

POLYNOMIAL EMBEDDINGS OF UNILATERAL WEIGHTED SHIFTS INTO 2-VARIABLE WEIGHTED SHIFTS

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ABSTRACT. Given a bounded sequence ω of positive numbers, and its associated unilateral weighted shift W_ω acting on the Hilbert space $\ell^2(\mathbb{Z}_+)$, we consider natural representations of W_ω as a 2-variable weighted shift, acting on the Hilbert space $\ell^2(\mathbb{Z}_+^2)$. Alternatively, we seek to examine the various ways in which the sequence ω can give rise to a 2-variable weight diagram, corresponding to a 2-variable weighted shift. Our best (and more general) embedding arises from looking at two polynomials p and q nonnegative on a closed interval $I \subseteq \mathbb{R}_+$ and the double-indexed moment sequence $\{\int p(r)^k q(r)^\ell d\sigma(r)\}_{k,\ell \in \mathbb{Z}_+}$, where W_ω is assumed to be subnormal with Berger measure σ such that $\text{supp } \sigma \subseteq I$; we call such an embedding a (p, q) -embedding of W_ω . We prove that every (p, q) -embedding of a subnormal weighted shift W_ω is (jointly) subnormal, and we explicitly compute its Berger measure. We apply this result to answer three outstanding questions: (i) Can the Bergman shift A_2 be embedded in a subnormal 2-variable spherically isometric weighted shift $W_{(\alpha,\beta)}$? If so, what is the Berger measure of $W_{(\alpha,\beta)}$? (ii) Can a contractive subnormal unilateral weighted shift be always embedded in a spherically isometric 2-variable weighted shift? (iii) Does there exist a (jointly) hyponormal 2-variable weighted shift $\Theta(W_\omega)$ (where $\Theta(W_\omega)$ denotes the classical embedding of a hyponormal unilateral weighted shift W_ω) such that some integer power of $\Theta(W_\omega)$ is not hyponormal? As another application, we find an alternative way to compute the Berger measure of the Agler j -th shift A_j ($j \geq 2$). Our research uses techniques from the theory of disintegration of measures, Riesz functionals, and the functional calculus for the columns of the moment matrix associated to a polynomial embedding.

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1. INTRODUCTION

Given a bounded sequence ω of positive numbers, and its associated unilateral weighted shift W_ω acting on the Hilbert space $\ell^2(\mathbb{Z}_+)$, we consider natural representations of W_ω as a 2-variable weighted shift, acting on the Hilbert space $\ell^2(\mathbb{Z}_+^2)$. Alternatively, we seek to examine the various ways in which the sequence ω can give rise to a 2-variable weight diagram, corresponding to a 2-variable weighted shift $W_{(\alpha,\beta)}$.¹ Our best (and more general) embedding arises from looking at two polynomials p and q nonnegative on a closed interval $I \subseteq \mathbb{R}_+$ and the double-indexed moment sequence $\{\int p(r)^k q(r)^\ell d\sigma(r)\}_{k,\ell \in \mathbb{Z}_+}$, where W_ω is assumed to be subnormal with Berger measure σ such that $\text{supp } \sigma \subseteq I$; we call such an embedding a (p, q) -embedding of W_ω , and denoted as $W_{(\alpha(\omega), \beta(\omega))}$. When $p(r) \equiv q(r) := r$ ($r \geq 0$), we recover the *classical embedding* discussed in [13] and [19]; when $p(r) := r$ and $q(r) := 1 - r$ ($r \in [0, 1]$), we obtain the 2-variable *spherically isometric embedding*, studied in [14], [19] and [20] (here $\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1$ for all $\mathbf{k} \in \mathbb{Z}_+^2$); and when $p(r) := r^2$ and $q(r) := r^3$ we generate a *Neil parabolic embedding*.

We first prove that every (p, q) -embedding of a subnormal weighted shift W_ω is (jointly) subnormal, and we explicitly compute its Berger measure. Next, we use polynomial embeddings to give answers to several open problems regarding spherically quasinormal 2-variable weighted shifts constructed from W_ω . We do these by appealing to techniques from the theory of disintegration of measures, Riesz functionals, and the functional calculus for the columns of the moment matrix associated to a polynomial embedding.

In a different direction, we recall that for a unilateral weighted shift W_ω it is well known that the hyponormality of W_ω implies the hyponormality of every integer power W_ω^m ($m \geq 1$). We use our theory of polynomial embeddings to build a (jointly) hyponormal 2-variable weighted shift $\Theta(W_\omega)$ such that some integer power of it is not hyponormal. As another application, we find an alternative way to compute the Berger measure of the Agler j -th shift A_j ($j \geq 2$).

¹ Hereafter, \mathbb{Z}_+ denotes the set of nonnegative integers $0, 1, 2, 3, \dots$.

One of the by-products of our research is the fact that some questions pertaining to unilateral weighted shifts (and therefore within the realm of single-variable operator theory) can be answered after rephrasing them in the context of (p, q) -embeddings, a topic that rightly belongs in multivariable operator theory.

The organization of the paper is as follows. In Section 2 we discuss spherical quasinormality, both for commuting pairs of operators and more specifically for 2-variable weighted shifts. These topics require basic knowledge of unilateral and 2-variable weighted shifts, and of the basic construction of spherically isometric 2-variable weighted shifts; the interested reader will find a summary of each such topic in the Appendix (Section 7). In Section 3 we introduce polynomial embeddings and we prove one of our main results: every polynomial embedding of a subnormal unilateral weighted shift is subnormal. Section 4 is devoted to the specific case of the classical embedding. We then use the Neil parabolic embedding $r \mapsto (r^2, r^3)$ to study some examples linking the k -hyponormality of W_ω to the k -hyponormality of the $(2, 3)$ -power of $\Theta(W_\omega)$. In particular, we use polynomial embeddings to give an alternative proof that hyponormality of 2-variable weighted shifts is not preserved under powers (cf. [12], [14]). Along the way, we exhibit the connection between the respective Berger measures. In Section 5 we find a concrete formula for the Berger measure of the spherically isometric embedding of the Bergman shift; this solves Question 5.15 in [20]. In Section 6, we first give an alternative solution to Problem 1.1(ii) that employs techniques from the theory of recursively generated weighted shifts. We conclude Section 6 with two examples that illustrate our solution to Problem 1.1.

We now present a precise formulation of the central questions of this paper.

Problem 1.1. [20] *Consider a spherically quasinormal 2-variable weighted shift $W_{(\alpha, \beta)}$ and let σ be the Berger measure of $W_0 := \text{shift}(\alpha_{(0,0)}, \alpha_{(1,0)}, \dots)$. Since $W_{(\alpha, \beta)}$ is subnormal (by [20, Theorem 3.10]), let μ be its Berger measure.*

(i) *Describe μ in terms of σ .*

(ii) *Assume that W_0 is recursively generated; that is, σ is finitely atomic. It is known that*

μ is also finitely atomic, and that $\text{supp } \mu \subseteq \text{supp } \sigma \times (c - \text{supp } \sigma)$, where $c > 0$ is the constant of spherical quasynormality [20, Theorem 5.2]. What else can we say? Can we give a concrete formula for the atoms and densities of μ ?

Problem 1.2. [20] Assume that W_0 is the Bergman shift $B_+ := \text{shift}(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \dots)$, and use Subsection 2.3 to build a spherically quasynormal 2-variable weighted shift $W_{(\alpha, \beta)}$ (cf. Figure 1). Observe that the j -th row is identical to the j -column, for every $j \geq 0$. Note also that $W_{(\alpha, \beta)}$ is a close relative of the Drury-Arveson 2-variable weighted shift, in that the j -row of $W_{(\alpha, \beta)}$ is the Agler $(j + 2)$ -th shift A_{j+2} . What is the Berger measure of $W_{(\alpha, \beta)}$?

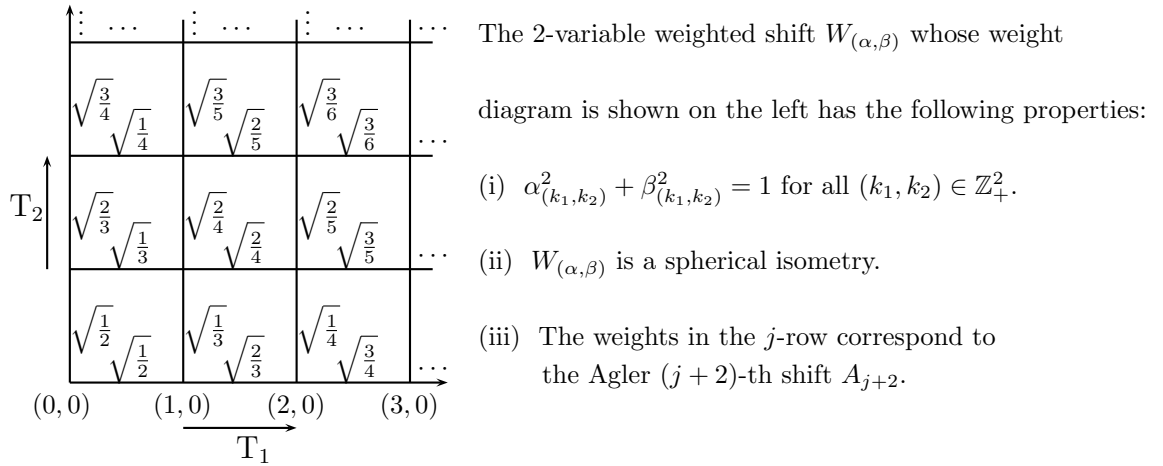


FIGURE 1. Weight diagram of the 2-variable weighted shift in Problem 1.2.

It is now natural to ask:

Problem 1.3. Given a subnormal unilateral weighted shift W_ω , can we always construct a spherically quasynormal 2-variable weighted shifts $W_{(\alpha(\omega), \beta(\omega))}$? In those instances when this is possible, we shall use the notation $\mathcal{SQNE}(\omega)$ for the spherically quasynormal embedding.

For the next question, recall that a subnormal unilateral weighted shift is recursively generated if and only if its Berger measure is finitely atomic (cf. Subsection 7.2 in the Appendix).

Problem 1.4. (i) Let W_ω be a recursively generated weighted shift, and let m be a positive integer. Is W_ω^m a direct sum of recursively generated weighted shifts?

(ii) Conversely, assume that W_ω^m is a direct sum of recursively generated weighted shifts for some $m \geq 1$. Is W_ω a recursively generated weighted shift?

We will prove that Problem 1.4 (i) has a positive answer, but (ii) does not. In fact, W_ω may not even be subnormal. We can then ask, under what conditions does Problem 1.4(ii) yield a positive answer?

For a subnormal 2-variable weighted shift $W_{(\alpha,\beta)}$ it is natural to say that it is *recursively generated* if its Berger measure is finitely atomic. We shall see in Theorem 1.8 that the property of being recursively generated transfers from W_ω to any (p, q) -embedding. We plan to build on this fact and study recursively generated 2-variable weighted shifts in detail in a forthcoming paper.

Problem 1.5. Let W_ω be a hyponormal unilateral weighted shift, and let $W_{\theta(\omega)} \equiv \Theta(W_\omega)$ denote the 2-variable weighted shift obtained via the classical embedding, that is, the (p, q) -embedding with $p(r) \equiv q(r) := r$ ($r \in [0, \|W_\omega\|^2]$). Also, let m and n be two positive integers, and consider the power $W_{\theta(\omega)}^{(m,n)}$. Is $W_{\theta(\omega)}^{(m,n)}$ hyponormal?

(Hereafter, we will ordinarily use the notation $W_{\theta(\omega)}$, for three reasons: (i) it easily signals the action of the classical embedding on 1-variable sequences; (ii) it more clearly denotes that it is a weighted shift; and (iii) it is more concise that allows for a succinct display of the powers.)

We conclude this section by stating our main results, which provide answers to the above mentioned problems.

Theorem 1.6. Let $W_{(\alpha(\omega),\beta(\omega))}$ be a (p, q) -embedding of W_ω , and let $f(r) := (p(r), q(r))$. If W_ω is subnormal with Berger measure σ such that $p|_{\text{supp } \sigma} \geq 0$ and $q|_{\text{supp } \sigma} \geq 0$, then $W_{(\alpha(\omega),\beta(\omega))}$ is subnormal, with Berger measure $\mu := \sigma \circ f^{-1}$.

