

# Subnormality for arbitrary powers of 2-variable weighted shifts whose restrictions to a large invariant subspace are tensor products

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## Abstract

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. We study LPCS within the class of commuting 2-variable weighted shifts  $\mathbf{T} \equiv (T_1, T_2)$  with subnormal components  $T_1$  and  $T_2$ , acting on the Hilbert space  $\ell^2(\mathbb{Z}_+^2)$  with canonical orthonormal basis  $\{e_{(k_1, k_2)}\}_{k_1, k_2 \geq 0}$ . The *core* of a commuting 2-variable weighted shift  $\mathbf{T}$ ,  $c(\mathbf{T})$ , is the restriction of  $\mathbf{T}$  to the invariant subspace generated by all vectors  $e_{(k_1, k_2)}$  with  $k_1, k_2 \geq 1$ ; we say that  $c(\mathbf{T})$  is of *tensor form* if it is unitarily equivalent to a shift of the form  $(I \otimes W_\alpha, W_\beta \otimes I)$ , where  $W_\alpha$  and  $W_\beta$  are subnormal unilateral weighted shifts. Given a 2-variable weighted shift  $\mathbf{T}$  whose core is of tensor form, we prove that LPCS is solvable for  $\mathbf{T}$  if and only if LPCS is solvable for any power  $\mathbf{T}^{(m, n)} := (T_1^m, T_2^n)$  ( $m, n \geq 1$ ).

*Keywords:* jointly hyponormal pairs, subnormal pairs, 2-variable weighted shifts, tensor form, core

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## 1. Introduction

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. In previous work ([6], [7], [8], [9], [14], [15], [16], [27]) we have studied LPCS from a number of different approaches. One such approach is to consider commuting pairs  $\mathbf{T} \equiv (T_1, T_2)$  of subnormal operators and to ask to what extent the existence of liftings for the powers  $\mathbf{T}^{(m,n)} := (T_1^m, T_2^n)$  ( $m, n \geq 1$ ) can guarantee a lifting for  $\mathbf{T}$ . For the class of 2-variable weighted shifts  $\mathbf{T}$ , it is often the case that the powers of  $\mathbf{T}$  are less complex than the initial pair; thus it becomes especially significant to unravel how subnormality behaves under the action  $(m, n) \mapsto \mathbf{T}^{(m,n)}$  ( $h, \ell \geq 1$ ).

Within the class of 2-variable weighted shifts, we consider the subclass  $\mathcal{TC}$  consisting of pairs whose cores are of tensor form; that is

$$\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0 : c(\mathbf{T}) \text{ is of tensor form}\}$$

This subclass has proved to be particularly attractive, since it is possible to separate, within it, subnormality from  $k$ -hyponormality; thus, results about LPCS for pairs in  $\mathcal{TC}$  are especially useful. The class  $\mathcal{TC}$  is small enough to allow for a simple description of its pairs, yet large enough to be used as test ground for many significant hypotheses.

Before we proceed, we briefly pause to establish our terminology. For  $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$  a bounded sequence of positive real numbers (called *weights*), let  $W_\alpha : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$  be the associated unilateral weighted shift, defined by  $W_\alpha e_n := \alpha_n e_{n+1}$  (all  $n \geq 0$ ), where  $\{e_n\}_{n=0}^\infty$  is the canonical orthonormal basis in  $\ell^2(\mathbb{Z}_+)$ . Similarly, consider double-indexed positive bounded sequences  $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$ ,  $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ , and let  $\{e_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$  be the canonical orthonormal basis in  $\ell^2(\mathbb{Z}_+^2)$ . We define the 2-variable weighted shift  $\mathbf{T} \equiv (T_1, T_2)$  acting on  $\ell^2(\mathbb{Z}_+^2)$  by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k} + \varepsilon_1} \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k} + \varepsilon_2},$$

where  $\varepsilon_1 := (1, 0)$  and  $\varepsilon_2 := (0, 1)$ . The *core* of a commuting 2-variable weighted shift  $\mathbf{T}$  (in symbols,  $c(\mathbf{T})$ ) is the restriction of  $\mathbf{T}$  to the invariant subspace generated by all vectors  $e_{(k_1, k_2)}$  with  $k_1, k_2 \geq 1$ ; we say that  $c(\mathbf{T})$  is of *tensor form* if it is unitarily equivalent to a shift of the form  $(I \otimes W_\alpha, W_\beta \otimes I)$ , where  $W_\alpha$  and  $W_\beta$  are subnormal unilateral weighted shifts. Figure 1 shows both the weight and Berger measure diagrams of a typical pair in  $\mathcal{TC}$ . As shown in [8], each  $\mathbf{T} \in \mathcal{TC}$  is completely determined by five parameters, i.e., the 1-variable

measures  $\sigma$ ,  $\tau$ ,  $\xi$  and  $\eta$ , and the positive number  $a \equiv \alpha_{(0,1)}$ . As we mentioned before,  $\mathcal{TC}$  is of substantial interest to us, since it provides a fertile ground to test results on subnormality and  $k$ -hyponormality, and in particular about the solubility of LPCS.

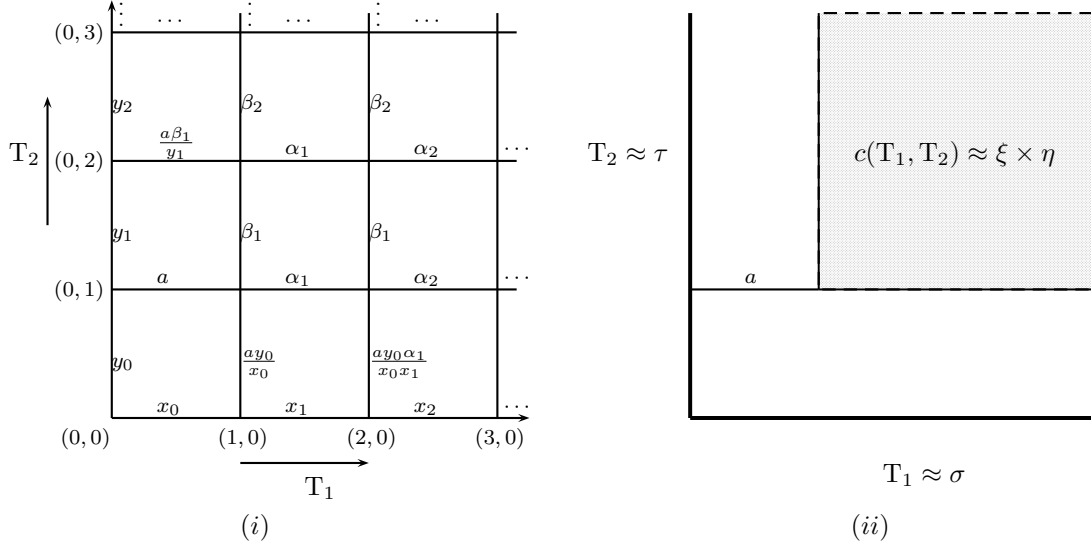


Figure 1: Weight and Berger measure diagrams of a typical 2-variable weighted shift in  $\mathcal{TC}$ .

Let us now denote the class of commuting pairs of subnormal operators on Hilbert space by  $\mathfrak{H}_0$ , the class of subnormal pairs by  $\mathfrak{H}_\infty$ , and for an integer  $k \geq 1$ , the class of  $k$ -hyponormal pairs in  $\mathfrak{H}_0$  by  $\mathfrak{H}_k$ . Clearly,  $\mathfrak{H}_\infty \subseteq \dots \subseteq \mathfrak{H}_k \subseteq \dots \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$ ; the main results in [14] and [6] show that these inclusions are all proper; moreover, examples illustrating these proper inclusions can be found in  $\mathcal{TC}$ .

