Norm and numerical radius of single operators through tools and techniques from multivariable operator theory

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Abstract
We employ tools and techniques from multivariable operator theory to obtain new proofs and extensions of well-known inequalities regarding the norm and the numerical radius of elementary operators defined on the C∗–algebra of all bounded operators on Hilbert space, or on the *–ideal of Hilbert-Schmidt operators. In the process, we provide new insights on the study of Heinz-type inequalities related to the arithmetic-geometric mean inequality, and generalize results of several authors, including R. Bhatia, G. Corach, C. Davis, F. Kittaneh, and M.S. Moslehian. To estimate the norm, our approach exploits, in particular, the Spectral Mapping Theorem for the Taylor spectrum, and Ky Fan’s Dominance Theorem. For the numerical radius, we use S. Hildebrandt’s description of the numerical range of an operator in terms of the norm of its translates.

Keywords: Taylor spectrum, Numerical radius, Spectral Mapping Theorem, Ky Fan’s Dominance Theorem
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1. Introduction

Over the last several decades, substantial and significant work has been done in the study of operator norm inequalities related to the arithmetic-geometric mean inequality, first obtained by R. Bhatia and C. Davis in [4]. The list of subsequent contributions is extensive, and it includes the work of J. Anderson and C. Foiaș [1], G. Corach, H. Porta and L. Recht [6], R. Bhatia and F. Kittaneh [5], O. Hirzallah, F. Kittaneh and K. Shebrawi [16], R. Kaur, M.S. Moslehian, M. Singh and C. Conde [19], and F. Kittaneh [20]. Concretely, given $n \times n$ complex matrices $A, B, X$, the arithmetic-geometric mean inequality states that

$$2 \|AXB\| \leq \|A^*AX + XBB^*\|,$$

for every unitarily invariant norm $\|\cdot\|$ on the normed linear space of matrices $M_n(\mathbb{C})$. In the present paper, we consider several analogs of this inequality when $\mathbb{C}^n$ is replaced by an infinite dimensional separable Hilbert space $\mathcal{H}$, and $A, B, X$ are bounded operators on $\mathcal{H}$.

For $A$ and $B$ bounded and invertible self-adjoint operators on $\mathcal{H}$, we first consider the 4-tuple of left and right multiplication operators $(L_A, L_{A^{-1}}, R_B, R_{B^{-1}})$ acting on the $C^*$–algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, or on the two-sided $*$–ideal of Hilbert-Schmidt operators $C_2$, equipped with the Hilbert-Schmidt norm
\(\|\cdot\|_{HS}\), also known as the Frobenius norm (cf. Appendix). We employ tools and techniques from multivariable operator theory to obtain new proofs and extensions of well-known inequalities regarding the norm and the numerical radius of various elementary operators determined by \(A\) and \(B\), their inverses, and suitable powers. Although our main goal is to demonstrate the value of the multivariable approach to calculate norms of single operators, we are also able to extend to the infinite dimensional case results that were originally proved for \(n \times n\) complex matrices.

To analyze an elementary operator acting on \(\mathcal{B}(\mathcal{H})\), our overarching idea is to study first the properties of the associated 4–tuple \((L_A, L_{A^{-1}}, R_B, R_{B^{-1}})\) when restricted to the ideal \(C_2\), which is in itself a Hilbert space. At that level, we can connect the spectral theory of the 4–tuple to the \(C_2\)–norm of the elementary operator using the Spectral Mapping Theorem for the Taylor spectrum. We can transfer the information for the Hilbert-Schmidt norm \(\|\cdot\|_{HS}\) to the operator norm \(\|\cdot\|\) (or, more generally, to a unitarily invariant norm) by taking advantage of Ky Fan’s Dominance Theorem in the case of \(\dim \mathcal{H} < \infty\), and then by extending the desired inequality to the case of \(\dim \mathcal{H} = \infty\) using a generalized version of the approximation-of-the-identity approach in [6].

Thus, anytime an elementary operator inequality can be traced back to a pair of self-adjoint and invertible operators, the above mentioned approach works, and leads to the desired inequality for an infinite dimensional Hilbert space provided it holds for \(n \times n\) complex matrices. This allows us to generalize a number of well-known inequalities to the infinite dimensional case.

