_{a,b} \) is exact, the kernel of \( D_{a,b}^{1} \) must equal the range of \( D_{a,b}^{0} \), that is, \( 0 \bigoplus xz = au \bigoplus ub \) for some \( u \in \mathcal{A} \). Then \( au = 0 \) and \( xz = ub \). It follows that \( xzy = uby = 0 \), contradicting (1).
Case (ii). Let \( c', d' \) be left inverses for \( a, b \), respectively. Then \( x' := 1 - ac' \) and \( y' := 1 - bd' \) are such that \( x'a = 0 \), \( y'b = 0 \), \( x' \neq 0 \) and \( y' \neq 0 \). As in the previous case, we can find \( z' \in \mathcal{A} \) such that \( x'z'y' \neq 0 \). Since \( D_{a,b}^1(z'y' \bigoplus 0) = z'y'b = 0 \), there exists \( u' \in \mathcal{A} \) such that \( z'y' \bigoplus 0 = au' \bigoplus u'b \). Then \( z'y' = au' \), which implies \( x'z'y' = x'au' = 0 \), a contradiction.

Remark 6. Although a detailed analysis of the case \( n = 1 \) in [Cu2, Problem 5.3] was sufficient for our purposes here, one wonders if a similar technique might also work for \( n > 1 \). We have attempted this for \( a = (a_1, a_2) \) and \( b = (b_1) \) without success: the argument related to exactness at the middle stage of \( K_{a,b} \) in the proof of Theorem 2 does not easily extend to either of the middle stages of \( K_{(a_1, a_2), b_1} \). At present, we are searching for alternative ways to deal with the intermediate stages of \( K_{(a_1, \ldots, a_n), (b_1, \ldots, b_n)} \).

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References


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