In this paper we show that for  $\mathbf{T} \in \mathcal{TC}$ , the subnormality of *any* power  $\mathbf{T}^{(m,n)}$  implies the subnormality of  $\mathbf{T}$ . To accomplish this, we first show that every power of  $\mathbf{T} \in \mathcal{TC}$  is the orthogonal direct sum of 2-variable weighted shifts in  $\mathcal{TC}$ . Since each 2-variable weighted shift in  $\mathcal{TC}$  is completely determined by five parameters, we then study how the five parameters of each direct summand in a power are related to the five parameters in the initial 2-variable weighted shift. Next, we recall from [8] that each  $\mathbf{T} \in \mathcal{TC}$  is associated with a pair of linear functionals  $\varphi \equiv \varphi(\mathbf{T})$  and  $\psi \equiv \psi(\mathbf{T})$  (each depending on the five parameters), and that  $\mathbf{T}$  is subnormal if and only if  $\varphi \geq 0$  and  $\psi \geq 0$ . With all of this at our disposal, we proceed to establish a connection between the pair  $(\varphi, \psi)$  associated with  $\mathbf{T}$  and those associated with the summands in the orthogonal direct sum decomposition of  $\mathbf{T}^{(m,n)}$ . This eventually leads to the proof of our main result (Theorem 3.1).

This result provides a complete generalization of Theorem 3.9 in [7]. At the

time we wrote [7], the techniques available to us allowed us to deal only with the quadratic powers  $\mathbf{T}^{(1,2)}$  and  $\mathbf{T}^{(2,1)}$ ; with the aid of a number of additional examples, together with the main result in [8], we have now been able to handle the case of arbitrary powers.

As an application of Theorem 3.1, we can exhibit a hyponormal 2-variable weighted shift such that none of its powers is subnormal. We describe the shift in Example 8.1. This provides a striking and concrete example of the big gap that exists between hyponormality and subnormality for 2-variable weighted shifts, even within a relatively simple class like  $\mathcal{TC}$ .

## 2. Notation and preliminaries

To describe our results in detail we need some notation; we further expand on our terminology later in this section. For  $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$ , we shall let  $\mathcal{M}_i$  (resp.  $\mathcal{N}_j$ ) be the subspace of  $\ell^2(\mathbb{Z}_+^2)$  which is spanned by the canonical orthonormal basis vectors associated to indices  $\mathbf{k}$  with  $k_1 \geq 0$  and  $k_2 \geq i$  (resp.  $k_1 \geq j$  and  $k_2 \geq 0$ ). The *core* of a 2-variable weighted shift  $\mathbf{T}$  is the restriction of  $\mathbf{T}$  to  $\mathcal{M}_1 \cap \mathcal{N}_1$ ; in symbols,  $c(\mathbf{T}) := \mathbf{T}|_{\mathcal{M}_1 \cap \mathcal{N}_1}$ . A 2-variable weighted shift  $\mathbf{T}$  is said to be of *tensor form* if  $\mathbf{T} \cong (I \otimes W_\xi, W_\eta \otimes I)$ , where  $W_\xi$  and  $W_\eta$  are unilateral weighted shifts. The class of all 2-variable weighted shifts  $\mathbf{T} \in \mathfrak{H}_0$  whose cores are of tensor form is denoted by  $\mathcal{TC}$ ; that is,

$$\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0 : c(\mathbf{T}) \text{ is of tensor form}\}$$

(see Figure 1(i)).

It is well known that the commutativity of a pair of subnormals is necessary but not sufficient for the existence of a lifting ([1], [20], [21], [22]), and it has recently been shown that the joint hyponormality of the pair is necessary but not sufficient [14]. Our previous work ([14], [15], [16], [17], [6], [7], [8], [9], [26], [27]) has revealed that the nontrivial aspects of the LPCS are best detected within the class  $\mathfrak{H}_0$ , especially within  $\mathcal{TC}$ ; we thus focus our attention on this class.

For a single operator  $T$ , the subnormality of all powers  $T^n$  ( $n \geq 2$ ) does not imply the subnormality of  $T$ , even if  $T$  is a unilateral weighted shift [24, pp. 378-379]. Thus one might guess that if we were to impose a further condition such as the subnormality of a restriction of  $T$  to an invariant subspace, e.g., the subnormality of  $T|_{\vee\{e_k \in \ell^2(\mathbb{Z}_+) : k \geq i\}}$  (for some  $i \geq 1$ ), that  $T$  would then be subnormal. However, even if we assume this for  $i = 1$ , the subnormality of  $T$  is not guaranteed. For example, let  $T := \text{shift}(\frac{1}{3}, \frac{1}{2}, 1, 1, \dots)$ . Then  $T|_{\vee\{e_k \in \ell^2(\mathbb{Z}_+) : k \geq 1\}} \equiv \text{shift}(\frac{1}{2}, 1, 1, \dots)$  is subnormal, and also all powers  $T^n$  ( $n \geq 2$ ) are subnormal, but  $T$  is not subnormal. As a matter of fact, no backward extension  $\text{shift}(\alpha_0, \frac{1}{2}, 1, 1, \dots)$  can be subnormal ([5, Corollary 6]). More generally, the necessary and sufficient conditions for a unilateral weighted shift  $W_\alpha$  to be subnormal when we assume that  $W_\alpha|_{\vee\{e_k \in \ell^2(\mathbb{Z}_+) : k \geq 1\}}$  is subnormal (with Berger measure  $\mu$ ) were obtained in [5, Proposition 8]:  $W_\alpha$  is subnormal if and only if  $\frac{1}{t} \in L^1(\mu)$  and  $\alpha_0^2 \left\| \frac{1}{t} \right\|_{L^1(\mu)} \leq 1$ .

In the multivariable case, the analogous results are highly nontrivial, if one further assumes that each component is subnormal. In 1-variable, the subspace  $\vee\{e_k \in \ell^2(\mathbb{Z}_+) : k \geq 1\}$  can be regarded as “the core of  $T$ ”; as we move into two variables it is therefore natural to consider the condition  $\mathbf{T} \in \mathcal{TC}$ .

To prove our results, we require a number of tools and techniques introduced in previous work, e.g., the Six-point Test (Lemma 4.1), the Backward Extension Theorem for 2-variable weighted shifts (Lemma 4.2) and the formula to reconstruct the Berger measure of a unilateral weighted shift (4.1), together with a new direct sum decomposition for powers of 2-variable weighted shifts which parallels the decomposition used in [11] to analyze  $k$ -hyponormality for powers of (1-variable) weighted shifts. Concretely, to analyze the power  $\mathbf{T}^{(m,n)}$  of a 2-variable weighted shift  $\mathbf{T} \equiv (T_1, T_2)$ , we split the ambient space  $\ell^2(\mathbb{Z}_+^2)$  as an orthogonal direct sum  $\bigoplus_{p=0}^{m-1} \bigoplus_{q=0}^{n-1} \mathcal{H}_{(p,q)}^{(m,n)}$ , where for  $p = 0, 1, \dots, m-1$ , and  $q = 0, 1, \dots, n-1$ ,

$$\mathcal{H}_{(p,q)}^{(m,n)} := \vee\{e_{(m\ell+p, nk+q)} : k, \ell \geq 0\}. \quad (2.1)$$

Each of the subspaces  $\mathcal{H}_{(p,q)}^{(m,n)}$  reduces  $T_1^m$  and  $T_2^n$ , and  $\mathbf{T}^{(m,n)}$  is subnormal if and only if each summand  $\mathbf{T}^{(m,n)}|_{\mathcal{H}_{(p,q)}^{(m,n)}}$  is subnormal. For a set of pairs  $\mathcal{X}$ , let  $\bigoplus \mathcal{X}$  denote the set of pairs that can be written as orthogonal sums of pairs in  $\mathcal{X}$ . We will show in Section 5 that  $\bigoplus \mathcal{TC}$  is invariant under the action  $(m, n) \mapsto \mathbf{T}^{(m,n)}$  ( $m, n \geq 1$ ).