Theorem 1.7. *The Bergman shift B_+ admits a spherically isometric embedding whose Berger measure is normalized arclength on the line segment $\{(s, t) \in \mathbb{R}_+^2 : s + t = 1\}$.*

Theorem 1.8. *Let $W_{(\alpha(\omega), \beta(\omega))}$ be a (p, q) -embedding of W_ω , where W_ω is subnormal with finitely atomic Berger measure $\sigma = \sum_{i=0}^{\ell-1} \rho_i \delta_{r_i}$, $\sum_{i=0}^{\ell-1} \rho_i = 1$, $\rho_i > 0$, and $0 \leq r_0 < r_1 < \dots < r_{\ell-1}$. Then, the Berger measure μ of $W_{(\alpha(\omega), \beta(\omega))}$ is*

$$\mu = \sum_{i=0}^{\ell-1} \rho_i \delta_{(p(r_i), q(r_i))}. \quad (1.1)$$

As a consequence, we can generalize Theorem 1.7, as follows.

Theorem 1.9. *Let $W_{(\alpha(\omega), \beta(\omega))}$ be a (p, q) -embedding of a subnormal weighted shift W_ω , and denote by μ and σ the Berger measures of $W_{(\alpha(\omega), \beta(\omega))}$ and W_ω , respectively. Assume that σ is the Lebesgue measure on $[0, 1]$. Then $d\mu(p(r), q(r))$ is normalized arclength on the curve $C := \{(p(r), q(r)) : 0 \leq r \leq 1\}$.*

Related to Problem 1.4, we have:

Theorem 1.10. *If W_ω is subnormal and if W_ω^m is a direct sum of recursively generated weighted shifts, then W_ω is a recursively generated weighted shift.*

2. NOTATION AND PRELIMINARIES

2.1. Spherically quasinormal n -tuples of operators. Let \mathcal{H} be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on \mathcal{H} . We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *quasinormal* if T commutes with T^*T , i.e., $TT^*T = T^*T^2$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$.

For the 1-variable case, we clearly have that

$$\text{normal} \implies \text{quasinormal} \implies \text{subnormal} \implies \text{hyponormal}.$$

For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S, T] := ST - TS$. We say that an n -tuple $\mathbf{T} \equiv (T_1, \dots, T_n)$ of operators on \mathcal{H} is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & \cdots & [T_n^*, T_1] \\ \vdots & \ddots & \vdots \\ [T_1^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of \mathcal{H} (cf. [1], [6]). The n -tuple \mathbf{T} is said to be *normal* if \mathbf{T} is commuting and each T_i is normal and *subnormal* if \mathbf{T} is the restriction of a normal n -tuple to a common invariant subspace. A commuting n -tuple \mathbf{T} is *k-hyponormal* if the tuple $(T_1, \dots, T_n, T_1^2, T_1T_2, \dots, T_n^2, \dots, T_1^k, T_1^{k-1}T_2, \dots, T_n^k)$ is jointly hyponormal.

Following A. Athavale, S. Podder [3] and J. Gleason [22], we say that

- (i) \mathbf{T} is *matricially quasinormal* if T_i commutes with $T_j^*T_k$ for all $i, j, k = 1, \dots, n$;
- (ii) \mathbf{T} is *(jointly) quasinormal* if T_i commutes with $T_j^*T_j$ for all $i, j = 1, \dots, n$; and
- (iii) \mathbf{T} is *spherically quasinormal* if T_i commutes with

$$P := T_1^*T_1 + \cdots + T_n^*T_n,$$

for $i = 1, \dots, n$. Then

$$\begin{aligned} \text{normal} &\implies \text{matricially quasinormal} \implies \text{(jointly) quasinormal} \\ &\implies \text{spherically quasinormal} \implies \text{subnormal ([3, Proposition 2.1])} \\ &\implies \text{k-hyponormal} \implies \text{hyponormal.} \end{aligned} \tag{2.1}$$

On the other hand, results of R.E. Curto, S.H. Lee and J. Yoon (cf. [14]), and of J. Gleason [22] show that the reverse implications in (2.1) do not necessarily hold.

In [20, Theorem 2.2], R.E. Curto and J. Yoon showed that the spherically quasinormal commuting pairs are precisely the fixed points of the spherical Aluthge transform. In [19]

it was also shown that every spherically quasinormal 2-variable weighted shift is a positive multiple of a spherical isometry (see Theorem 2.3). In order to state this result, we need a brief discussion of unilateral and 2-variable weighted shifts; we refer the reader to Subsections 7.1 and 7.3 of the Appendix.

2.2. Spherically Quasinormal 2-variable Weighted Shifts. In this section we present a characterization of spherical quasinormality for 2-variable weighted shifts, which was announced in [18] and proved in [19] and [20]. Before we state it, we list some simple facts about quasinormality for 2-variable weighted shifts.

Remark 2.1. (cf. [14]) We first observe that no 2-variable weighted shift can be matricially quasinormal, as a simple calculation shows. Also, a 2-variable weighted shift $W_{(\alpha,\beta)}$ is (jointly) quasinormal if and only if $\alpha_{(k_1,k_2)} = \alpha_{(0,0)}$ and $\beta_{(k_1,k_2)} = \beta_{(0,0)}$ for all $k_1, k_2 \geq 0$. This can be seen via a simple application of (7.5) and (7.6). As a result, up to a scalar multiple in each component, a quasinormal 2-variable weighted shift is identical to the so-called Helton-Howe shift. This fact is entirely consistent with the one-variable result: a unilateral weighted shift W_ω is quasinormal if and only if $W_\omega = c \cdot U_+$ for some $c > 0$.

We now recall the class of spherically isometric commuting pairs of operators (cf. [2], [3], [22]). A commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is a *spherical isometry* if $T_1^*T_1 + T_2^*T_2 = I$.

Lemma 2.2. [2, Proposition 2] *Any spherical isometry is subnormal.*

Theorem 2.3. ([14, Theorem 3.1]; cf. [19, Lemma 10.3]) *For a commuting 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$, the following statements are equivalent:*

- (i) $W_{(\alpha,\beta)}$ is a spherically quasinormal 2-variable weighted shift;
- (ii) (algebraic condition) $T_1^*T_1 + T_2^*T_2 = cI$, for some constant c ;
- (iii) (weight condition) for all $(k_1, k_2) \in \mathbb{Z}_+^2$, $\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2 = c$, for some constant $c > 0$;
- (iv) (moment condition) for all $(k_1, k_2) \in \mathbb{Z}_+^2$, $\gamma_{(k_1+1,k_2)} + \gamma_{(k_1,k_2+1)} = c\gamma_{(k_1,k_2)}$, for some constant $c > 0$.

Corollary 2.4. *Any spherically quasinormal 2-variable weighted shift is subnormal.*

It is now natural to ask:

Problem 2.5. *Given a subnormal unilateral weighted shift W_ω , can we always construct a spherically quasinormal 2-variable weighted shifts $W_{(\alpha(\omega),\beta(\omega))}$?*

In particular, if $W_\omega = B_+$, the Bergman shift, or if $W_\omega = \text{shift}(a, b, b, \dots)$ with $0 < a < b < 1$, can we construct a spherically quasinormal 2-variable weighted shift $W_{(\alpha(\omega),\beta(\omega))}$? For the case of the Bergman shift, Figure 1 shows the desired spherical isometry. Thus, in the case of the Bergman shift, we can obtain a spherically isometric embedding, which we will denote by $\mathcal{SIE}(\omega)$. While this can be established via a detailed calculation, we shall prove in Section 3 a much more general result, linking concretely the Berger measures of W_ω and $W_{(\alpha(\omega),\beta(\omega))}$. For the second shift, observe that its Berger measure is $(1 - (\frac{a}{b})^2)\delta_0 + (\frac{a}{b})^2\delta_{b^2}$. From this one can show that W_ω does admit a spherically isometric extension, whose Berger measure is $(1 - (\frac{a}{b})^2)\delta_{(0,1)} + (\frac{a}{b})^2\delta_{(b^2,1-b^2)}$.

2.3. Construction of Spherically Quasinormal 2-variable Weighted Shifts. As observed in [20, Section 4], within the class of 2-variable weighted shifts there is a simple description of spherical isometries, in terms of the weight sequences $\alpha \equiv \{\alpha_{(k_1,k_2)}\}$ and $\beta \equiv \{\beta_{(k_1,k_2)}\}$. Indeed, since spherical isometries are (jointly) subnormal, we know that the unilateral weighted shift associated with each row is subnormal, and in particular the 0-th row in the weight diagram corresponds to a contractive subnormal unilateral weighted shift; let us denote its weights by $\omega \equiv \{\alpha_{(k,0)}\}_{k=0,1,2,\dots}$ and its Berger measure by σ . Also, in view of Theorem 2.3 we can assume that $c = 1$. We may also assume, without loss of generality, that ω is strictly increasing. Using the identity

$$\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1 \quad (\mathbf{k} \in \mathbb{Z}_+^2), \quad (2.2)$$

we immediately see that $\beta_{(k,0)} = \sqrt{1 - \alpha_{k,0}^2}$ for $k = 0, 1, 2, \dots$. With these new values at our disposal, we can use the commutativity property (7.5) to generate the values of α in the

first row; that is,

$$\alpha_{(k,1)} = \alpha_{(k,0)}\beta_{(k+1,0)}/\beta_{(k,0)}. \quad (2.3)$$

Let us briefly pause to observe that, starting with the sequence $\{\alpha_{(k,0)}\}_{k \geq 0}$ for the 0-th row, we have been able to generate the sequence $\tau \equiv \{\alpha_{(k,1)}\}_{k \geq 0}$ for the first row. Now, the 0-row sequence came with a Berger measure σ ; does the new sequence have a Berger measure?

Let us look at the moments of τ : $\gamma_0(W_\tau) = 1$ and, for $k \geq 1$,

$$\begin{aligned} \gamma_k(W_\tau) &= \alpha_{(0,1)}^2 \alpha_{(1,1)}^2 \cdots \alpha_{(k-1,1)}^2 \\ &= \frac{1}{\beta_{(0,0)}^2} \alpha_{(0,0)}^2 \alpha_{(1,0)}^2 \cdots \alpha_{(k-1,0)}^2 \beta_{(k,0)}^2 \quad (\text{by (2.3)}) \\ &= \frac{1}{\beta_{(0,0)}^2} \gamma_k(W_\omega) \beta_{(k,0)}^2 \\ &= \frac{1}{\beta_{(0,0)}^2} \gamma_k(W_\omega) (1 - \alpha_{(k,0)}^2) \quad (\text{by (2.2)}) \\ &= \frac{1}{\beta_{(0,0)}^2} (\gamma_k(W_\omega) - \gamma_{k+1}(W_\omega)) \\ &= \frac{1}{\beta_{(0,0)}^2} \left(\int r^k d\sigma(r) - \int r^{k+1} d\sigma(r) \right) \\ &= \frac{1}{\beta_{(0,0)}^2} \int r^k (1 - r) d\sigma(r). \end{aligned}$$

It is now evident that τ does have a Berger measure ζ , given by $d\zeta(r) = \frac{1}{\beta_{(0,0)}^2} (1 - r) d\sigma(r)$. This is entirely consistent with the fact that in any subnormal 2-variable weighted shift with Berger measure $d\mu(s, t)$, the Berger measures ξ_1 and ξ_0 of the first and 0-th rows, respectively, are related by the equation $d\xi_1(s) = \frac{1}{\gamma_{(0,1)}} t \xi_0(s)$. Also, note that for the new sequence, $\alpha_{(k,1)} < \alpha_{(k,0)} < 1$ for all $k \geq 0$.

We can now repeat the algorithm, and calculate the weights $\beta_{(k,1)}$ for $k = 0, 1, 2, \dots$, again using the identity (2.2). This in turn leads to the α weights for the second row, and so on. For more on this construction, the reader is referred to [20]. In particular, it is worth noting that the construction may stall if the sequence $\{\alpha_{(k,0)}\}_{k \geq 0}$ is not strictly increasing (see Proposition 2.6). Moreover, the fact that we can determine concretely the Berger

measures of each row (and each column, if we were to start with the 0-column and apply the algorithm!) does not guarantee the existence of a Berger measure for the 2-variable weighted shift. That comes from either Corollary 2.4 or the more general Theorem 1.6 (proved in Section 3). As we know from [16], it is possible for a commuting 2-variable weighted shift to have rows and columns with mutually absolutely continuous Berger measures, and still not be (jointly) subnormal. Thus, in the case of 2-variable weighted shifts, the Lifting Problem for Commuting Subnormals entails more than the subnormality of rows and columns (see for instance [7], [15], [16]).

Proposition 2.6. (*[7, Proposition 12.14]*) *Let*

$$\alpha_{(0,0)} := \sqrt{p}, \quad \alpha_{(1,0)} := \sqrt{q}, \quad \alpha_{(2,0)} := \sqrt{r} \quad \text{and} \quad \alpha_{(3,0)} := \sqrt{r},$$

and assume that $0 < p < q < r < 1$. Then the algorithm described in this section fails at some stage. As a consequence, there does not exist a spherical isometry interpolating these initial data.