In brief, there are three key components to our construction:

(i) a general result about the norm of certain elementary operators in terms of the Taylor spectrum of their associated tuples;

(ii) an application of Ky Fan’s Dominance Theorem, to transfer the information for the norm \(\|\cdot\|_{HS}\) to the operator norm \(\|\cdot\|\) for \(n \times n\) complex matrices (and more generally, to a unitarily invariant norm \(\|\cdot\|\)); and

(iii) a result about preservation of inequalities when using a net of orthogonal projections ascending to the identity operator.

The organization of the paper is as follows. In Section 2 we state our main results; the proofs are given in Section 6. In Section 3 we study the connection between the norm of suitable elementary operators and the Taylor spectrum of the 4–tuple \((L_A, L_{A^{-1}}, R_B, R_{B^{-1}})\). In Section 4 we address the transition of an inequality in the norm of \(C_2\) to the operator norm and, more generally, to a unitarily invariant norm. We reserve Section 5 for the extension of an inequality from a finite dimensional Hilbert space to an infinite dimensional one. Finally, Section 7 contains a collection of foundational results needed in the rest of the paper.

For the reader’s convenience, the following table shows the logical dependence
of the various results in this paper.

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2. Main results

We now state the main results in this paper. We provide detailed proofs in Section 6.

To start, in Theorem 2.1(i) we present a natural generalization of a result obtained by R. Bhatia and C. Davis for \( n \times n \) complex matrices and \( \ell = \frac{1}{2} \) [4].

**Theorem 2.1.** Let \( A \) and \( B \) be self-adjoint and invertible operators on a Hilbert space \( \mathcal{H} \), let \( X \) be an arbitrary operator on \( \mathcal{H} \), and fix real numbers \( r \) and \( \ell \). Then

\[
(i) \quad \left\| A^\ell XB^\ell \right\| \leq \left\| \frac{A^r XB^{2\ell-r} + A^{2\ell-r} XB^r}{2} \right\|;
\]

\[
(ii) \quad w\left( A^\ell XB^\ell \right) \leq w\left( \frac{A^r XB^{2\ell-r} + A^{2\ell-r} XB^r}{2} \right),
\]

where \( w(\cdot) \) denotes the numerical radius.

**Theorem 2.2.** Let \( A \) and \( B \) be contractive, self-adjoint and invertible operators on a Hilbert space \( \mathcal{H} \), let \( \delta := \text{dist} (\sigma(A), \sigma(B)) \), let \( X \) be an arbitrary operator on \( \mathcal{H} \), and fix a real number \( \alpha \). Then

\[
(i) \quad 2\left\| A^{\alpha+\frac{1}{2}} XB^{\alpha+\frac{1}{2}} \right\| \leq \left\| A^\alpha XB^{1+\alpha} + A^{1+\alpha} XB^\alpha \right\| \leq \|AX + XB\|;
\]

\[
(ii) \quad 2\delta \cdot \|X\| \leq \|AXB^{-1} - A^{-1}XB\|.
\]
Theorem 2.3. Let $A$ and $B$ be selfadjoint and invertible operators on a Hilbert space $\mathcal{H}$, let $X$ be an arbitrary operator on $\mathcal{H}$, and fix real numbers $\alpha$ and $\beta$. Then
\[
2 \cdot w \left( A^{\frac{\alpha+\beta}{2}} XB^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} XB^{\frac{\alpha+\beta}{2}} \right) \leq w(A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha \\
+ A^\beta XB^{1-\beta} + A^{1-\beta}XB^\beta).
\]

Theorem 2.4. Let $A$ and $B$ be selfadjoint and invertible operators on a Hilbert space $\mathcal{H}$, let $\delta := \text{dist}(\sigma(A), \sigma(B))$, and let $X$ be an arbitrary operator on $\mathcal{H}$. Then
\[
2\delta \cdot w(X) \leq w(AXB^{-1} - A^{-1}XB).
\]

Remark 2.5. Recall that, given an arbitrary operator $A \in \mathcal{B}(\mathcal{H})$, it is always true that $w(A) \leq \|A\| \leq 2w(A)$ ([13, Theorem 2]); that is, $w(\cdot)$ and $\|\cdot\|$ are equivalent norms. One is then tempted to claim that Theorem 2.4 and Theorem 2.2(ii) are logically related. However, a moment’s thought reveals that the bounds in the asserted inequalities are different, so the above mentioned equivalence of norms does not directly yield the correct bounds, and a different approach must be used.