Briefly stated, our strategy to prove our main result (Theorem 3.1) is as follows: (i) if  $\mathbf{T} \in \mathcal{TC}$  then each power  $\mathbf{T}^{(m,n)} \in \bigoplus \mathcal{TC}$ ; (ii) without loss of generality, we can always assume  $m = 1$ ; (iii) the pair  $(\varphi, \psi)$  associated with  $\mathbf{T}$  is directly related to the pairs  $(\varphi^{(p,q)}, \psi^{(p,q)})$  associated to the direct summands in the orthogonal decomposition of  $\mathbf{T}^{(1,n)}$ ; (iv) if a power  $\mathbf{T}^{(1,n)}$  is subnormal, the functionals  $\varphi^{(0,0)}$  and  $\psi^{(0,1)}$  are both positive; and (v) it then follows that  $\varphi$  and  $\psi$  are both positive, and therefore  $\mathbf{T}$  is subnormal.

We devote the rest of this section to establishing some additional terminology and notation. Let  $\mathcal{H}$  be a complex Hilbert space and let  $\mathcal{B}(\mathcal{H})$  denote the algebra of bounded linear operators on  $\mathcal{H}$ . Recall that a bounded linear operator  $T \in \mathcal{B}(\mathcal{H})$  is *normal* if  $T^*T = TT^*$ , and *subnormal* if  $T = N|_{\mathcal{H}}$ , where  $N$  is normal and  $N(\mathcal{H}) \subseteq \mathcal{H}$ . An operator  $T$  is said to be *hyponormal* if  $T^*T \geq TT^*$ . For  $S, T \in \mathcal{B}(\mathcal{H})$ , let  $[S, T] := ST - TS$ . An  $n$ -tuple  $\mathbf{T} := (T_1, \dots, T_n)$  of operators on  $\mathcal{H}$  is said to be (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive semidefinite on the direct sum of  $n$  copies of  $\mathcal{H}$  (cf. [2], [10], [12], [13], [23]). For instance, if  $n = 2$ ,

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}.$$

The  $n$ -tuple  $\mathbf{T} \equiv (T_1, T_2, \dots, T_n)$  is said to be *normal* if  $\mathbf{T}$  is commuting and each  $T_i$  is normal, and  $\mathbf{T}$  is *subnormal* if  $\mathbf{T}$  is the restriction of a normal  $n$ -tuple to a common invariant subspace. In particular, a commuting pair  $\mathbf{T} \equiv (T_1, T_2)$  is said to be *k-hyponormal* ( $k \geq 1$ ) [6] if

$$\mathbf{T}(k) := (T_1, T_2, T_1^2, T_2 T_1, T_2^2, \dots, T_1^k, T_2 T_1^{k-1}, \dots, T_2^k)$$

is hyponormal, or equivalently

$$[\mathbf{T}(k)^*, \mathbf{T}(k)] = ((T_2^q T_1^p)^*, T_2^m T_1^n)_{\substack{1 \leq n+m \leq k \\ 1 \leq p+q \leq k}} \geq 0.$$

Clearly, normal  $\Rightarrow$  subnormal  $\Rightarrow$   $k$ -hyponormal. For  $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$  a bounded sequence of positive real numbers (called *weights*), let  $W_\alpha : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$  be the associated unilateral weighted shift, defined by  $W_\alpha e_n := \alpha_n e_{n+1}$  (all  $n \geq 0$ ), where  $\{e_n\}_{n=0}^\infty$  is the canonical orthonormal basis in  $\ell^2(\mathbb{Z}_+)$ . For a weighted shift  $W_\alpha$ , the *moments of  $\alpha$*  are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1, & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2, & \text{if } k > 0 \end{cases}. \quad (2.2)$$

It is easy to see that  $W_\alpha$  is never normal, and that it is hyponormal if and only if  $\alpha_0 \leq \alpha_1 \leq \dots$ . Similarly, consider double-indexed positive bounded sequences  $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$ ,  $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ . We define the 2-variable weighted shift  $\mathbf{T} \equiv (T_1, T_2)$  by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where  $\varepsilon_1 := (1, 0)$  and  $\varepsilon_2 := (0, 1)$ . Clearly,

$$T_1 T_2 = T_2 T_1 \iff \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2). \quad (2.3)$$

In an entirely similar way one can define multivariable weighted shifts. Given  $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ , the *moment of  $(\alpha, \beta)$  of order  $\mathbf{k}$*  is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \begin{cases} 1 & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

We remark that, due to the commutativity condition (2.3),  $\gamma_{\mathbf{k}}$  can be computed using any nondecreasing path from  $(0, 0)$  to  $(k_1, k_2)$ .

We now recall a well known characterization of subnormality for multivariable weighted shifts [19] (due to C. Berger [3, II.6.10] and independently established by R. Gellar and L.J. Wallen [18] in the single variable case):  $\mathbf{T}$  admits a commuting normal extension if and only if there is a probability measure  $\mu$  (which we call the Berger measure of  $\mathbf{T}$ ) defined on the 2-dimensional rectangle  $R = [0, a_1] \times [0, a_2]$  (where  $a_i := \|T_i\|^2$ ) such that  $\gamma_{\mathbf{k}} = \int_R s^{k_1} t^{k_2} d\mu(s, t)$ , for all  $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ .

The following well known result, which links the Berger measure of a subnormal unilateral weighted shift with the Berger measure of its restriction to a suitable invariant subspace, will be needed in Section 5.

**Lemma 2.1.** ([14, p. 5140]) *Let  $W_\alpha$  be a subnormal unilateral weighted shift and let  $\xi$  denote its Berger measure. For  $n \geq 1$  let  $\mathcal{L}_n := \vee\{e_h : h \geq n\}$  denote the invariant subspace obtained by removing the first  $n$  vectors in the canonical orthonormal basis of  $\ell^2(\mathbb{Z}_+)$ . Then the Berger measure of  $W_\alpha|_{\mathcal{L}_n}$  is*

$$d\xi_n(s) := \frac{s^n}{\gamma_n} d\xi(s), \quad (2.4)$$

where  $\gamma_n$  is the  $n$ -th moment of  $\alpha$ , given by (2.2).

We will occasionally write  $\text{shift}(\alpha_0, \alpha_1, \dots)$  to denote the weighted shift with weight sequence  $\{\alpha_k\}_{k=0}^\infty$ . We also denote by  $U_+ := \text{shift}(1, 1, \dots)$  the (unweighted) unilateral shift, and for  $0 < a < 1$  we let  $S_a := \text{shift}(a, 1, 1, \dots)$ . Observe that  $U_+$  and  $S_a$  are subnormal, with Berger measures  $\delta_1$  and  $(1 - a^2)\delta_0 + a^2\delta_1$ , respectively, where  $\delta_p$  denotes the point-mass probability measure with support the singleton  $\{p\}$ .