Remark 2.7. (a) In Proposition 2.6 the reader may have noticed that the 0-th row is not subnormal; for, it is well known that, up to a constant, the only subnormal unilateral weighted shifts with two equal weights are U_+ and S_a ([29, Theorem 6]). Thus, save for these two special (trivial) cases, assuming subnormality of the 0-th row will automatically guarantee that $\alpha_{(k,0)}$ is strictly increasing; therefore, in the sequel we will always assume that the 0-th row is subnormal.

(b) While the 0-th row of the shift in Proposition 2.6 is not subnormal (not even quadratically hyponormal!), it is possible to prove that the algorithm also stalls with a 0-th row of the form $\frac{3}{4} < \sqrt{\frac{2}{3}} < \sqrt{\frac{3}{4}} < \sqrt{\frac{4}{5}} < \dots$, which is a rank-one perturbation of B_+ . We shall prove in Theorem 3.5 that the construction algorithm always works when the 0-th row is subnormal.

3. POLYNOMIAL EMBEDDINGS OF UNILATERAL WEIGHTED SHIFTS

Let I be a closed interval in \mathbb{R}_+ . Consider a one-parametric representation of a planar curve C described by a function $f : I \rightarrow \mathbb{R}_+^2$ given by $f(r) \equiv (s, t) := (p(r), q(r))$, where p, q are nonnegative real polynomials on I and f is injective. Assume that W_ω is a subnormal unilateral weighted shift with Berger measure σ and such that $\text{supp } \sigma \subseteq I$, that is,

$$\gamma_k(W_\omega) := \omega_0^2 \omega_1^2 \cdots \omega_{k-1}^2 = \int_I r^k d\sigma(r). \quad (3.1)$$

Definition 3.1. *The polynomial embedding $\mathcal{PE}(\omega; f)$ of W_ω via the pair $f \equiv (p, q)$ is the 2-variable weighted shift $W_{(\alpha(\omega), \beta(\omega))}$ with moments*

$$\gamma_{(k_1, k_2)} := \int_I p(r)^{k_1} q(r)^{k_2} d\sigma(r). \quad (3.2)$$

If we knew that $\mathcal{PE}(\omega; f)$ is subnormal, with Berger measure μ , the moment equations would be

$$\int s^{k_1} t^{k_2} d\mu(s, t) = \gamma_{(k_1, k_2)}[\mu] = \int_I p(r)^{k_1} q(r)^{k_2} d\sigma(r) = \int s^{k_1} t^{k_2} d\sigma(f^{-1}(s, t)),$$

after using the change of variables $s = p(r)$, $t = q(r)$. This gives the heuristics for the following result.

Theorem 3.2. *(cf. Problems 1.1 and 1.3) Let $W_{(\alpha(\omega), \beta(\omega))}$ be a (p, q) -embedding of W_ω . If W_ω is subnormal with Berger measure σ , then $W_{(\alpha(\omega), \beta(\omega))}$ is subnormal, with Berger measure $\mu := \sigma \circ f^{-1}$.*

Proof. Define a probability measure μ on \mathbb{R}_+^2 by $\mu(E) := \sigma(f^{-1}(E))$, for all Borel sets E in \mathbb{R}_+^2 . Now compute the moments of μ and compare them to those in 3.2. □

The Classical Embedding. We now recall the 2-variable weighted shift embedding introduced and studied in [13] and [19]: Given a 1-variable unilateral weighted shift W_ω we

embed ω into $\ell^\infty(\mathbb{Z}_+^2)$ as follows:

$$\alpha_{(k_1, k_2)} \equiv \beta_{(k_1, k_2)} := \omega_{k_1 + k_2} \quad (k_1, k_2 \geq 0). \quad (3.3)$$

We denote the associated 2-variable weighted by $W_{\theta(\omega)}$ (see Figure 2(i)); we will also use the notation $\mathcal{CE}(\omega)$. We also recall that a probability measure ϵ on $X \times X$ is said to be *diagonal* if $\text{supp } \epsilon \subseteq \{(s, s) : s \in X\}$.

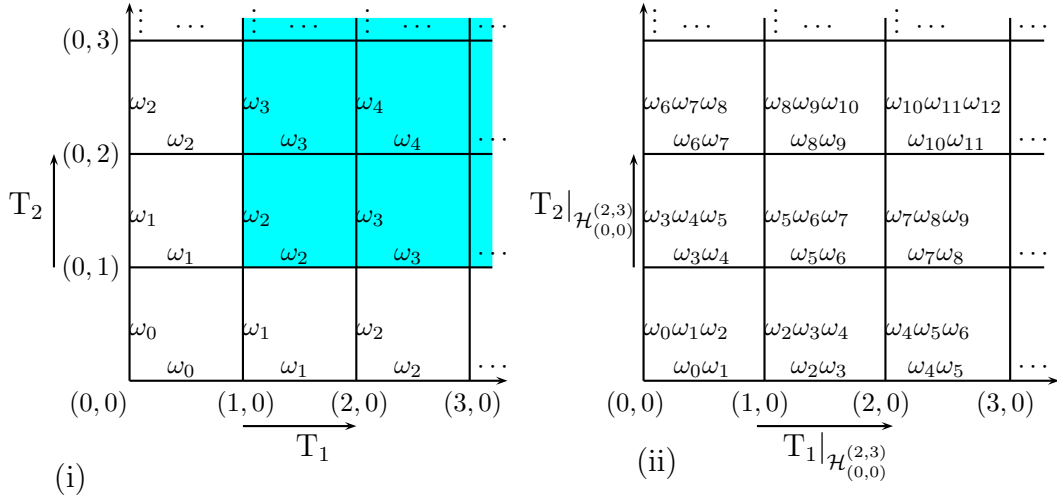


FIGURE 2. (i) Weight diagram of the 2-variable weighted shift $W_{\theta(\omega)}$, highlighting its restriction to $\mathcal{M} \cap \mathcal{N}$ (see page 24 for the definitions of \mathcal{M} and \mathcal{N}); (ii) weight diagram used in Example 3.10.

Lemma 3.3. [13] (a) Let W_ω be a unilateral weighted shift, let $W_{\theta(\omega)}$ be the classical embedding of ω , and let $k \geq 1$. Then

$$W_\omega \text{ is } k\text{-hyponormal if and only if } W_{\theta(\omega)} \text{ is } k\text{-hyponormal.} \quad (3.4)$$

(b) W_ω is subnormal if and only if $W_{\theta(\omega)}$ is subnormal; in this case, the Berger measure ϵ of $W_{\theta(\omega)}$ is diagonal. Moreover, ϵ^X is the Berger measure of W_ω .

Example 3.4. Let W_ω , σ , $W_{(\alpha(\omega), \beta(\omega))}$ and $\gamma_{(k_1, k_2)}$ be as in (3.2). If $p(r) = r$ and $q(r) = r$, then the (p, q) -embedding $W_{(\alpha(\omega), \beta(\omega))}$ of W_ω is the embedding considered in [13]; that is,

$W_{(\alpha(\omega),\beta(\omega))} = W_{\theta(\omega)}$, or equivalently,

$$\beta_{(k_1,k_2)} = \alpha_{(k_1,k_2)} = \omega_{k_1+k_2} \quad (\text{for all } k_1, k_2 = 0, 1, 2, \dots)$$

(see Figure 2(i)). To prove this, observe that

$$\gamma_{(k_1,k_2)}(W_{(\alpha(\omega),\beta(\omega))}) = \int r^{k_1} r^{k_2} d\sigma(r) = \int r^{k_1+k_2} d\sigma(r) = \gamma_{k_1+k_2}(W_\omega),$$

and therefore $\beta_{(k_1,k_2)} = \alpha_{(k_1,k_2)} = \omega_{k_1+k_2}$ (all $k_1, k_2 = 0, 1, 2, \dots$), as desired. \square

The Spherically Quasinormal Embedding.

We now state and prove that, starting with a non-flat subnormal shift W_ω , the construction described in Subsection 2.3 always leads to a spherically quasinormal embedding.

Theorem 3.5. *(cf. Problem 1.1(i), and Problems 1.2 and 1.3) Let W_ω be a subnormal unilateral weighted shift, with Berger measure σ , and let $\mu := \sigma \circ f^{-1}$, where $f(r) := (r, c-r)$, where $c := \|W_\omega\|^2$. Then the 2-variable weighted shift $\mathcal{PE}(\omega; f)$ is a spherically quasinormal embedding of W_ω .*

Proof. We know that $\mathcal{PE}(\omega; f)$ is subnormal, and that $\text{supp } \mu \subseteq \{(r, c-r) : r \in [0, c]\}$.

Then

$$\begin{aligned} \gamma_{(k_1+1,k_2)}[\mu] + \gamma_{(k_1,k_2+1)}[\mu] &= \int \int s^{k_1+1} t^{k_2} d\mu(s, t) + \int \int s^{k_1} t^{k_2+1} d\mu(s, t) \\ &= \int \int (s+t) s^{k_1} t^{k_2} d\mu(s, t) = \int \int c s^{k_1} t^{k_2} d\mu(s, t) \\ &= c \gamma_{(k_1,k_2)}[\mu], \end{aligned}$$

for all $k_1, k_2 \in \mathbb{Z}_+$. By Theorem 2.3(iv), we see that $\mathcal{PE}(\omega; f)$ is spherically quasinormal, as desired. \square

(p, q) -embeddings of Recursively Generated Weighted Shifts.

Theorem 3.6. (cf. Problem 1.1(ii)) Let $W_{(\alpha(\omega), \beta(\omega))}$ be a (p, q) -embedding 2-variable weighted shift, where W_ω is subnormal with finitely atomic Berger measure $\sigma = \sum_{i=0}^{\ell-1} \rho_i \delta_{r_i}$, $\sum_{i=0}^{\ell-1} \rho_i = 1$, $\rho_i > 0$, and $0 \leq r_0 < r_1 < \dots < r_{\ell-1} \leq 1$. Then the Berger measure μ of $W_{(\alpha(\omega), \beta(\omega))}$ is

$$\mu = \sum_{i=0}^{\ell-1} \rho_i \delta_{(p(r_i), q(r_i))}. \quad (3.5)$$

Proof. A moment's thought reveals that, without loss of generality, we can assume that $p(r) \equiv r^m$ and $q(r) \equiv r^n$, where m and n are positive integers. We first observe that $\text{supp } \mu \subseteq \{(p(r), q(r)) : 0 \leq r \leq 1\}$. Since

$$\begin{aligned} \int \int s^{k_1} t^{k_2} (t^m - s^n) d\mu(s, t) &= \int \int (s^{k_1} t^{k_2+m} - s^{k_1+n} t^{k_2}) d\mu(s, t) \\ &= \int_{[0,1]} (r^{mk_1} r^{n(k_2+m)} - r^{m(k_1+n)} r^{nk_2}) d\sigma(r) = 0, \end{aligned}$$

for all $k_1, k_2 \geq 0$, we easily see that $t^m = s^n$ a.e. μ . Thus, $\text{supp } \mu \subseteq \{(r^m, r^n) : 0 \leq r \leq 1\}$.

The rest of the proof will follow once we establish the next two Claims.

Claim 1: If $\sigma = \delta_{s_0}$, then $\mu = \delta_{(s_0^m, s_0^n)}$.

Proof of Claim 1: Observe that for all $(k_1, k_2) \in \mathbb{Z}_+^2$,

$$\gamma_{mk_1+nk_2}(W_\omega) = \int_{[0,1]} r^{mk_1+nk_2} d\sigma(r) = \int_{[0,1]} p(r)^{k_1} q(r)^{k_2} d\sigma(r) = \gamma_{(k_1, k_2)}(W_{(\alpha(\omega), \beta(\omega))}).$$

It follows that

$$\begin{aligned} \int \int s^{k_1} t^{k_2} d\delta_{(s_0, s_0)}(s, t) &= s_0^{mk_1} s_0^{nk_2} = s_0^{mk_1+nk_2} \\ &= \int r^{mk_1+nk_2} d\delta_{s_0}(r) = \gamma_{mk_1+nk_2}[\sigma] \\ &= \gamma_{(k_1, k_2)}(W_{(\alpha(\omega), \beta(\omega))}). \end{aligned}$$

Thus, we must have $\mu = \delta_{(s_0^m, s_0^n)}$.

Claim 2: If $\sigma = \sum_{i=0}^{\ell-1} \rho_i \delta_{s_i}$, then $\mu = \sum_{i=0}^{\ell-1} \rho_i \delta_{(s_i^m, s_i^n)}$.