The next two results, which rely on multivariable spectral theory tools and techniques, are new even in the finite dimensional setting. Moreover, Theorem 2.6 generalizes portions of [19, Theorem 4.1].

Theorem 2.6. Let $A, B, X \in \mathcal{B}(\mathcal{H})$ with $A, B$ positive, invertible and contractive, and let $s, t \in \mathbb{R}$. Then
\[
\|A^s XB^{1+t} + A^{1+s}XB^t\| \leq \|AX + XB\|. \tag{2.1}
\]

Theorem 2.7. Let $A, B, X \in \mathcal{B}(\mathcal{H})$ with $A, B$ positive and invertible. Then, for any real numbers $\alpha, \beta$,
\[
2 \left\| A^{\frac{\alpha+\beta}{2}} XB^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} XB^{\frac{\alpha+\beta}{2}} \right\| \\
\leq \|A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha + A^\beta XB^{1-\beta} + A^{1-\beta}XB^\beta\| \leq 2 \|AX + XB\|. \tag{4}
\]

Remark 2.8. In the special case of complex matrices $A, B$ and $X$, the inequality (3) in theorem 2.7 follows from the generalized arithmetic-geometric inequality in [4], using the change of variables $Y := A^\alpha XB^\alpha$. On the other hand, we believe that the inequality (4) in theorem 2.7 is new even for complex matrices.
3. Norm and Joint Spectrum

We first present a familiar fact about normal operators; we include a proof for the sake of completeness.

**Lemma 3.1.** For any normal operator \( N \) on a Hilbert space \( (\mathcal{K}, \| \cdot \|) \) and for all \( x \in \mathcal{K} \),

\[
\inf_{\lambda \in \sigma(N)} |\lambda| \cdot \|x\| \leq \|Nx\|.
\]  

PROOF. Without loss of generality, assume that \( x \neq 0 \), and let

\[
\mathcal{M} := \bigvee \{ p(N, N^*) x : p \in \mathbb{C}[z, z]\}.
\]

By construction, \( \mathcal{M} \) is a \(*\)-cyclic reducing subspace for \( N \). Write \( \mathcal{K} = \mathcal{M} \oplus \mathcal{M}^\perp \); then \( N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \). Clearly, \( \sigma(N_1) \subseteq \sigma(N) \) and \( \|N x\| = \|N_1 x\| \). Note that

\[
\inf_{\lambda \in \sigma(N)} |\lambda| \leq \inf_{\lambda \in \sigma(N_1)} |\lambda|.
\]  

Since \( \mathcal{M} \) is \(*\)-cyclic, it follows that \( N_1 \) is unitarily equivalent to a multiplication operator \( M_z \) on \( L^2(\sigma(N_1), \mu) \) for some positive Borel measure \( \mu \). Thus, \( M_z = U^* N_1 U \), where \( U : L^2(\sigma(N_1), \mu) \rightarrow \mathcal{M} \) is a unitary operator (i.e., \( U \) maps \( L^2(\sigma(N_1), \mu) \) isometrically onto \( \mathcal{M} \)). We observe that, for \( x \in \mathcal{M} \),

\[
\|N_1 x\|^2 = \|U^* N_1 x\|^2 = \|M_z f\|^2 = \int_{\sigma(N_1)} |zf|^2 \, d\mu \geq \inf_{\lambda \in \sigma(N_1)} |\lambda| \|f\|^2 \int_{\sigma(N_1)} |f|^2 \, d\mu = (\inf_{\lambda \in \sigma(N_1)} |\lambda|)^2 \|f\|^2.
\]

Thus, \( \|N_1 x\| \geq \inf_{\lambda \in \sigma(N_1)} |\lambda| \|f\|_2 = \inf_{\lambda \in \sigma(N_1)} |\lambda| \cdot \|x\| \). Hence we conclude that

\[
\|Nx\| = \|N_1 x\| \geq \inf_{\lambda \in \sigma(N_1)} |\lambda| \cdot \|x\| \geq \inf_{\lambda \in \sigma(N)} |\lambda| \cdot \|x\| \text{ (by eq. (3.2))},
\]

as desired.