Given integers  $i$  and  $m$  ( $m \geq 1, 0 \leq i \leq m-1$ ), consider  $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+) = \vee\{e_n : n \geq 0\}$  and define  $\mathcal{H}_i := \vee\{e_{mj+i} : j \geq 0\}$ , so  $\mathcal{H} = \bigoplus_{i=0}^{m-1} \mathcal{H}_i$ . For a sequence  $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ , let  $\alpha(m:i) := \{\prod_{k=0}^{m-1} \alpha_{mj+i+k}\}_{j=0}^\infty$ , that is,  $\alpha(m:i)$  denotes the sequence of products of numbers in adjacent packets of size  $m$ , beginning with the product  $\alpha_i \cdots \alpha_{i+m-1}$ . For example,  $\alpha(2:0) : \alpha_0\alpha_1, \alpha_2\alpha_3, \alpha_4\alpha_5, \dots$ , and  $\alpha(3:2) : \alpha_2\alpha_3\alpha_4, \alpha_5\alpha_6\alpha_7, \dots$ . Then for  $m \geq 1$  and  $0 \leq i \leq m-1$ ,  $W_{\alpha(m:i)}$  is unitarily equivalent to  $W_\alpha^m|_{\mathcal{H}_i}$ . Therefore,  $W_\alpha^m$  is unitarily equivalent to  $\bigoplus_{i=0}^{m-1} W_{\alpha(m:i)}$ . This analysis naturally leads to the following result, which will be needed in Section 6.

**Lemma 2.2.** ([11, Theorem 2.9]) *Let  $W_\alpha$  be a subnormal unilateral weighted shift with Berger measure  $\mu$ . Then  $W_{\alpha(m,i)}$  is subnormal with Berger measure  $\mu_{(m,i)}$ , where*

$$d\mu_{(m,0)}(s) = d\mu(s^{\frac{1}{m}}) \text{ and } d\mu_{(m,i)}(s) = \frac{s^{\frac{i}{m}}}{\gamma_i} d\mu(s^{\frac{1}{m}}) \text{ for } 1 \leq i \leq m-1.$$

### 3. Statement of the main result

In ([7]) we showed that if  $\mathbf{T} \in \mathcal{TC}$  (see Figure 1(i)), then

$$\mathbf{T}^{(1,2)} \in \mathfrak{H}_\infty \Leftrightarrow \mathbf{T}^{(2,1)} \in \mathfrak{H}_\infty \Leftrightarrow \mathbf{T} \in \mathfrak{H}_\infty. \quad (3.1)$$

The main result in this paper, which follows, is based on (3.1) and a myriad of examples that have arisen in our previous research.

**Theorem 3.1.** *Let  $\mathbf{T} \in \mathcal{TC}$ . The following statements are equivalent.*

- (i)  $\mathbf{T} \in \mathfrak{H}_\infty$ ;
- (ii)  $\mathbf{T}^{(m,n)} \in \bigoplus \mathfrak{H}_\infty$  for all  $m, n \geq 1$ ;
- (iii)  $\mathbf{T}^{(m,n)} \in \bigoplus \mathfrak{H}_\infty$  for some  $m, n \geq 1$ .

#### 4. Some basic facts

For the reader's convenience, in this section we list several well known auxiliary results and definitions which are needed for the proof of the main result. First, to detect hyponormality for 2-variable weighted shifts we use a simple criterion involving a base point  $\mathbf{k}$  in  $\mathbb{Z}_+^2$  and its five neighboring points in  $\mathbf{k} + \mathbb{Z}_+^2$  at path distance at most 2.

**Lemma 4.1.** ([4]) (*Six-point Test*) *Let  $\mathbf{T} \equiv (T_1, T_2)$  be a 2-variable weighted shift, with weight sequences  $\alpha$  and  $\beta$ . Then*

$$[\mathbf{T}^*, \mathbf{T}] \geq 0 \iff$$

$$H_{(k_1, k_2)}(1) := \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0 \text{ (all } \mathbf{k} \in \mathbb{Z}_+^2\text{)}.$$

Next, we present a criterion to detect the subnormality of 2-variable weighted shifts. First, we need some definitions.

(i) Let  $\mu$  and  $\nu$  be two positive measures on  $\mathbb{R}_+$ . We say that  $\mu \leq \nu$  on  $X := \mathbb{R}_+$ , if  $\mu(E) \leq \nu(E)$  for all Borel subset  $E \subseteq \mathbb{R}_+$ ; equivalently,  $\mu \leq \nu$  if and only if  $\int f d\mu \leq \int f d\nu$  for all  $f \in C(X)$  such that  $f \geq 0$  on  $\mathbb{R}_+$ .

(ii) Let  $\mu$  be a probability measure on  $X \times Y$ , and assume that  $\frac{1}{t} \in L^1(\mu)$ . The *extremal measure*  $\mu_{ext}$  (which is also a probability measure) on  $X \times Y$  is given by  $d\mu_{ext}(s, t) := (1 - \delta_0(t)) \frac{1}{t \|\frac{1}{t}\|_{L^1(\mu)}} d\mu(s, t)$ .

(iii) Given a measure  $\mu$  on  $X \times Y$ , the *marginal measure*  $\mu^X$  is given by  $\mu^X := \mu \circ \pi_X^{-1}$ , where  $\pi_X : X \times Y \rightarrow X$  is the canonical projection onto  $X$ . Thus  $\mu^X(E) = \mu(E \times Y)$ , for every  $E \subseteq X$ .

To state the following result, recall the notation in (2.1), and let  $\mathcal{M} := \mathcal{M}_1 \equiv \mathcal{H}_{(0,1)}^{(1,1)}$ .

**Lemma 4.2.** ([14, Proposition 3.10]) (*Subnormal backward extension*) *Let  $\mathbf{T} \equiv (T_1, T_2)$  be a 2-variable weighted shift, and assume that  $\mathbf{T}|_{\mathcal{M}}$  is subnormal with associated measure  $\mu_{\mathcal{M}}$  and that  $W_0 := \text{shift}(\alpha_{00}, \alpha_{10}, \dots)$  is subnormal with associated measure  $\sigma$ . Then  $\mathbf{T}$  is subnormal if and only if*

- (i)  $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$ ;
- (ii)  $\beta_{00}^2 \leq (\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})})^{-1}$ ;
- (iii)  $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \sigma$ .

Moreover, if  $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} = 1$ , then  $(\mu_{\mathcal{M}})_{ext}^X = \sigma$ . In the case when  $\mathbf{T}$  is subnormal, its Berger measure  $\mu$  is given by

$$\begin{aligned} d\mu(s, t) &= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s, t) \\ &+ \left( d\sigma(s) - \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s) \right) d\delta_0(t). \end{aligned} \quad (4.1)$$