Proof of Claim 2: This is straightforward. □

Corollary 3.7. *Let $W_{(\alpha(\omega), \beta(\omega))}$ be a polynomial embedding of W_ω , and let μ, σ and $f := (p, q)$ be as in Theorem 3.2; that is,*

$$d\mu(p(r), q(r)) = d\sigma(r).$$

Then

$$d(\mu(p(r), q(r)))^X = d\sigma(p(r))$$

and

$$d(\mu(p(r), q(r)))^Y = d\sigma(q(r))$$

where for a measure ν , $(\nu)^X$ and $(\nu)^Y$ denote the marginal measures of ν . Alternatively, recall that $\mu := \sigma \circ f^{-1}$. Then

$$\mu^X = \sigma \circ p^{-1} \tag{3.6}$$

and

$$\mu^Y = \sigma \circ q^{-1}.$$

As a consequence,

$$\mu^X \circ p = \sigma$$

and

$$\mu^Y \circ q = \sigma,$$

so that

$$\mu^X \circ p = \mu^Y \circ q.$$

In the case when σ is the Lebesgue measure on $[0, 1]$, we see that

$$d\mu^X(s) = d(p^{-1}(s)) = (p^{-1})'(s)ds.$$

and

$$d\mu^Y(t) = d(q^{-1}(t)) = (q^{-1})'(t)dt.$$

(For additional details on marginal measures, the reader is referred to Subsection 7.4 in the Appendix.)

Proof. We focus on the marginal measure μ^X . From Subsection 7.4, we know that

$$\mu^X = \mu \circ \pi_X^{-1}.$$

Therefore,

$$d\mu^X(p(r)) = d\mu(p(r), q(r)) = d\sigma(r)$$

(where we have used the injectivity of f to claim that there is a unique point $(p(r), q(r))$ projecting onto $p(r)$). Then

$$\mu^X \circ p = \sigma,$$

or

$$\mu^X = \sigma \circ p^{-1},$$

as desired.

For the reader's convenience, we also provide an alternative proof, that uses techniques from convex analysis. First, observe that the result is obvious if σ is a point mass δ_r in (3.6). As a consequence, the result is also true for finitely atomic measures. Moreover, we know that these point masses are the extreme points of the unit ball of the space of probability Borel measures on either a closed interval in \mathbb{R} or a closed rectangle in \mathbb{R}^2 . Next, each such probability Borel measure is the w^* -limit of a net of finitely atomic measures. Since the map that sends μ to its marginal measures μ^X and μ^Y is w^* -continuous, it follows that Corollary 3.7 now holds for all probability Borel measures. \square

It is well known that, for continuous functions on a closed interval, the Lebesgue integral can be calculated as a limit of Riemann sums over partitions whose norms tend to zero [28]. We shall use this fact to find explicitly the Berger measure μ of a (p, q) -embedding from the Berger measure σ of the original unilateral weighted shift, when the latter is the Bergman

shift. Concretely, we prove that the Berger measure of every (p, q) -embedding of B_+ is normalized arclength on the graph of (p, q) .

Theorem 3.8. (cf. Problem 1.2) *Let $W_{(\alpha(\omega), \beta(\omega))}$ be a (p, q) -embedding of a subnormal weighted shift W_ω , and denote by μ and σ the Berger measures of $W_{(\alpha(\omega), \beta(\omega))}$ and W_ω , respectively. Assume that σ is the Lebesgue measure on $[0, 1]$. Then $d\mu(p(r), q(r))$ is normalized arclength on the curve $C := \{(p(r), q(r)) : 0 \leq r \leq 1\}$.*

Proof. Consider a partition $\mathcal{P} \equiv \{0 \leq r_0 < r_1 < \dots < r_{N-1}\}$, and let $\|\mathcal{P}\|$ denote the norm of \mathcal{P} . The net $\{\sum_{i=0}^{N-1} (r_{i+1} - r_i) \delta_{\tilde{r}_i}\}_{\mathcal{P}}$, running over the partially ordered set of all partitions \mathcal{P} , converges in the w^* -topology to the Lebesgue measure on $[0, 1]$. (As usual, the points \tilde{r}_i belong to the interval $[r_i, r_{i+1}]$.) For a continuous function g on $[0, 1]$, a typical Riemann sum is of the form $\sum_{i=0}^{N-1} (r_{i+1} - r_i) g(\tilde{r}_i)$; thus, it can be interpreted as integration with respect to the 2-variable finitely atomic measure

$$\sum_{i=0}^{N-1} (r_{i+1} - r_i) \delta_{(p(\tilde{r}_i), q(\tilde{r}_i))}, \quad (3.7)$$

given by Theorem 1.8. We also know that the support of μ is contained in the curve C . Thus, the set $\mathcal{P}(C) := \{(p(\tilde{r}_0), q(\tilde{r}_0)), \dots, (p(\widetilde{r_{N-1}}), q(\widetilde{r_{N-1}}))\}$ is a partition in the curve C , which can be used to approximate the Lebesgue measure on C . Moreover, since the Riemann sum over the original partition \mathcal{P} (thought of as an integral with respect to a finitely atomic measure) has total mass equal to 1, the finitely atomic measure associated to the partition $\mathcal{P}(C)$ is a probability measure.

It follows that the finitely atomic measures involved in (3.7) are probability measures, as will be any w^* -limit of that net. If we could establish that the norm of $\mathcal{P}(C)$, $\|\mathcal{P}(C)\|$ is majorized by a constant multiple of $\|\mathcal{P}\|$, it would then follow that the Riemann sums in (3.7) w^* -converge to normalized Lebesgue measure on C ; that is, it would w^* -converge to normalized arclength.

To complete the proof, we observe that the distance between two consecutive points in $\mathcal{P}(C)$ is at most $\sqrt{2} \max\{\|p'\|_\infty, \|q'\|_\infty\} \|\mathcal{P}\|$, and this quantity majorizes $\|\mathcal{P}(C)\|$, where

p' and q' denote the derivatives of p and q , respectively. (Observe that the norm $\|\cdot\|_\infty$ is calculated over the interval $[0, 1]$.) \square

Example 3.9. On the closed interval $[0, 1]$, consider the (p, q) -embedding $W_{(\alpha(\omega), \beta(\omega))}$ generated by the nonnegative polynomials $p(r) := r$ and $q(r) := r(1 - r)$. By Theorem 3.2, the Berger measure μ of the embedding is related to the Berger measure σ of W_ω by the identity

$$d\mu(r, r(1 - r)) = d\sigma(r) \quad (r \in [0, 1]).$$

It follows that the moments of μ and σ are related by the identity

$$\begin{aligned} \gamma_{(k, \ell)}[\mu] &= \int_{[0, 1]} r^k (r(1 - r))^\ell d\sigma(r) = (-1)^\ell \int_{[0, 1]} r^{k+\ell} (r - 1)^\ell d\sigma(r) \\ &= (-1)^\ell \int r^{k+\ell} \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} r^i d\sigma(r) = \int \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} r^{k+\ell+i} d\sigma(r) \\ &= \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \gamma_{k+\ell+i}[\sigma]. \end{aligned} \tag{3.8}$$

If we now let $\sigma \equiv \lambda$, the Lebesgue measure on $[0, 1]$, we know by Theorem 3.8 that μ is normalized arclength on the curve $t = s - s^2$, for $0 \leq s \leq 1$. The moments of μ are

$$\begin{aligned} \gamma_{(k, \ell)}[\mu] &= \int_{[0, 1]} r^{k+\ell} (1 - r)^\ell dr = B(k + \ell + 1, \ell + 1) = \frac{\Gamma(k + \ell + 1)\Gamma(\ell + 1)}{\Gamma(k + 2\ell + 2)} \\ &= \frac{(k + \ell)! \ell!}{(k + 2\ell + 1)!}, \end{aligned}$$

where B and Γ denote the classical Beta and Gamma functions, respectively. On the other hand, (3.8) says that

$$\gamma_{(k, \ell)}[\mu] = \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \gamma_{k+\ell+i}[\lambda] = \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \frac{1}{k + \ell + i + 1},$$

and therefore

$$\sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \frac{1}{k + \ell + 1 + i} = \frac{(k + \ell)! \ell!}{(k + 2\ell + 1)!}. \tag{3.9}$$

In particular,

$$\lim_{\ell \rightarrow \infty} \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \frac{1}{k + \ell + 1 + i} = 0.$$

While (3.9) is known (see for instance [23, vol. 3, (3.8)]), it is interesting that it can be obtained as a by-product of a calculation of moments for the $(r, r(1-r))$ -embedding.

The Neil Parabolic Embedding. The *Neil parabola*

$$NP := \{(r^2, r^3) : r \in [0, 1]\}$$

(also known as the *semicubical parabola*) has been studied in various contexts; see for instance [26] and [27]. Here we will use it to produce a simple example, which will lead to an answer to Problem 1.5. First, we recall that given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \rightarrow X$ is the canonical projection onto X . Thus $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$ (cf. Subsection 7.4).

Example 3.10. Let W_ω be a subnormal unilateral weighted shift, with Berger measure σ , let $p(r) = r^2$ and $q(r) = r^3$, and let $f := (p, q)$. Then the (p, q) -embedding $\mathcal{PE}(\omega; f)$ is subnormal, with Berger measure μ supported on NP , such that

$$d\mu(r^2, r^3) = d\sigma(r).$$

Furthermore, $W_{(\alpha, \beta)}$ has the weight diagram shown in Figure 2(ii). By direct calculation, one observes that the weight diagram for this embedding is intimately related to the weight diagram for the integer powers of $W_{\theta(\omega)}$. Since $W_{\theta(\omega)}$ is subnormal, let us denote by ν its Berger measure; we know that $\text{supp } \nu \subseteq \{(u, u) : u \in \mathbb{R}_+\}$. For $(s, t) \in NP$, we then have

$$d\mu(s, t) = d\nu\left(s^{\frac{1}{2}}, t^{\frac{1}{3}}\right).$$

Moreover, the last expression also represents the Berger measure of $W_{\theta(\omega)}$ restricted to the reducing subspace $\mathcal{H}_{(0,0)}^{(2,3)}$ (see (4.6) in Section 4 for the precise definition of this subspace).

In the special case of $W_\omega = B_+$, we know that σ is Lebesgue measure on $[0, 1]$, and therefore

$$d\mu(r^2, r^3) = dr.$$

By Corollary 3.7,

$$d(\mu^X(s)) = d\left(s^{\frac{1}{2}}\right) = \frac{1}{2\sqrt{s}} ds$$

and

$$d(\mu^Y(t)) = d\left(t^{\frac{1}{3}}\right) = \frac{1}{3t^{\frac{2}{3}}} dt.$$

A simple calculation of moments reveals that this is consistent with the information in Figure 2(ii). □

4. APPLICATION 1: THE CLASSICAL CASE – k -HYPONORMALITY IN THE CLASS

$$\{\mathcal{CE}(\omega) : W_\omega \text{ A SUBNORMAL UNILATERAL WEIGHTED SHIFT}\}$$

In this section, we consider the class of 2-variable weighted shifts obtained as classical embeddings of unilateral weighted shifts. We ask whether a power of $W_{\theta(\omega)}$ can be (jointly) k -hyponormal, under the assumption that W_ω is k -hyponormal ($k \geq 1$). The following examples will answer this question. First, recall that for every positive integer m we have

$$W_\omega^m \cong W_{\omega(m:0)} \oplus W_{\omega(m:1)} \oplus \cdots \oplus W_{\omega(m:m-1)}, \quad (4.1)$$

where the unilateral weighted shift $W_{\omega(m:i)}$ is associated with the sequence $\omega(m:i) := \{\omega_{i+k}\omega_{i+k+1} \cdots \omega_{i+k+m-1}\}_{k \geq 0}$, for $i = 0, 1, \dots, m-1$ (cf. [11]).

If W_ω is subnormal with Berger measure σ , then each summand $W_{\omega(m:i)}$ is subnormal ($0 \leq i \leq m-1$). Let σ_i be the Berger measure for $W_{\omega(m:i)}$ ($0 \leq i \leq m-1$). In what follows, we will need the following result.