We now recall the definitions of the operators \( L_A \) and \( R_B \). Let \( A, B \in \mathcal{B}(\mathcal{H}) \), and let \( L_A \) (called **left multiplication**) and \( R_B \) (called **right multiplication**) be the operators on \( \mathcal{B}(\mathcal{H}) \) defined by \( X \mapsto AX \) and \( X \mapsto XB \) (\( X \in \mathcal{B}(\mathcal{H}) \)), respectively. In the sequel, we will denote by \( \sigma_T \) the Taylor spectrum of a commuting \( n \)-tuple of operators on a Hilbert space \( \mathcal{H} \). For \( A = (A_1, \ldots, A_n) \) and \( B = (B_1, \ldots, B_n) \) two commuting \( n \)-tuples of normal bounded operators on \( \mathcal{H} \), and for \( C(\sigma_T(L_A, R_B)) \) the \( C^* \)-algebra of continuous functions on the Taylor
spectrum \(\sigma_T(L_A, R_B)\), we shall let \(\Phi : C(\sigma_T(L_A, R_B)) \to \mathcal{B}(\mathcal{H})\) denote the continuous functional calculus obtained from the Spectral Theorem applied to the \((2n)\)-commuting tuple \((L_A, R_B)\), acting on the Hilbert-Schmidt class \(C_2\); that is, \(\Phi(f) := f(L_A, R_B)\) \((f \in C(\sigma_T(L_A, R_B)))\).

**Proposition 3.3.** Let \(A = (A_1, \ldots, A_n)\) and \(B = (B_1, \ldots, B_n)\) be commuting \(n\)-tuples of normal operators acting on a Hilbert space \(\mathcal{H}\), and let \(L_A|_{C_2}\) and \(R_B|_{C_2}\) denote the restrictions of the associated left and right multiplication operators to the Hilbert-Schmidt class \((C_2, \|\cdot\|_{HS})\). Let \(f \in C(\sigma_T(A) \times \sigma_T(B))\). Then, for all \(X \in C_2\) we have

\[
\|f(L_A|_{C_2}, R_B|_{C_2})(X)\|_{HS} \geq \inf_{(x,y)\in\sigma_T(A)\times\sigma_T(B)} \{ |f(x,y)| \} \cdot \|X\|_{HS}. \tag{3.3}
\]

**Proof.** Recall, from theorem 7.1, that on \(C_2\) we have

\[
(L_A)^* = L_{A^*} \quad (i = 1, \ldots, n) \quad \text{and} \quad (R_B)^* = R_{B^*} \quad (j = 1, \ldots, n).
\]

As a consequence, given \(f \in C(\sigma_T(A) \times \sigma_T(B))\), we know that \(f(L_A|_{C_2}, R_B|_{C_2})\) is normal as an operator on \(C_2\). By theorem 3.1 and eq. (7.1), we conclude that

\[
\|f(L_A|_{C_2}, R_B|_{C_2})(X)\|_{HS} \geq \inf_{\lambda \in \sigma(f(L_A|_{C_2}, R_B|_{C_2}))} \{ |\lambda| \cdot \|X\|_{HS} \}
\]

\[
\geq \inf_{(x,y)\in\sigma_T(A)\times\sigma_T(B)} \{ |f(x,y)| \} \cdot \|X\|_{HS},
\]

as desired.

We now obtain an appropriate lower bound for a key elementary operator acting on \(C_2\), built from two self-adjoint and invertible operators acting on \(\mathcal{H}\).