## 5. The structure of powers of 2-variable weighted shifts in $\mathcal{TC}$

Consider a 2-variable weighted shift  $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$  (see Figure 1(i)). Since  $T_1$  (resp.  $T_2$ ) is subnormal, we know that  $\text{shift}(\alpha_1, \alpha_2, \alpha_3 \cdots)$  (resp.  $\text{shift}(\beta_1, \beta_2, \beta_3 \cdots)$ ) is subnormal; let  $\xi$  (resp.  $\eta$ ) be its Berger measure. Similarly, let  $\sigma$  (resp.  $\tau$ ) denote the Berger measure of  $\text{shift}(x_0, x_1, x_2, \cdots)$  (resp.  $\text{shift}(y_0, y_1, y_2, \cdots)$ ). Finally, let  $\tau_1$  be the Berger measure of  $\text{shift}(y_1, y_2, y_3, \cdots) \equiv \text{shift}(y_0, y_1, y_2, \cdots)|_{\vee\{e_k : k \geq 1\}}$ . Figure 1 shows the general form of a pair in  $\mathcal{TC}$ , and that it is uniquely determined by the five parameters  $\sigma$ ,  $\tau$ ,  $a$ ,  $\xi$  and  $\eta$ . Thus, in what follows we will identify a pair  $\mathbf{T} \in \mathcal{TC}$  with the 5-tuple  $\langle \sigma, \tau, a, \xi, \eta \rangle$ . We shall also let  $[a, \xi]$  denote the Berger measure of the subnormal unilateral weighted shift  $W$  whose 0-th weight is  $a$  and with  $\xi$  as the Berger measure of  $W|_{\mathcal{L}_1}$ , where  $\mathcal{L}_1 := \vee\{e_k : k \geq 1\}$ . For instance, in Figure 1(i) the Berger measure for the first row is  $[a, \xi]$ , and for the second row is  $\left[\frac{a\beta_1}{y_1}, \xi\right]$ . Finally, we shall let  $z_j \equiv z_j(\eta)$  denote the  $j$ -th weight of the unilateral weighted shift whose Berger measure is  $\eta$ ; that is,  $\text{shift}(z_0, z_1, \cdots)$  has Berger measure  $\eta$ .

**Lemma 5.1.** *Let  $\mathbf{T} \in \mathcal{TC}$ , and let  $m, n \geq 1$ . Then  $\mathbf{T}^{(m,n)} \in \bigoplus \mathcal{TC}$ .*

PROOF. First recall from (2.1) that  $\ell^2(\mathbb{Z}_+^2)$  can be written as an orthogonal direct sum  $\bigoplus_{p=0}^{m-1} \bigoplus_{q=0}^{n-1} \mathcal{H}_{(p,q)}^{(m,n)}$ , where for  $p = 0, 1, \dots, m-1$  and  $q = 0, 1, \dots, n-1$  we have  $\mathcal{H}_{(p,q)}^{(m,n)} := \vee\{e_{(m\ell+p, nk+q)} : k, \ell \geq 0\}$ . Now write  $\mathbf{T} \equiv \langle \sigma, \tau, a, \xi, \eta \rangle$ . We shall establish that

$$\mathbf{T}^{(m,n)} = \bigoplus_{p=0}^{m-1} \bigoplus_{q=0}^{n-1} \left\langle \sigma^{(p,q)}, \tau^{(p,q)}, a^{(p,q)}, \xi^{(p,q)}, \eta^{(p,q)} \right\rangle, \quad (5.1)$$

where  $\langle \sigma^{(p,q)}, \tau^{(p,q)}, a^{(p,q)}, \xi^{(p,q)}, \eta^{(p,q)} \rangle$  is the 5-tuple associated to the restriction of  $\mathbf{T}^{(m,n)}$  to the reducing subspace  $\mathcal{H}_{(p,q)}^{(m,n)}$ . Since  $\mathbf{T}^{(m,n)} = (\mathbf{T}^{(m,1)})^{(1,n)}$ , and since  $\mathbf{T}^{(m,1)} \in \bigoplus \mathcal{TC}$  if and only if  $\mathbf{T}^{(1,m)} \in \bigoplus \mathcal{TC}$ , it suffices to prove (5.1) in the case  $m = 1$ . The proof is simple but a bit tedious, and it entails careful diagram chasing in Figure 3. Visual inspection of that Figure reveals that when  $m = 1$  we have  $\sigma^{(0,0)} = \sigma$ ,  $\sigma^{(0,1)} = [a, \xi]$ ,  $\dots$ ,  $\sigma^{(0,n-1)} = \left[\frac{az_0 \cdots z_{n-3}}{y_1 \cdots y_{n-2}}, \xi\right]$  ( $n \geq 3$ ); moreover,  $\tau^{(0,q)} = \tau_{(n,q)}$ , where the latter notation was introduced in Lemma 2.2. Still looking at Figure 1, we observe that  $a^{(0,q)} = \frac{az_0 \cdots z_{n-2+q}}{y_1 \cdots y_{n-1+q}}$  and that  $\xi^{(0,q)} = \xi$ ,  $\eta^{(0,0)} = \eta_{(n,n-1)}$  and  $\eta^{(0,q)} = (\eta_{(n,q-1)})_1$  ( $q \geq 1$ ). This completes the proof.  $\square$

**Corollary 5.2.** *For  $m, n \geq 1$  we have  $(\bigoplus \mathcal{TC})^{(m,n)} \subseteq \bigoplus \mathcal{TC}$ .*

We now restate the main result in [8]. First, recall that if  $\tau$  is the Berger measure of  $\text{shift}(y_0, y_1, \cdots)$ , we denote by  $\tau_1$  the Berger measure of  $\text{shift}(y_1, y_2, \cdots)$ . As described in Lemma 2.1, we know that  $d\tau_1(t) \equiv \frac{t}{y_0^2} d\tau(t)$ .

**Lemma 5.3.** (*[8, Theorem 2.3]*) Let  $\mathbf{T} \equiv \langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$  be as in Figure 2(i) and let

$$\begin{aligned} \psi &: = \tau_1 - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \eta \\ \varphi &: = \sigma - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 - a^2 y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\eta)} \frac{\xi}{s}, \end{aligned} \quad (5.2)$$

where  $y_0 \equiv \beta_{00} := \sqrt{\int t \, d\tau(t)}$ . Then  $\mathbf{T}$  is subnormal if and only if  $\psi \geq 0$  and  $\varphi \geq 0$ .

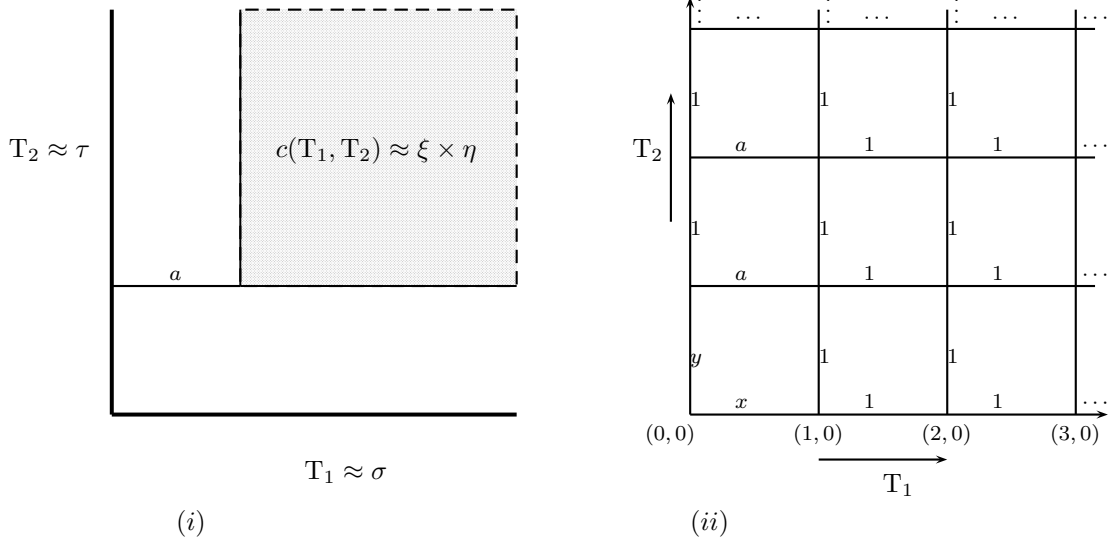


Figure 2: Berger measure diagram and weight diagram of the 2-variable weighted shifts in Lemma 5.3 and Example 5.4, respectively.