Theorem 4.1. ([11, Theorem 2.9])

$$d\sigma_0(r) = d\sigma\left(r^{\frac{1}{m}}\right)$$

and

$$d\sigma_i(r) = \frac{r^i}{\omega_0^2 \cdots \omega_{i-1}^2} d\sigma\left(r^{\frac{1}{m}}\right) \quad (1 \leq i \leq m-1).$$

Related to the decomposition (4.1) and Theorem 4.1, we now prove:

Theorem 4.2. *Let r be a nonnegative real number and $0 \leq i \leq m-1$.*

(i) *if $r \in \text{supp } \sigma$ and $r \neq 0$, then $r^m \in \text{supp } \sigma_i$ ($1 \leq i \leq m-1$);*

(ii) *if $r \in \text{supp } \sigma_i$ and $r \neq 0$, then $r^{\frac{1}{m}} \in \text{supp } \sigma$ ($1 \leq i \leq m-1$);*

(iii) *$0 \in \text{supp } \sigma$ if and only if $0 \in \text{supp } \sigma_0$.*

Proof. Let $\text{Bo}(\mathbb{R}_+)$ denote the Borel σ -algebra on \mathbb{R}_+ . For $N \in \text{Bo}(\mathbb{R}_+)$, we let $N^{\frac{1}{m}} := \{r^{\frac{1}{m}} : r \in N\}$ and $N^m := \{r^m : r \in N\}$.

(i): Since $r \in \text{supp } \sigma$, there exists $N_r \in \text{Bo}(\mathbb{R}_+)$ such that $r \in N_r$ and $\sigma(N_r) > 0$. Note that

$$\sigma_0(N_r^m) > 0 \iff \sigma(N_r) > 0 \quad (4.2)$$

and

$$\sigma_i(N_r^m) > 0 \iff \sigma(N_r) > 0. \quad (4.3)$$

Since $N_r^m \in \text{Bo}(\mathbb{R}_+)$, by the definition of N_r^m and (4.3) we must have $r^m \in \text{supp } \sigma_i$, as desired.

(ii): Since $r \in \text{supp } \sigma_i$, there exists $N_r \in \text{Bo}(\mathbb{R}_+)$ such that $r \in N_r$ and $\sigma_i(N_r) > 0$. Note that

$$\sigma_0(N_r) > 0 \iff \sigma\left(N_r^{\frac{1}{m}}\right) > 0 \quad (4.4)$$

and

$$\sigma_i(N_r) > 0 \iff \sigma\left(N_r^{\frac{1}{m}}\right) > 0. \quad (4.5)$$

Since $N_r^{\frac{1}{m}} \in \text{Bo}(\mathbb{R}_+)$, by the definition of $N_r^{\frac{1}{m}}$ and (4.5) we must have $r^{\frac{1}{m}} \in \text{supp } \sigma$, as desired.

(iii): Since $d\sigma_0(r) = d\sigma\left(r^{\frac{1}{m}}\right)$, the equivalences (4.2) and (4.4) give the desired conclusion. \square

We now establish:

Theorem 4.3. (cf. Problem 1.4(ii)) *Let W_ω be a subnormal unilateral weighted shift, and let m be a positive integer. Assume that W_ω^m is a direct sum of recursively generated weighted shifts. Then W_ω is a recursively generated weighted shift.*

Proof. This follows easily from Theorem 4.2 and the fact that a subnormal unilateral weighted shift W_ω is recursively generated if and only if its Berger measure σ is finitely atomic, that is, $\text{card}(\text{supp } \sigma) < +\infty$ (cf. Subsection 7.2 in the Appendix). \square

By analogy with the decomposition used in [11], we now introduce a decomposition for a commuting 2-variable weighted shift $W_{(\alpha,\beta)}$ which will allow us to analyze its integer powers. Concretely, we split the ambient space $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+^2)$ as an orthogonal direct sum $\mathcal{H} \equiv \bigoplus_{p=0}^{m-1} \bigoplus_{q=0}^{n-1} \mathcal{H}_{(p,q)}^{(m,n)}$, where

$$\mathcal{H}_{(p,q)}^{(m,n)} := \vee \{e_{(mi+p, nj+q)} : i = 0, 1, 2, \dots, j = 0, 1, 2, \dots\} \quad \begin{cases} (0 \leq p \leq m-1) \\ (0 \leq q \leq n-1) \end{cases} \quad (4.6)$$

(see Figure 3).

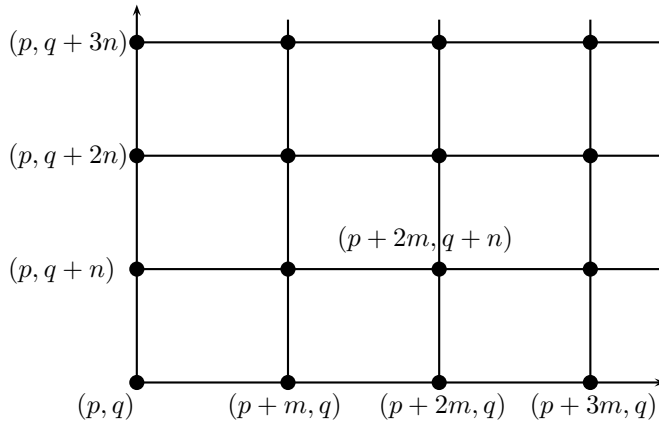


FIGURE 3. Schematic diagram of the subspace $\mathcal{H}_{(p,q)}^{(m,n)}$

We now let $\mathcal{M} := \vee\{e_{(k_1, k_2)} : k_1 \geq 0, k_2 \geq 1\}$ and $\mathcal{N} := \vee\{e_{(k_1, k_2)} : k_1 \geq 1, k_2 \geq 0\}$.

Each of the subspaces $\mathcal{H}_{(p, q)}^{(m, n)}$ reduces T_1^m and T_2^n , and $W_{(\alpha, \beta)}^{(m, n)}$ is subnormal if and only if each $W_{(\alpha, \beta)}^{(m, n)} \Big|_{\mathcal{H}_{(p, q)}^{(m, n)}}$ is subnormal. Moreover, each $W_{(\alpha, \beta)}^{(m, n)} \Big|_{\mathcal{H}_{(p, q)}^{(m, n)}}$ is a 2-variable weighted shift.

We now recall a simple criterion to detect the k -hyponormality of 2-variable weighted shifts.

Lemma 4.4. ([12, Theorem 2.4]) *The following statements are equivalent:*

- (i) $W_{(\alpha, \beta)}$ is k -hyponormal;
- (ii) $(\gamma_{(u_1, u_2) + (n, m) + (p, q)})_{\substack{0 \leq n+m \leq k \\ 0 \leq p+q \leq k}} \geq 0$ for all $(u_1, u_2) \in \mathbb{Z}_+^2$.

Example 4.5. Consider $W_{\theta(\omega)} \equiv (T_1, T_2)$ given by Figure 4(ii), that is, $\alpha_{(k_1, k_2)} = \beta_{(k_1, k_2)} = \gamma_{k_1+k_2}(\omega)$ for all $k_1, k_2 \geq 0$. Assume that

$$W_\omega = \text{shift}(\alpha_{(0,0)}, \alpha_{(1,0)}, \dots) = \text{shift}(\sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \dots).$$

(W_ω is a rank-one perturbation of B_+ , and it was previously studied in [5].) Then, we have:

(i) W_ω is hyponormal if and only if $W_{\theta(\omega)}$ is (jointly) hyponormal if and only if $0 < x \leq \frac{2}{3} \simeq 0.667$.

(ii) W_ω is 2-hyponormal if and only if $W_{\theta(\omega)}$ is (jointly) 2-hyponormal if and only if $0 < x \leq \frac{9}{16} \simeq 0.563$.

(iii) W_ω is 3-hyponormal if and only if $W_{\theta(\omega)}$ is (jointly) 3-hyponormal if and only if $0 < x \leq \frac{8}{15} \simeq 0.533$.

(iv) $W_{(\alpha, \beta)}^{(2,3)} = W_{\theta(\omega)}^{(2,3)}$ is (jointly) hyponormal if and only if $0 < x \leq \frac{2}{3} \simeq 0.667$.

(v) $W_{\theta(\omega)}^{(2,3)} \Big|_{\mathcal{H}_{(0,0)}^{(2,3)}}$ is (jointly) 2-hyponormal if and only if $0 < x \leq \frac{49}{90} \simeq 0.544$.

As a consequence, we have:

(vi) $W_{\theta(\omega)}$ is (jointly) 2-hyponormal but $W_{\theta(\omega)}^{(2,3)}$ is not (jointly) 2-hyponormal, when $\frac{49}{90} < x \leq \frac{9}{16}$.

We establish (i)–(vi) by direct calculations using (4.6), Lemma 4.4, and *Mathematica* [30]. □

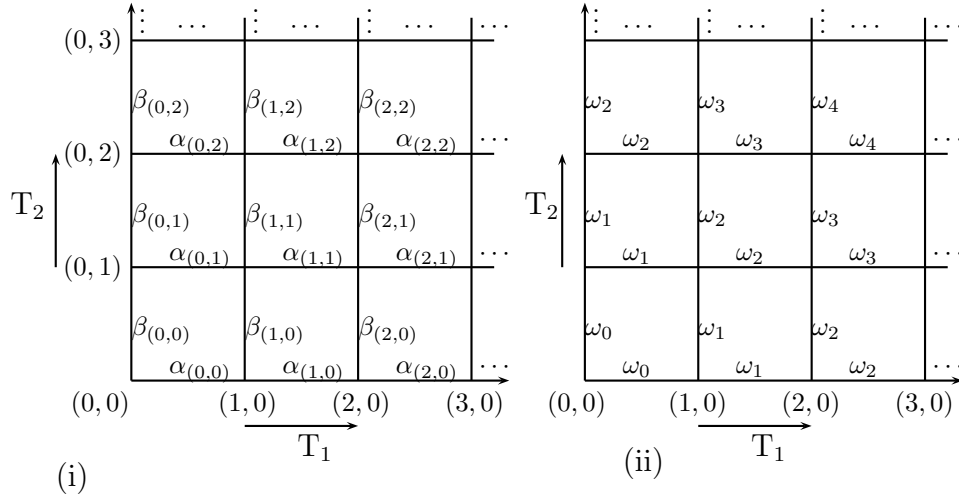


FIGURE 4. (i) Weight diagram of a generic 2-variable weighted shift; (ii) weight diagram of a generic classical embedding $W_{\theta(\omega)} \equiv (T_1, T_2)$.

Example 4.6. For $x \in [\frac{1}{2}, \frac{2}{3}]$, let $\omega(x) := \text{shift}(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \dots)$. Then

- (i) $W_{\theta(\omega(x))}$ is hyponormal, but $W_{\theta(\omega(x))}^{(2,3)}$ is not hyponormal.
- (ii) $W_{\theta(\omega(x))}$ is not 2-hyponormal.
- (iii) $W_{\theta(\omega(x))}|_{\mathcal{M} \cap \mathcal{N}}$ and $[W_{\theta(\omega(x))}|_{\mathcal{M} \cap \mathcal{N}}]^{(2,2)}$ are *not* 2-hyponormal.
- (iv) $[W_{\theta(\omega(x))}|_{\mathcal{M} \cap \mathcal{N}}]^{(3,3)}$ and $[W_{\theta(\omega(x))}|_{\mathcal{M} \cap \mathcal{N}}]^{(4,4)}$ are 2-hyponormal.

The statements (i)–(iv) are established by direct calculations using (4.6) and Lemma 4.4. □

We recall the following open question in [17, Problem 1.1]: Let T be an operator and let $k \geq 2$. Does the k -hyponormality of T imply the k -hyponormality of T^2 ?

Remark 4.7. (i) Consider the classical embedding $W_{\theta(\omega)}$, as in Example 4.5. By Lemma 3.3, we know that W_ω is k -hyponormal if and only if $W_{\theta(\omega)}$ is (jointly) k -hyponormal. Moreover, we might expect that Example 4.5(vi) will provide some clues on the solution of the above mentioned open problem.

(ii) For $p, q \geq 1$, let $\mathcal{M}_q := \vee\{e_{(k_1, k_2)} : k_1 \geq 0 \text{ and } k_2 \geq q\}$ and $\mathcal{N}_p := \vee\{e_{(k_1, k_2)} : k_1 \geq p \text{ and } k_2 \geq 0\}$. The reader may have noticed that for the classical embedding $W_{\theta(\omega)}$, the restriction $W_{\theta(\omega(x))}|_{\mathcal{M}_q \cap \mathcal{N}_p}$ is unitarily equivalent to the restriction $W_{\theta(\omega(x))}|_{\mathcal{M}_{p+q}}$ and also to the restriction $W_{\theta(\omega(x))}|_{\mathcal{N}_{p+q}}$. This strengthens the role of the classical embedding as true one-parametric representation of W_ω inside a 2-variable weighted shift.

(iii) For a general operator T on Hilbert space, it is well known that the hyponormality of T does not imply the hyponormality of T^2 [24, Problem 209]. However, for a unilateral weighted shift W_ω , the hyponormality of W_ω (detected by the condition $\omega_k \leq \omega_{k+1}$ for all $k \geq 0$) clearly implies the hyponormality of every power W_ω^m ($m \geq 1$). For 2-variable weighted shifts, one is thus tempted to expect that a similar result would hold, especially if we restrict attention to the class of classical embeddings. Somewhat surprisingly, it is actually possible to build a hyponormal 2-variable weighted shift $W_{\theta(\omega)}$ such that $W_{\theta(\omega)}^{(2,3)}$ is not hyponormal, as shown in Example 4.6(i).

(iv) It is not hard to prove that the classical embedding Θ of W_ω^m (thought of as the orthogonal direct sum of the classical embeddings of each summand in (4.1)) is unitarily equivalent to the power $(m, 1)$ or $(1, m)$ of the classical embedding of W_ω ; that is,

$$\Theta(W_\omega^m) := \bigoplus_{i=0}^{m-1} \Theta(W_{\omega^{(m:i)}}) \cong W_{\theta(\omega)}^{(m,1)} \cong W_{\theta(\omega)}^{(1,m)}.$$

Thus, when it comes to detecting the behavior of positive integer powers of a unilateral weighted shift, to a great extent we can rely on the information provided by the powers

$(m, 1)$ or $(1, m)$ of its classical embedding.

(v) Consider the 2-variable weighted shift $W_{(\alpha, \beta)} \equiv W_{\theta(\omega)}$, as in Example 4.6. By (iv) above, we observe that $W_{(\alpha, \beta)}|_{\mathcal{M} \cap \mathcal{N}}$ and $(W_{(\alpha, \beta)}|_{\mathcal{M} \cap \mathcal{N}})^{(3,3)}$ are both also classical embeddings. By Lemma 3.3, we know that $W_\omega|_{\mathcal{L}_1}$ is k -hyponormal if and only if $W_{\theta(\omega)}|_{\mathcal{M}_1}$ is (jointly) k -hyponormal (all $k \geq 1$), where \mathcal{L}_1 denotes the subspace of $\ell^2(\mathbb{Z}_+)$ generated by the orthonormal basis vectors e_1, e_2, \dots . Example 4.5(v) provides a clear signal that it might be possible to find an operator T such that the 2-hyponormality of T^2 and T^3 does not imply the 2-hyponormality of T .

5. APPLICATION 2: THE CASE OF THE BERGMAN SHIFT

For an answer to Problem 1.2, we recall K. Zhu's lemma. For $n \geq 2$, let \mathbb{S}_n (resp. \mathbb{B}_n) be unit sphere (resp. ball) in \mathbb{C}^{n-1} . We let dv denote the normalized volume measure on \mathbb{B}_n , so that $v(\mathbb{B}_n) = 1$. The normalized surface measure on \mathbb{S}_n will be denoted by $d\sigma$.

Lemma 5.1. [32] *For $n \geq 2$, let \mathbb{S}_n (resp. \mathbb{B}_n) be unit sphere (resp. ball) in \mathbb{C}^{n-1} , $\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathbb{Z}_+^n$, and $\alpha > -1$. Let $\mathbf{m}! = m_1!m_2!\dots m_n!$, $|\mathbf{m}| = m_1 + m_2 + \dots + m_n$, and Γ be the gamma function. Then*

$$\int_{\mathbb{S}_n} |\rho^{\mathbf{m}}|^2 d\sigma(\rho) = \frac{(n-1)!\mathbf{m}!}{(n-1+|\mathbf{m}|)!} \quad (5.1)$$

and

$$\int_{\mathbb{B}_n} |z^{\mathbf{m}}|^2 dv_\alpha(z) = \frac{\mathbf{m}!\Gamma(n+\alpha+1)}{\Gamma(n+|\mathbf{m}|+\alpha+1)},$$

where $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z)$ and c_α is the a normalizing constant so that $v_\alpha(\mathbb{B}_n) = 1$ and $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$.

Theorem 5.2. (cf. Problem 1.2) *The Berger measure of $\mathcal{SIE}(B_+)$ is normalized arclength on the line segment $L := \{(r, 1-r) : 0 \leq r \leq 1\}$, and $\mathcal{SIE}(B_+)$ is unitarily equivalent to the commuting pair (M_{z_1}, M_{z_2}) of multiplication operators on the Hardy space of the unit sphere \mathbb{S}_3 .*

Proof. From Figure 1 we easily see that

$$\gamma_{(k_1, k_2)}(\mathcal{SIE}(B_+)) = \frac{k_1!k_2!}{(k_1 + k_2 + 1)!}.$$

On the other hand, these moments agree with those in (5.1), when $n = 2$. It follows that the weight diagram in Figure 1 corresponds to the commuting pair (M_{z_1}, M_{z_2}) of multiplication operators on the Hardy space of the unit sphere \mathbb{S}_3 . Now recall from Theorem 3.8 that the Berger measure associated with the weight diagram in Figure 1 is normalized arclength in the line segment L . As a consequence, we observe that

$$\int_{\mathbb{S}_n} |\rho^{\mathbf{m}}|^2 d\sigma(\rho) = \frac{\mathbf{m}!}{(|\mathbf{m}| + 1)!} = \int_{[0,1]} r^{m_1} (1 - r)^{m_2} dr.$$

□

An Alternative Calculation of the Berger Measure of the Agler Shifts. We conclude this section with finding an alternative approach to the calculation of the Berger measure of the Agler j -th shift A_j ($j \geq 2$). For this, we recall that each component T_i of a subnormal 2-variable weighted shift $W_{(\alpha, \beta)} \equiv (T_1, T_2)$ is subnormal. For instance, $T_1 \cong \bigoplus_{k_2=0}^{\infty} W_{k_2}$ (resp. $T_2 \cong \bigoplus_{k_1=0}^{\infty} V_{k_1}$). In this case, we let $\{\xi_{k_2}\}_{k_2=0}^{\infty}$ and $\{\eta_{k_1}\}_{k_1=0}^{\infty}$ be the 1-variable Berger measures for $\{W_{k_2}\}_{k_2=0}^{\infty}$ and $\{V_{k_1}\}_{k_1=0}^{\infty}$, respectively. Fix $k_2 \geq 0$ and observe that the moments of ξ_{k_2} are given by

$$\begin{aligned} \int_{[0, \|T_1\|]} s^{k_1} d\xi_{k_2}(s) &= \alpha_{0, k_2}^2 \cdots \alpha_{k_1-1, k_2}^2 = \frac{\gamma_{(k_1, k_2)}}{\gamma_{(0, k_2)}} \\ &= \frac{1}{\gamma_{(0, k_2)}} \int_{[0, \|T_1\|] \times [0, \|T_2\|]} s^{k_1} t^{k_2} d\mu(s, t) \quad (\text{all } k_1 \geq 0). \end{aligned} \quad (5.2)$$

We also know that if μ is the Berger measure of $W_{(\alpha, \beta)}$, then $\xi_{k_2} = \mu_{k_2}^X$, where $d\mu_{k_2}(s, t) = \frac{1}{\gamma_{0k_2}} t^{k_2} d\mu(s, t)$ ([16, Theorem 3.1]).

Recall now the Drury-Arveson 2-variable weighted shift DA , whose weight sequences are given by

$$\alpha_{(k_1, k_2)} := \sqrt{\frac{k_1 + 1}{k_1 + k_2 + 1}} \quad (\text{for all } k_1, k_2 \geq 0) \quad (5.3)$$

$$\beta_{(k_1, k_2)} := \alpha_{(k_2, k_1)} \quad (\text{for all } k_1, k_2 \geq 0). \quad (5.4)$$

If we denote the successive rows of the weight diagram of DA by R_0, R_1, R_2, \dots , it is easy to see that $R_0 = A_1$, the (unweighted) unilateral shift, $R_1 = A_2 \equiv B_+$, the Bergman shift and, more generally, $R_{k_2-1} = A_{k_2}$, the Agler k_2 -th shift ($k_2 \geq 2$); in particular, all rows and columns are subnormal weighted shifts. For $k_2 \geq 2$, it is well known that the Berger measure of A_{k_2} is $(k_2 - 1)(1 - r)^{k_2-2}dr$ on the closed interval $[0, 1]$ (cf. [8, paragraph immediately preceding Theorem 1.1]). However, DA is *not* subnormal (not even hyponormal!), so we may not directly use the above argument. On the other hand, DA is a close relative of the spherically isometric embedding $\mathcal{SIE}(A_2)$ of the Bergman shift. That is, each row R_j ($j \geq 1$) in DA appears as row R_{j-1} in $\mathcal{SIE}(A_2)$. We thus obtain an alternative way to calculate the Berger measure of the Agler shifts.

Corollary 5.3. *For $k_2 \geq 2$, the Berger measure of A_{k_2} is $(k_2 - 1)(1 - r)^{k_2-2}dr$ on $[0, 1]$.*

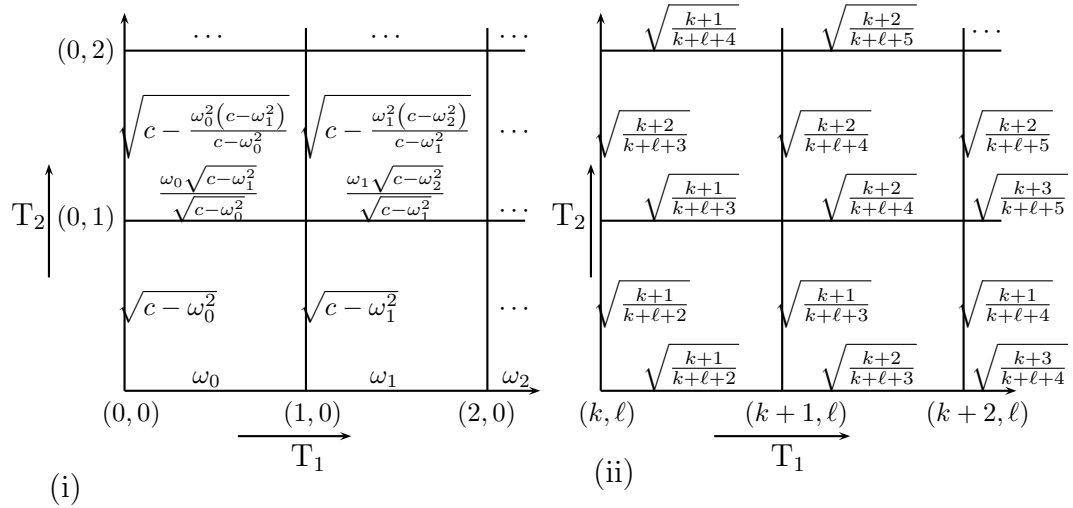


FIGURE 5. (i) Weight diagram of a spherically quasinormal embedding of W_ω ; (ii) weight diagram of $W_{\theta(\omega)}|_{\mathcal{M}_k \cap \mathcal{N}_\ell}$ when W_ω is the Bergman shift.

Proof. Consider Figure 5(ii) and Figure 1 with $W_\omega = B_+$. Since $W_{(\alpha(\omega),\beta(\omega))}$ is spherically quasinormal, it is also subnormal. Let μ be the Berger measure of $W_{(\alpha(\omega),\beta(\omega))}$ and let σ_{k_2} be the Berger measure of A_{k_2} , respectively. Recall that

$$\sigma_{k_2} = \mu_{k_2-2}^X, \quad \text{where} \quad d\mu_{k_2}(s, t) = \frac{1}{\gamma_{0k_2}} t^{k_2} d\mu(s, t).$$

Also, for $k_2 \geq 2$, $\sigma_{k_2} = \xi_{k_2-2}$. By Theorem 1.6, Figure 5(ii) and Figure 1, for all $k_2 \geq 0$ and $(s, t) \in G(f) = \{(r, 1-r) : r \in [0, 1]\}$, we can see that

$$d\mu_{k_2}(s, t) = \frac{1}{\gamma_{0k_2}} t^{k_2} d\mu(s, t) = (k_2 + 1) t^{k_2} d\mu(s, t) = (k_2 + 1) (1-r)^{k_2} d\mu(r, 1-r).$$

Thus, we have

$$d\xi_{k_2}(r) = d(\mu_{k_2}(s, t))^X = (k_2 + 1) \left((1-r)^{k_2} d\mu(r, 1-r) \right)^X = (k_2 + 1) (1-r)^{k_2} d\sigma(r), \quad (5.5)$$

where we have used Corollary 3.7 to calculate the marginal measure. Therefore, the Berger measure of A_{k_2} is $(k_2 + 1)(1-r)^{k_2} dr$ on the closed interval $[0, 1]$. This completes the proof. \square

Remark 5.4. (i) In the Proof of Corollary 5.3 we chose to calculate the Berger measure of the Agler shifts by using the rows of the spherically isometric embedding. However, since the weight diagram in Figure 1 is symmetric with respect to the diagonal, we could have chosen columns instead of rows, to perform the calculation. In that case, one gets

$$d\eta_{k_1}(r) = d(\mu_{k_1}(s, t))^Y = (k_1 + 1) (r^{k_1} d\mu(r, 1-r))^Y = (k_1 + 1) r^{k_1} d\sigma(1-r).$$

The substitution $u := 1-r$ shows that this answer is identical to that in (5.5).