**Example 5.4.** Assume the very simple case of  $\mathbf{T} \equiv \langle \sigma, \tau, a, \xi, \eta \rangle$ , where  $\sigma := [x, \delta_1]$ ,  $\tau := [y, \delta_1]$ ,  $0 < a < 1$ ,  $\xi := \delta_1$  and  $\eta := \delta_1$  (cf. Figure 2(ii)). Then  $\psi = \delta_1 - a^2 \delta_1 = (1 - a^2) \delta_1$  and  $\varphi = (1 - x^2) \delta_0 + x^2 \delta_1 - y^2 (1 - a^2) \delta_0 - a^2 y^2 \delta_1 = \{(1 - x^2) - y^2 (1 - a^2)\} \delta_0 + (x^2 - a^2 y^2) \delta_1$ . Thus,  $\mathbf{T}$  is subnormal if and only if  $(1 - x^2) - y^2 (1 - a^2) \geq 0$ , a condition identical to that in [14, Proposition 2.11].

## 6. The pairs $(\psi, \varphi)$ for $\langle \sigma, \tau, a, \xi, \eta \rangle$ and $\langle \sigma^{(p,q)}, \tau^{(p,q)}, a^{(p,q)}, \xi^{(p,q)}, \eta^{(p,q)} \rangle$

In this section we establish a direct relationship between the pair  $(\psi, \varphi)$  associated to  $\langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$  and some of the pairs  $(\psi^{(p,q)}, \varphi^{(p,q)})$  associated to the direct summands  $\langle \sigma^{(p,q)}, \tau^{(p,q)}, a^{(p,q)}, \xi^{(p,q)}, \eta^{(p,q)} \rangle$  in  $\langle \sigma, \tau, a, \xi, \eta \rangle^{(m,n)}$ .

**Proposition 6.1.** Let  $\langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$ , and let  $n \geq 2$ . Consider the decomposition  $\langle \sigma, \tau, a, \xi, \eta \rangle^{(1,n)} = \bigoplus_{q=0}^{n-1} \langle \sigma^{(0,q)}, \tau^{(0,q)}, a^{(0,q)}, \xi^{(0,q)}, \eta^{(0,q)} \rangle$ , and let  $(\psi, \varphi)$  (resp.  $(\psi^{(0,q)}, \varphi^{(0,q)})$ ) be the associated pair in Lemma 5.3. Then  
(i)  $\psi^{(0,1)} \geq 0 \iff \psi \geq 0$ ; and  
(ii)  $\varphi^{(0,0)} \geq 0 \iff \varphi \geq 0$ .

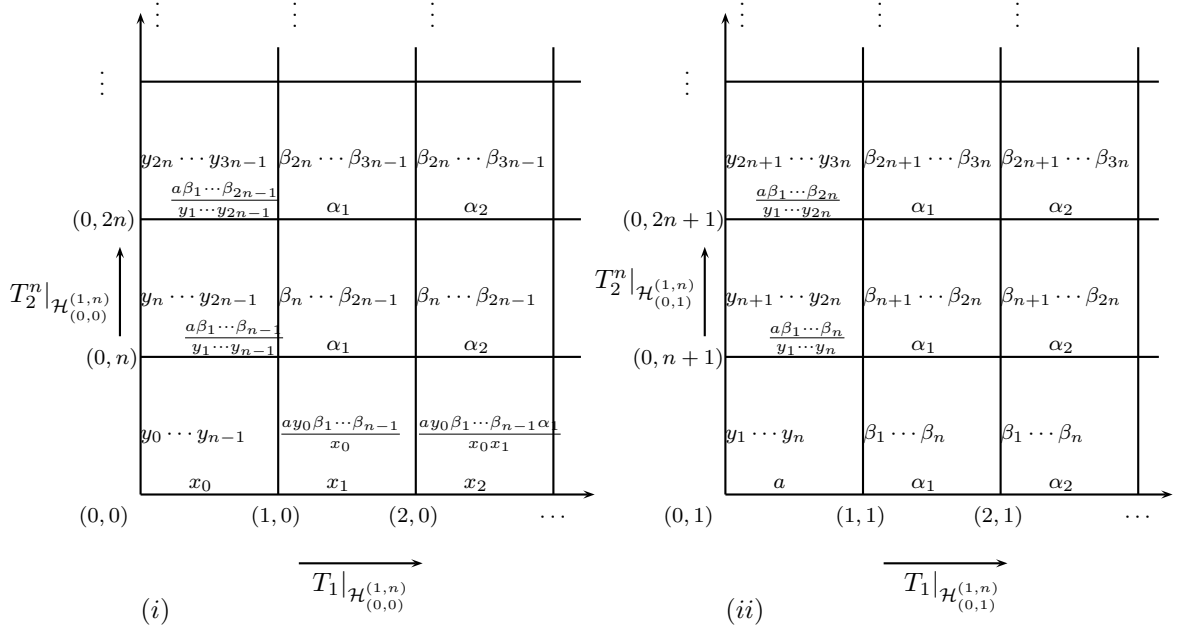


Figure 3: Weight diagrams of  $\mathbf{T}^{(1,n)}|_{\mathcal{H}_{(0,0)}^{(1,n)}}$  and  $\mathbf{T}^{(1,n)}|_{\mathcal{H}_{(0,1)}^{(1,n)}}$ .

PROOF. We refer the reader to Figure 3. By Lemma 5.1, we have  $\sigma^{(0,0)} = \sigma$ ,  $\sigma^{(0,1)} = [a, \xi]$ ,  $\tau^{(0,0)} = \tau_{(n,0)}$ ,  $\tau^{(0,1)} = \tau_{(n,1)}$ ,  $a^{(0,0)} = \frac{az_0 \cdots z_{n-2}}{y_1 \cdots y_{n-1}}$ ,  $a^{(0,1)} = \frac{az_0 \cdots z_{n-1}}{y_1 \cdots y_n}$ ,  $\xi^{(0,0)} = \xi^{(0,1)} = \xi$ ,  $\eta^{(0,0)} = \eta_{(n,n-1)}$  and  $\eta^{(0,1)} = (\eta_{(n,0)})_1$ . Then

$$\begin{aligned} \psi^{(0,1)} &= (\tau^{(0,1)})_1 - (a^{(0,1)})^2 \left\| \frac{1}{s} \right\|_{L^1(\xi^{(0,1)})} \eta^{(0,1)} \\ &= (\tau_{(n,1)})_1 - \frac{a^2 z_0^2 \cdots z_{n-1}^2}{y_1^2 \cdots y_n^2} \left\| \frac{1}{s} \right\|_{L^1(\xi)} (\eta_{(n,0)})_1. \end{aligned} \quad (6.1)$$

We now calculate  $(\tau_{(n,1)})_1$  and  $(\eta_{(n,0)})_1$ . From Lemma 2.2 we know that

$$d\tau_{(n,1)}(t) = \frac{t^{\frac{1}{n}}}{y_0^2} d\tau(t^{\frac{1}{n}}),$$

so that

$$d(\tau_{(n,1)})_1(t) = \frac{t}{y_1^2 \cdots y_n^2} d\tau_{(n,1)}(t) = \frac{t^{1+\frac{1}{n}}}{y_0^2 y_1^2 \cdots y_n^2} d\tau(t^{\frac{1}{n}}) \text{ (by Lemma 2.1).}$$