(ii) The reader may have noticed that our calculation of the Berger measure of the Agler shifts in Corollary 5.3 is based on the theory of spherically isometric embeddings and the disintegration-of-measure techniques from [16]. While this approach bypasses K. Zhu's result (Lemma 5.1), it does not provide a formula for the measure μ (as we did in Theorem 5.2).

6. APPLICATION 3: MONOMIAL PAIRS

In this section we first present an alternative solution to Problem 1.1(ii), using the theory of recursively generated weighted shifts. We then discuss two examples that illustrate the solution to Problem 1.1.

Theorem 6.1. *(cf. Problem 1.1(ii)) Let W_0 , σ and μ be given as in Problem 1.1, assume that W_0 is recursively generated, with*

$$\sigma = \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{s_{r-1}}$$

with $0 \leq s_0 < \cdots < s_{r-1} \leq c$, $\rho_0, \cdots, \rho_{r-1} > 0$, and $\rho_0 + \cdots + \rho_{r-1} = 1$. Then

$$\mu = \rho_0 \delta_{(s_0, c-s_0)} + \cdots + \rho_{r-1} \delta_{(s_{r-1}, c-s_{r-1})}.$$

Proof. Since $W_0 := \text{shift}(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$ is a recursively generated weighted shift, by Lemma 7.1 there exists a generating function $g_0(s)$ such that the zeros of g_0 are the atoms of the Berger measure of W_0 . That is, if we let

$$g_0(s) = s^r - (\varphi_{r-1} s^{r-1} + \varphi_{r-2} s^{r-2} + \cdots + \varphi_1 s + \varphi_0), \quad (6.1)$$

then $g_0(s)$ has r distinct real roots $0 < s_0 < s_1 < \cdots < s_{r-1} \leq c$. Since $\alpha_{(k_1, k_2)}^2 + \beta_{(k_1, k_2)}^2 = c > 0$ for every $(k_1, k_2) \in \mathbb{Z}_+^2$, we have

$$\gamma_{(k_1+1, k_2)}(W_{(\alpha, \beta)}) + \gamma_{(k_1, k_2+1)}(W_{(\alpha, \beta)}) = c \cdot \gamma_{(k_1, k_2)}(W_{(\alpha, \beta)}).$$

It follows that we have a generating function $h(s, t)$ of $W_{(\alpha, \beta)}$, given by

$$h(s, t) := s + t - c. \quad (6.2)$$

Thus, the atoms of the Berger measure μ for $W_{(\alpha, \beta)}$ are also zeros of h . Let τ be the Berger measure of $V_0 := \text{shift}(\beta_{(0,0)}, \beta_{(0,1)}, \beta_{(0,2)}, \cdots)$. By [20, Corollary 5.6]

$$\text{supp } \tau = c - \text{supp } \sigma := \{c - s : s \in \text{supp } \sigma\}.$$

It follows that the atoms of μ must be of the form $\{(s_i, c - s_i) : 0 \leq i \leq r - 1\}$. We now denote the densities of μ by ξ_i , so that

$$\mu = \xi_0 \delta_{(s_0, c-s_0)} + \cdots + \xi_{r-1} \delta_{(s_{r-1}, c-s_{r-1})}, \quad (6.3)$$

where $\xi_0, \xi_1, \dots, \xi_{r-1}$ are positive real numbers which add up to 1. We thus have

$$\begin{aligned} \xi_0 + \xi_1 + \cdots + \xi_{r-1} &= \gamma_{(0,0)} = 1; \\ \xi_0 s_0 + \xi_1 s_1 + \cdots + \xi_{r-1} s_{r-1} &= \gamma_{(1,0)}; \\ \xi_0 s_0^2 + \xi_1 s_1^2 + \cdots + \xi_{r-1} s_{r-1}^2 &= \gamma_{(2,0)}; \\ &\vdots \\ \xi_0 s_0^{r-1} + \xi_1 s_1^{r-1} + \cdots + \xi_{r-1} s_{r-1}^{r-1} &= \gamma_{(r-1,0)}, \end{aligned} \quad (6.4)$$

Observe that from (6.4) we obtain an invertible $r \times r$ matrix

$$V := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ s_0 & s_1 & \cdots & s_{r-1} \\ s_0^2 & s_1^2 & \cdots & s_{r-1}^2 \\ s_0^3 & s_1^3 & \cdots & s_{r-1}^3 \\ \vdots & \vdots & \ddots & \vdots \\ s_0^{r-1} & s_1^{r-1} & \cdots & s_{r-1}^{r-1} \end{pmatrix}_{(r \times r)}.$$

Then, we have

$$\begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{r-1} \end{pmatrix} = V^{-1} \begin{pmatrix} \gamma_{(0,0)} \\ \gamma_{(1,0)} \\ \gamma_{(2,0)} \\ \gamma_{(3,0)} \\ \vdots \\ \gamma_{(r-1,0)} \end{pmatrix}_{(1 \times r)} = V^{-1} \begin{pmatrix} \gamma_0[\sigma] \\ \gamma_1[\sigma] \\ \gamma_2[\sigma] \\ \gamma_3[\sigma] \\ \vdots \\ \gamma_{r-1}[\sigma] \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{r-1} \end{pmatrix}.$$

Thus, we have

$$\mu = \rho_0 \delta_{(s_0, c-s_0)} + \cdots + \rho_{r-1} \delta_{(s_{r-1}, c-s_{r-1})}. \quad (6.5)$$

Therefore, (6.5) provides an answer to Problem 1.1 (both (i) and (ii)). \square

Example 6.2. Consider a contractive spherically quasinormal 2-variable weighted shift $W_{(\alpha, \beta)} \equiv (T_1, T_2)$ given by Figure 5(i), where $W_\omega := \text{shift}(\omega_0, \omega_1, \dots)$ is subnormal with Berger measure $\sigma := \frac{1}{3}\delta_{\frac{1}{3}} + \frac{1}{3}\delta_{\frac{1}{2}} + \frac{1}{3}\delta_1$. Then $W_{(\alpha, \beta)}$ is subnormal with 3-atomic Berger measure μ which is given by

$$\mu = \frac{1}{3}\delta_{(\frac{1}{3}, \frac{2}{3})} + \frac{1}{3}\delta_{(\frac{1}{2}, \frac{1}{2})} + \frac{1}{3}\delta_{(1, 0)},$$

where $c = 1$.

To see this, note that

$$V = \begin{pmatrix} 1 & 1 & 1 \\ s_0 & s_1 & s_2 \\ s_0^2 & s_1^2 & s_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{1}{9} & \frac{1}{4} & 1 \end{pmatrix} \text{ and } V^{-1} = \begin{pmatrix} \frac{9}{2} & -\frac{27}{2} & 9 \\ -4 & 16 & -12 \\ \frac{1}{2} & -\frac{5}{2} & 3 \end{pmatrix}$$

Thus, we have

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} & -\frac{27}{2} & 9 \\ -4 & 16 & -12 \\ \frac{1}{2} & -\frac{5}{2} & 3 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{11}{18} \\ \frac{49}{36} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix},$$

as desired. \square

Our last example shows how the Berger measure of a spherically quasinormal embedding can sometimes be calculated by elementary methods, bypassing Theorem 1.8.

Example 6.3. Consider the spherically quasinormal 2-variable weighted shift $W_{(\alpha, \beta)} \equiv (T_1, T_2)$ given by Figure 5(i), where $W_\omega := \text{shift}(\omega_0, \omega_1, \dots)$ is subnormal with Berger measure $\sigma := x\delta_a + y\delta_b + (1-x-y)\delta_1$ ($0 \leq a < b < 1$, $0 < x, y < 1$, and $x+y < 1$). In Figure 5(i), assume that $c \geq 1$. Then $W_{(\alpha, \beta)}$ is subnormal with a three-atomic Berger measure μ

given by

$$\mu = x\delta_{(a,c-a)} + y\delta_{(b,c-b)} + (1-x-y)\delta_{(1,c-1)}.$$

Indeed, it follows from (6.2) that $W_{(\alpha,\beta)}$ is subnormal with three-atomic Berger measure.

Thus, we have

$$\mu = \rho_0\delta_{(a,c-a)} + \rho_1\delta_{(b,c-b)} + (1-\rho_0-\rho_1)\delta_{(1,c-1)}, \quad (6.6)$$

and recall that $0 < x, y < 1$ and $x + y < 1$. Since $0 \leq a < b < 1$, (7.9) and (6.6) imply that

$$\lim_{k_1 \rightarrow \infty} \gamma_{(k_1,0)}(W_{(\alpha,\beta)}) = 1 - \rho_0 - \rho_1 = \lim_{k_1 \rightarrow \infty} \gamma_{k_1}(W_\omega) = 1 - x - y. \quad (6.7)$$

It follows readily that

$$\rho_0 + \rho_1 = x + y. \quad (6.8)$$

By (6.6) and (6.8), for all $k_1 \geq 0$, we have

$$\gamma_{(k_1,0)}(W_{(\alpha,\beta)}) = \gamma_{k_1}(W_\omega) \implies (\rho_0 - x)a^{k_1} + (\rho_1 - y)b^{k_1} = 0.$$

If $\rho_0 - x \neq 0$, then

$$0 = \lim_{k_1 \rightarrow \infty} \left(\frac{a}{b}\right)^{k_1} = \frac{y-n}{m-x} \implies n = y,$$

which is a contradiction to (6.8). Thus, we must have $\rho_0 = x$, and a fortiori $\rho_1 = y$. It follows that

$$\mu = x\delta_{(a,c-a)} + y\delta_{(b,c-b)} + (1-x-y)\delta_{(1,c-1)}, \quad (6.9)$$

as desired. Note in passing that the marginal measure μ^X is σ . \square

7. APPENDIX

7.1. Unilateral weighted shifts. For $\omega \equiv \{\omega_n\}_{n=0}^\infty$ a bounded sequence of positive real numbers (called *weights*), let $W_\omega \equiv \text{shift}(\omega_0, \omega_1, \dots) : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\omega e_n := \omega_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^\infty$ is the

canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of $\omega \equiv \{\omega_n\}_{n=0}^\infty$ are given as

$$\gamma_k \equiv \gamma_k(W_\omega) := \begin{cases} 1, & \text{if } k = 0 \\ \omega_0^2 \cdots \omega_{k-1}^2, & \text{if } k \geq 1. \end{cases} \quad (7.1)$$

The (unweighted) unilateral shift is $U_+ := \text{shift}(1, 1, 1, \dots)$, and for $0 < a < 1$ we let

$$S_a := \text{shift}(a, 1, 1, \dots).$$

We now recall a well known characterization of subnormality for unilateral weighted shifts, due to C. Berger (cf. [4, III.8.16]) and independently established by Gellar and Wallen ([21]): W_ω is subnormal if and only if there exists a probability measure σ supported in $[0, \|W_\omega\|^2]$ (called the *Berger measure* of W_ω) such that

$$\gamma_k(W_\omega) = \omega_0^2 \cdots \omega_{k-1}^2 = \int t^k d\sigma(t) \quad (k \geq 1).$$

Observe that U_+ and S_a are subnormal, with Berger measures δ_1 and $(1 - a^2)\delta_0 + a^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support the singleton set $\{p\}$. On the other hand, the Berger measure of the Bergman shift B_+ (acting on $A^2(\mathbb{D})$, and with weights $\omega_n := \sqrt{\frac{n+1}{n+2}}$ ($n \geq 0$)) is the Lebesgue measure on the interval $[0, 1]$.

7.2. Recursively generated unilateral weighted shifts. We close this section with recalling some terminology and basic results from [9] and [10]. A subnormal unilateral weighted shift W_ω is said to be *recursively generated* if the sequence of moments γ_n admits a finite-step recursive relation; that is, if there exists an integer $k \geq 1$ and real coefficients $\varphi_0, \varphi_1, \dots, \varphi_{k-1}$ such that

$$\gamma_{n+k} = \varphi_0 \gamma_n + \varphi_1 \gamma_{n+1} + \cdots + \varphi_{k-1} \gamma_{n+k-1} \quad (\text{all } n \geq 0). \quad (7.2)$$

In conjunction with (7.2) we consider the generating function

$$g_\omega(s) := s^k - (\varphi_0 + \varphi_1 s + \cdots + \varphi_{k-1} s^{k-1}). \quad (7.3)$$

The following result characterizes recursively generated subnormal unilateral weighted shifts.

Lemma 7.1. [10] *Let W_ω be a subnormal unilateral weighted shift. The following statements are equivalent.*

(i) W_ω is recursively generated.

(ii) The Berger measure σ of W_ω is finitely atomic, and $g_\omega(s) = \prod_i (s - s_i)$, where $\text{supp } \sigma = \{s_0, \dots, s_{k-1}\}$.

7.3. 2-variable weighted shifts. Consider now two double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by \mathbb{Z}_+^2 . (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $W_{(\alpha, \beta)} \equiv (T_1, T_2)$ as a pair of operators given by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k} + \varepsilon_1} \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k} + \varepsilon_2}, \quad (7.4)$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \iff \beta_{\mathbf{k} + \varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k} + \varepsilon_2} \beta_{\mathbf{k}} \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2). \quad (7.5)$$

Moreover, for $\mathbf{k} \in \mathbb{Z}_+^2$ we have

$$T_1^* e_{0, k_2} = 0 \quad \text{and} \quad T_1^* e_{\mathbf{k}} = \alpha_{\mathbf{k} - \varepsilon_1} e_{\mathbf{k} - \varepsilon_1} \quad (k_1 \geq 1); \quad (7.6)$$

$$T_2^* e_{k_1, 0} = 0 \quad \text{and} \quad T_2^* e_{\mathbf{k}} := \beta_{\mathbf{k} - \varepsilon_2} e_{\mathbf{k} - \varepsilon_2} \quad (k_2 \geq 1). \quad (7.7)$$

In an entirely similar way one can define multivariable weighted shifts. The weight diagram of a generic 2-variable weighted shift is shown in Figure 4(i).

When all weights are equal to 1 we obtain the so-called Helton-Howe shift; that is, the shift that corresponds to the pair of multiplications by the coordinate functions in the Hardy space

$H^2(\mathbb{T} \times \mathbb{T})$ of the 2-torus, with respect to normalized arclength measure on each unit circle \mathbb{T} (cf. [22]). This shift can also be represented as $(U_+ \otimes I, I \otimes U_+)$, acting on $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$.

We now recall the definition of *moments* for a commuting 2-variable weighted shift $W_{(\alpha, \beta)}$. Given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the moment of $W_{(\alpha, \beta)}$ of order \mathbf{k} is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(W_{(\alpha, \beta)}) := \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \quad (7.8)$$

We remark that, due to the commutativity condition (7.5), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2) .

To detect hyponormality, there is a simple criterion:

Theorem 7.2. ([6]) (*Six-point Test*) *Let $W_{(\alpha, \beta)}$ be a 2-variable weighted shift, with weight sequences α and β . Then*

$$W_{(\alpha, \beta)} \text{ is hyponormal} \Leftrightarrow \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}} \beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}} \beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0 \text{ (all } \mathbf{k} \in \mathbb{Z}_+^2 \text{)}.$$

A straightforward generalization of the above mentioned Berger-Gellar-Wallen result was proved in [25]. That is, a commuting pair $W_{(\alpha, \beta)}$ admits a commuting normal extension if and only if there is a probability measure μ (which we call the Berger measure of $W_{(\alpha, \beta)}$) defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ (where $a_i := \|T_i\|^2$) such that

$$\gamma_{\mathbf{k}}(W_{(\alpha, \beta)}) = \int t_1^{k_1} t_2^{k_2} d\mu(t_1, t_2) \text{ (all } \mathbf{k} \in \mathbb{Z}_+^2 \text{)}. \quad (7.9)$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems. (For more on 2-variable weighted shifts, see [7].)

7.4. Disintegration of measures. The theory of disintegration of measures is well known, and has been used in operator theory for more than forty years, especially in the study of composition operators, where the expectation operator has become a fundamental research tool. Most of the discussion in this Subsection is taken from [4, VII.2, pp. 317-319], [16] and [31]. Let X and Z be compact metric spaces and let μ be a positive regular Borel measure on Z and the space of all Borel measures on Z is denoted by $M(Z)$. For $\phi : Z \rightarrow X$ a Borel mapping, let ν be the Borel measure $\mu \circ \phi^{-1}$ on X ; that is,

$$\nu(A) := \mu(\phi^{-1}(A))$$

for all Borel sets A contained in X . Let $\mathcal{L}^1(\mu) := \{f : f \text{ is Borel function on } Z \text{ such that } \int |f| d\mu < \infty\}$, and let $L^1(\mu) := \{[f] : f \in \mathcal{L}^1(\mu)\}$, where $[f] := \{g \in \mathcal{L}^1(\mu) : \int |f - g| d\mu = 0\}$. The map

$$\psi \rightarrow \int_Z (\psi \circ \phi) f d\mu \quad (f \in L^1(\mu))$$

defines a bounded linear functional on $L^\infty(\nu)$. If attention is restricted to characteristic functions χ_A in $L^\infty(\nu)$,

$$A \rightarrow \int_Z (\chi_A \circ \phi) f d\mu = \int_{\phi^{-1}(A)} f d\mu$$

is a countably additive measure defined on Borel sets in X , that is absolutely continuous with respect to ν . Hence there is a unique element $E(f)$ in $L^1(\nu)$ such that

$$\int_Z (\chi_A \circ \phi) f d\mu = \int_X \chi_A E(f) d\nu$$

for all Borel subsets A of X . By an approximation argument one can show that

$$\int_Z (\psi \circ \phi) f d\mu = \int_X \psi E(f) d\nu$$

for all ψ in $L^\infty(\nu)$. This defines a map

$$E : \mathcal{L}^1(\mu) \rightarrow L^1(\nu)$$

called the *expectation operator*.

Definition 7.3. A disintegration of the measure μ with respect to ϕ is a function $t \mapsto \Phi_t$ from X to $M(Z)$, such that

- (i) for each t in X , Φ_t is a probability measure;
- (ii) if $f \in \mathcal{L}^1(\mu)$, $E(f)(t) = \int_Z f d\Phi_t$ a.e. $[\nu]$.

In the sequel we always assume that X and Y are compact metric spaces; in case reference is made to weighted shifts, we assume that $X, Y \subseteq \mathbb{R}_+$.

Definition 7.4. Given a measure μ on $X \times Y$, the marginal measure μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \rightarrow X$ is the canonical projection onto X . Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. Observe that if μ is a probability measure, then so is μ^X .

Then, we have:

Theorem 7.5. [31] Let μ and Φ_t be as in Definition 7.3. Define

$$\mu^Y(G) := \mu(X \times G) \text{ (all } G \subseteq Y\text{)}.$$

Then for almost every t in Y (with respect to μ_Y) and all continuous functions $\phi(s, t)$ on the product space $X \times Y$, the measures μ^Y and Φ_t satisfy

$$\int_{X \times Y} \phi d\mu(s, t) = \int_Y \left(\int_X \phi(s, t) d\Phi_t(s) \right) d\mu^Y(t). \quad (7.10)$$

Theorem 7.6. [16] Let μ be the Berger measure of a subnormal 2-variable weighted shift, and for $j \geq 0$ let ξ_j be the Berger measure of the associated j -th horizontal 1-variable weighted shift $W_j = W_{\alpha^{(j)}}$. Then $\xi_j = \mu_j^X$ (cf. Definition 7.4), where $d\mu_j(s, t) := \frac{1}{\gamma_{0j}} t^j d\mu(s, t)$; more precisely,

$$d\xi_j(s) = \left\{ \frac{1}{\gamma_{0j}} \int_Y t^j d\Phi_s(t) \right\} d\mu^X(s), \quad (7.11)$$

where $d\mu(s, t) \equiv d\Phi_s(t)d\mu^X(s)$ is the disintegration of μ by vertical slices. A similar result holds for the Berger measure η_i of the associated i -th vertical 1-variable weighted shifts $W_{\beta^{(i)}} (i \geq 0)$.

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REFERENCES

- [1] A. Athavale, On joint hyponormality of operators, *Proc. Amer. Math. Soc.* 103(1988), 417–423.
- [2] A. Athavale, On the intertwining of joint isometries, *J. Operator Theory*, **23**(1990), 339–350.
- [3] A. Athavale and S. Podder, On the reflexivity of certain operator tuples, *Acta Math. Sci. (Szeged)* 81(2015), 285–291.
- [4] J. Conway, *The Theory of Subnormal Operators*, Mathematical Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.
- [5] R. Curto, Quadratically hyponormal weighted shifts, *Integral Equations Operator Theory* 13(1990), 49–66.
- [6] R. Curto, Joint hyponormality: A bridge between hyponormality and subnormality, *Proc. Symposia Pure Math.* 51(1990), 69–91.
- [7] R.E. Curto, Two-variable weighted shifts in multivariable operator theory, in *Handbook of Analytic Operator Theory*, K. Zhu (ed.), CRC Press, 2019; 17–63.
- [8] R.E. Curto and G.R. Exner, Berger measure for some transformations of subnormal weighted shifts, *Integral Equations Operator Theory* 84(2016), 429–450.
- [9] R. Curto and L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, *Integral Equations Operator Theory* 17(1993), 202–246.
- [10] R. Curto and L. Fialkow, Solution of the truncated complex moment problem with flat data, *Memoirs Amer. Math. Soc.* no. 568, Amer. Math. Soc., Providence, 1996.
- [11] R. Curto and S. Park, k -hyponormality of powers of weighted shifts via Schur products, *Proc. Amer. Math. Soc.* 131(2003), 2761–2769.
- [12] R. Curto, S.H. Lee and J. Yoon, k -hyponormality of multivariable weighted shifts, *J. Funct. Anal.* 229(2005), 462–480.

- [13] R. Curto, S.H. Lee and J. Yoon, Subnormality of 2-variable weighted shifts with diagonal core, *C. R. Acad. Sci. Paris*, 351(2013) 203–207.
- [14] R. Curto, S.H. Lee and J. Yoon, Quasinormality of powers of commuting pairs of bounded operators, *J. Funct. Anal.* 278(2020), art. 108342.
- [15] R. Curto and J. Yoon, Jointly hyponormal pairs of subnormal operators need not be jointly subnormal, *Trans. Amer. Math. Soc.* 358(2006), 5139–5159.
- [16] R. Curto and J. Yoon, Disintegration-of-measure techniques for commuting multivariable weighted shifts, *Proc. London Math. Soc.*, 93(2006), 381–402.
- [17] R. Curto and J. Yoon, When is hyponormality for 2-variable weighted shifts invariant under powers?, *Indiana Univ. Math. J.* 60(2011), 997–1032.
- [18] R. Curto and J. Yoon, Toral and spherical Aluthge transforms for 2-variable weighted shifts, *C. R. Acad. Sci. Paris* 354(2016), 1200–1204.
- [19] R. Curto and J. Yoon, Aluthge transforms of 2-variable weighted shifts, *Integral Equations Operator Theory*, 90(2018), Art. 52, 32 pp.
- [20] R. Curto and J. Yoon, Spherical Aluthge transforms and quasinormality for commuting pairs of operators, in *Analysis of Operators on Function Spaces (The Serguei Shimorin Memorial Volume)*, *Trends in Math.*, Birkhäuser, 2019; 213–237.
- [21] R. Gellar and L.J. Wallen, Subnormal weighted shifts and the Halmos-Bram criterion, *Proc. Japan Acad.* 46(1970), 375–378.
- [22] J. Gleason, Quasinormality of Toeplitz tuples with analytic symbols, *Houston J. Math.* 32(2006) 293–298.
- [23] H.W. Gould, *Tables of Combinatorial Identities*, vol. 1–8, Edited and Compiled by Jocelyn Quaintance, 2010; <https://math.wvu.edu/~hgould/>.
- [24] P.R. Halmos, *A Hilbert space problem book*, Springer-Verlag, Berlin and New York.
- [25] N.P. Jewell and A.R. Lubin, Commuting weighted shifts and analytic function theory in several variables, *J. Operator Theory* 1(1979), 207–223.
- [26] G. Knese, Function theory on the Neil parabola, *Michigan Math. J.* 55(2007), 139–154.
- [27] C.A. Pickover, The Length of Neile’s Semicubical Parabola, *The Math Book: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics*, Sterling Publishing Company, Inc., 2009; p. 148.
- [28] R. Pihlström, *Lebesgue Theory: A Brief Overview*, U.U.D.M. Project Report 2016:26, Uppsala Universitet, 2016.

- [29] J. Stampfli, Which weighted shifts are subnormal, *Pacific J. Math.* **17**(1966), 367–379.
- [30] Wolfram Research, Inc., *Mathematica*, Version 12.0, Champaign, IL, 2019.
- [31] J. Yoon, Disintegration of measures and contractive 2-variable weighted shifts, *Integral Equations Operator Theory* **59**(2007), 281–298.
- [32] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005.

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