On the other hand, and again using Lemma 2.2, we have

$$d\eta_{(n,0)}(t) = d\eta(t^{\frac{1}{n}}),$$

so that

$$d(\eta_{(n,0)})_1(t) = \frac{t}{z_0^2 \cdots z_{n-1}^2} d\eta_{(n,0)}(t) = \frac{t}{z_0^2 \cdots z_{n-1}^2} d\eta(t^{\frac{1}{n}}) \text{ (again by Lemma 2.1).}$$

It follows from (6.1) that

$$\begin{aligned} d\psi^{(0,1)}(t) &= \frac{t^{1+\frac{1}{n}}}{y_0^2 y_1^2 \cdots y_n^2} d\tau(t^{\frac{1}{n}}) - \frac{a^2 z_0^2 \cdots z_{n-1}^2}{y_1^2 \cdots y_n^2} \left\| \frac{1}{s} \right\|_{L^1(\xi)} \cdot \frac{t}{z_0^2 \cdots z_{n-1}^2} d\eta(t^{\frac{1}{n}}) \\ &= \frac{t}{y_1^2 \cdots y_n^2} \left\{ \frac{t^{\frac{1}{n}}}{y_0^2} d\tau(t^{\frac{1}{n}}) - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} d\eta(t^{\frac{1}{n}}) \right\} \\ &= \frac{t}{y_1^2 \cdots y_n^2} \left\{ d\tau_1(t^{\frac{1}{n}}) - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} d\eta(t^{\frac{1}{n}}) \right\} \\ &= \frac{t}{y_1^2 \cdots y_n^2} d\psi(t^{\frac{1}{n}}). \end{aligned}$$

It now readily follows that  $\psi^{(0,1)} \geq 0$  if and only if  $\psi \geq 0$ , which establishes (i).

To prove (ii), we begin by calculating  $\psi^{(0,0)}$ . As in (6.1), we have

$$\begin{aligned} d\psi^{(0,0)}(t) &= d(\tau^{(0,0)})_1(t) - (a^{(0,0)})^2 \left\| \frac{1}{s} \right\|_{L^1(\xi^{(0,0)})} d\eta^{(0,0)}(t) \\ &= d(\tau_{(n,0)})_1(t) - \frac{a^2 z_0^2 \cdots z_{n-2}^2}{y_1^2 \cdots y_{n-1}^2} \left\| \frac{1}{s} \right\|_{L^1(\xi)} d\eta_{(n,n-1)}(t) \\ &= \frac{t}{y_0^2 \cdots y_{n-1}^2} d\tau(t^{\frac{1}{n}}) - \frac{a^2 z_0^2 \cdots z_{n-2}^2}{y_1^2 \cdots y_{n-1}^2} \left\| \frac{1}{s} \right\|_{L^1(\xi)} \frac{t^{\frac{n-1}{n}}}{z_0^2 \cdots z_{n-2}^2} d\eta(t^{\frac{1}{n}}) \\ &\quad \text{(by Lemmas 2.1 and 2.2)} \\ &= \frac{1}{y_0^2 y_1^2 \cdots y_{n-1}^2} \left\{ t d\tau(t^{\frac{1}{n}}) - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} t^{\frac{n-1}{n}} d\eta(t^{\frac{1}{n}}) \right\}. \end{aligned}$$

It follows that

$$\begin{aligned} y_0^2 y_1^2 \cdots y_{n-1}^2 \int \frac{1}{t} d\psi^{(0,0)}(t) &= \int d\tau(t^{\frac{1}{n}}) - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \int t^{-\frac{1}{n}} d\eta(t^{\frac{1}{n}}) \\ &= 1 - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
y_0^2 \int \frac{1}{t} d\psi(t) &= y_0^2 \left\{ \int \frac{1}{t} d\tau_1(t) - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \int \frac{1}{t} d\eta(t) \right\} \\
&= y_0^2 \int \frac{1}{t} \cdot \frac{t}{y_0^2} d\tau(t) - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \\
&= 1 - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)}.
\end{aligned}$$

Thus,

$$y_0^2 y_1^2 \cdots y_{n-1}^2 \left\| \frac{1}{t} \right\|_{L^1(\psi^{(0,0)})} = y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)}. \quad (6.2)$$

Consider now

$$\varphi^{(0,0)} = \sigma^{(0,0)} - y_0^2 \cdots y_{n-1}^2 \left\| \frac{1}{t} \right\|_{L^1(\psi^{(0,0)})} \delta_0 - (a^{(0,0)})^2 y_0^2 \cdots y_{n-1}^2 \left\| \frac{1}{t} \right\|_{L^1(\eta^{(0,0)})} \frac{\xi^{(0,0)}}{s}.$$

We know that  $\sigma^{(0,0)} = \sigma$ , that  $a^{(0,0)} = \frac{az_0 \cdots z_{n-2}}{y_1 \cdots y_{n-1}}$  and that  $\xi^{(0,0)} = \xi$ , so using (6.2) we obtain

$$\varphi^{(0,0)} = \sigma - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 - a^2 y_0^2 z_0^2 \cdots z_{n-2}^2 \left\| \frac{1}{t} \right\|_{L^1(\eta^{(0,0)})} \frac{\xi}{s}.$$

Since  $\varphi = \sigma - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 - a^2 y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\eta)} \frac{\xi}{s}$ , it is easy to see that it suffices to prove that  $z_0^2 \cdots z_{n-2}^2 \left\| \frac{1}{t} \right\|_{L^1(\eta^{(0,0)})} = \left\| \frac{1}{t} \right\|_{L^1(\eta)}$ . We know that  $\eta^{(0,0)} = \eta_{(n,n-1)}$ , so

$$\begin{aligned}
z_0^2 \cdots z_{n-2}^2 \left\| \frac{1}{t} \right\|_{L^1(\eta^{(0,0)})} &= z_0^2 \cdots z_{n-2}^2 \int \frac{1}{t} d\eta_{(n,n-1)}(t) \\
&= z_0^2 \cdots z_{n-2}^2 \int \frac{1}{t} \frac{t^{\frac{n-1}{n}}}{z_0^2 \cdots z_{n-2}^2} d\eta(t^{\frac{1}{n}}) \\
&= \int t^{-\frac{1}{n}} d\eta(t^{\frac{1}{n}}) = \left\| \frac{1}{t} \right\|_{L^1(\eta)},
\end{aligned}$$

as desired.  $\square$

**Corollary 6.2.** *Let  $\langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$ , and let  $n \geq 2$ . Assume that  $\langle \sigma, \tau, a, \xi, \eta \rangle^{(1,n)}$  is subnormal. Then  $\langle \sigma, \tau, a, \xi, \eta \rangle$  is subnormal.*

PROOF. Assume that  $\langle \sigma, \tau, a, \xi, \eta \rangle^{(1,n)}$  is subnormal, and recall that the power of a 2-variable weighted shift splits as an orthogonal direct sum of 2-variable weighted shifts. Moreover, each summand is in  $\mathcal{TC}$  (because  $\langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$ ). The fact that  $\langle \sigma, \tau, a, \xi, \eta \rangle^{(1,n)}$  is subnormal readily implies that each direct summand is subnormal, and then Lemma 5.3 says that  $\psi^{(0,q)} \geq 0$  and  $\varphi^{(0,q)} \geq 0$  (all  $q \geq 0$ ). In particular,  $\psi^{(0,1)} \geq 0$  and  $\varphi^{(0,0)} \geq 0$ . It follows from Proposition 6.1 that  $\psi \geq 0$  and  $\varphi \geq 0$ . Applying Lemma 5.3 once again, we see that  $\langle \sigma, \tau, a, \xi, \eta \rangle$  is subnormal.  $\square$

**Corollary 6.3.** *Let  $\langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$ , and let  $m \geq 2$ . Assume that  $\langle \sigma, \tau, a, \xi, \eta \rangle^{(m,1)}$  is subnormal. Then  $\langle \sigma, \tau, a, \xi, \eta \rangle$  is subnormal.*

## 7. Proof of the main theorem

We are now ready to prove our main result, which we restate for the reader's convenience.

**Theorem 7.1.** *Let  $\mathbf{T} \in \mathcal{TC}$ . The following statements are equivalent.*

- (i)  $\mathbf{T} \in \mathfrak{H}_\infty$ ;
- (ii)  $\mathbf{T}^{(m,n)} \in \bigoplus \mathfrak{H}_\infty$  for all  $m, n \geq 1$ ;
- (iii)  $\mathbf{T}^{(m,n)} \in \bigoplus \mathfrak{H}_\infty$  for some  $m, n \geq 1$ .

PROOF. It is clear that (i)  $\implies$  (ii) and that (ii)  $\implies$  (iii). Assume that (iii) holds, with  $n \geq 2$ . Since  $\mathbf{T}^{(m,n)} = (\mathbf{T}^{(m,1)})^{(1,n)}$ , we can use Corollary 6.2 to conclude that  $\mathbf{T}^{(m,1)}$  is subnormal. If we now apply Corollary 6.3, we obtain that  $\mathbf{T}$  is subnormal, as desired.  $\square$

## 8. An application

In our previous work ([14], [15], [16], [17], [6], [7], [8], [9], [26], [27]), we have shown that there are many different families of commuting pairs of subnormal operators, jointly hyponormal but not admitting commuting normal extensions, that is,  $\mathbf{T} \in \mathfrak{H}_1$  but  $\mathbf{T} \notin \mathfrak{H}_\infty$  (all  $m, n \geq 1$ ). As a simple application of Theorem 7.1, we now show that  $\mathfrak{H}_1 \cap \mathcal{TC} \neq \mathfrak{H}_\infty \cap \mathcal{TC}$ ; moreover, there exists  $\mathbf{T} \in \mathcal{TC}$ , such that  $\mathbf{T} \in \mathfrak{H}_1$  but  $\mathbf{T}^{(m,n)} \notin \bigoplus \mathfrak{H}_\infty$  (all  $m, n \geq 1$ ). We recall that  $\text{shift}(x_0, x_1, \dots)$  and  $\text{shift}(y_0, y_1, \dots)$  are subnormal unilateral weighted shifts with Berger measures  $\sigma$  and  $\tau$ , respectively. Consider a contractive 2-variable weighted shift  $\mathbf{T} \in \mathfrak{H}_0$  whose weight diagram is given by Figure 4(i); that is, in the 5-tuple  $\langle \sigma, \tau, a, \xi, \eta \rangle$ , we have

- \*  $d\sigma(t) := (1 - \kappa^2)d\delta_0(t) + \frac{\kappa^2}{2}dt + \frac{\kappa^2}{2}d\delta_1(t)$ ,
- \*  $\tau$  is the Berger measure of  $\text{shift}(y_0, y_1, \dots)$ , with  $\tau_1$  the 2-atomic Berger measure  $\rho_0\delta_{t_0} + \rho_1\delta_{t_1}$  of the Stampfli subnormal completion of  $\sqrt{\omega_0} < \sqrt{\omega_1} < \sqrt{\omega_2}$ ,
- \*  $a$  is a positive number,
- \*  $\xi := \delta_1$ , and
- \*  $\eta := \delta_1$ .

**Example 8.1.** *Let  $\mathbf{T} \equiv \langle \sigma, \tau, a, \xi, \eta \rangle$  be the 2-variable weighted shift given by Figure 4(i), with  $\sigma, \tau_1, a, \xi$  and  $\eta$  as above. Then  $\mathbf{T} \in \mathfrak{H}_1$  and  $\mathbf{T}^{(m,n)} \notin \mathfrak{H}_\infty$  (all  $m, n \geq 1$ ) if and only if  $s(\kappa) < y_0 < h(\kappa)$ , where*

$$s(\kappa) := \min \left\{ \frac{\sqrt{t_1}}{a} \sqrt{\rho_1}, \sqrt{\frac{(1-\kappa^2)}{\|\frac{1}{t}\|_{L^1(\tau_1)} - \frac{a^2}{t_1}}}, \frac{\sqrt{t_1}}{a} \sqrt{\frac{\kappa^2}{2}}, \sqrt{\frac{1}{\|\frac{1}{t}\|_{L^1(\tau_1)}}} \right\}$$

and

$$h(\kappa) := \sqrt{\frac{x_0^2 y_1^2 (x_1^2 - x_0^2)}{x_0^2 (x_1^2 - x_0^2) + (a^2 - x_0^2)^2}}.$$

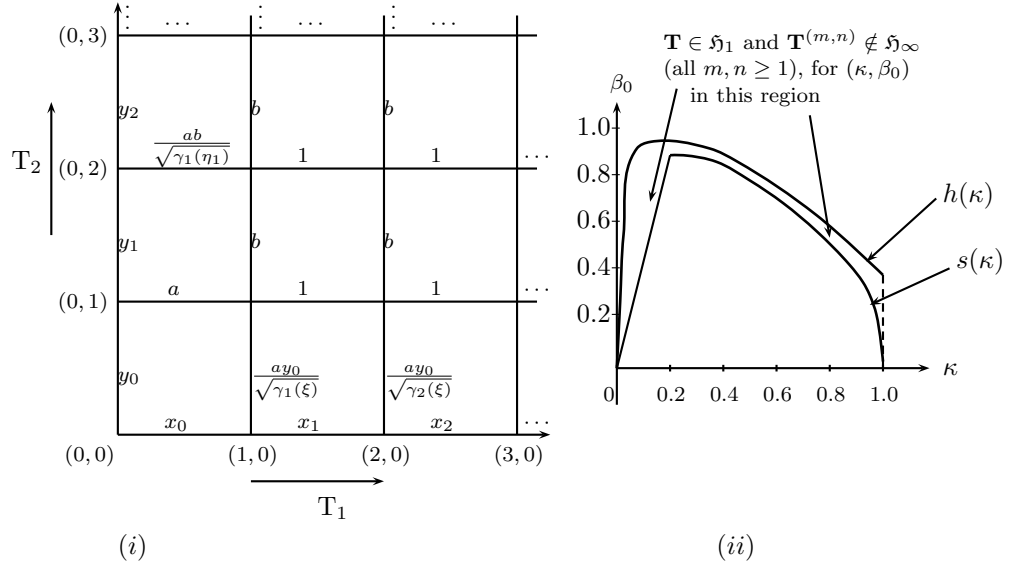


Figure 4: Weight diagram of the 2-variable weighted shift in Example 8.1 and graphs of  $s(\kappa)$  and  $h(\kappa)$  on the interval  $[0, 1]$ , respectively; here  $\omega_0 := \frac{5}{6}$ ,  $\omega_1 := \frac{6}{7}$ ,  $\omega_2 := \frac{7}{8}$  and  $y_0 := \beta_0$ .

Figure 4(ii) specifies a region in the  $(\kappa, \beta_0)$  plane where  $\mathbf{T}$  is hyponormal but none of its powers is subnormal. A detailed analysis of Example 8.1 and of other applications of Theorem 3.1 will be discussed elsewhere.

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