**Definition 3.5.** Let \(A, B \in \mathcal{B}(\mathcal{H})\) be self-adjoint and invertible operators on \(\mathcal{H}\). Define

\[
\tau : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \quad (\text{resp. } \tau : C_2 \longrightarrow C_2)
\]

by

\[
\tau(X) = \tau_{(A,B)}(X) := AXB^{-1} + A^{-1}XB \quad (X \in \mathcal{B}(\mathcal{H})) \quad \text{(resp. } X \in C_2). \tag{3.4}
\]

Since \(C_2\) is a two-sided \(*\)-ideal in \(\mathcal{B}(\mathcal{H})\), we will use the same symbol to denote the restriction of \(\tau = \tau_{(A,B)}\) to \(C_2\), unless confusion may arise. In any event, we will always specify whether the argument \(X\) is from \(\mathcal{B}(\mathcal{H})\) or from \(C_2\).
Theorem 3.6. Let $A, B \in B(H)$ be self-adjoint and invertible operators, and let 
$\tau = \tau_{(A,B)} : C_2 \to C_2$ be as in eq. (3.4). Then

$$2 \|X\|_{HS} \leq \|\tau(X)\|_{HS} = \|AXB^{-1} + A^{-1}XB\|_{HS}. \quad (3.5)$$

Proof. Consider the pairs $(A, A^{-1})$ and $(B^{-1}, B)$, and the 4-tuple

$$(L_A|c_2, L_{A^{-1}}|c_2, R_{B^{-1}}|c_2, R_B|c_2).$$

By Theorem 7.4, we know that

$$\sigma_T(L_A|c_2, L_{A^{-1}}|c_2, R_{B^{-1}}|c_2, R_B|c_2) = \sigma_T(A, A^{-1}) \times \sigma_T(B^{-1}, B)$$

$$= \{(\alpha, \alpha^{-1}) : \alpha \in \sigma(A)\} \times \{(\beta^{-1}, \beta) : \beta \in \sigma(B)\}.$$  \quad (3.6)

Let $p : \mathbb{C}^4 \to \mathbb{C}$ be given by

$$p(z_1, z_2, w_1, w_2) := z_1w_1 + z_2w_2.$$  

Observe that $\tau = p(L_A|c_2, L_{A^{-1}}|c_2, R_{B^{-1}}|c_2, R_B|c_2)$. By the Spectral Mapping

Theorem for the Taylor spectrum (eq. (3.6) and eq. (7.1)), we have:

$$\sigma(\tau) = \sigma(L_AR_{B^{-1}} + L_{A^{-1}}R_B)$$

$$= \sigma(p(L_A, L_{A^{-1}}, R_{B^{-1}}, R_B))$$

$$= p(\sigma_T(L_A, L_{A^{-1}}, R_{B^{-1}}, R_B))$$

$$= p(\{(\alpha, \alpha^{-1}) : \alpha \in \sigma(A)\} \times \{(\beta^{-1}, \beta) : \beta \in \sigma(B)\}) \quad \text{(by eq. (3.6))}$$

$$= \{\alpha \beta^{-1} + \alpha^{-1} \beta : \alpha \in \sigma(A), \beta \in \sigma(B)\}.$$  

It follows that $\lambda \in \sigma(\tau) \iff \lambda = \alpha \beta^{-1} + \alpha^{-1} \beta$ for $\alpha \in \sigma(A), \beta \in \sigma(B)$. Now,

$$\alpha \beta^{-1} + \alpha^{-1} \beta = \frac{\alpha^2 + \beta^2}{\alpha \beta}. \quad (3.7)$$

Since $A$ and $B$ are selfadjoint, we have $\alpha, \beta \in \mathbb{R}$ and

$$|\alpha \beta| \leq \frac{\alpha^2 + \beta^2}{2}. \quad (3.8)$$

Thus, by eq. (3.7) and eq. (3.8), if $\lambda \in \sigma(\tau)$ then

$$|\lambda| \geq \frac{(\alpha^2 + \beta^2)}{2 (\alpha^2 + \beta^2)} = 2. \quad (3.9)$$
for some $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$. It follows that

$$m(\tau) := \inf \{|\lambda| : \lambda \in \sigma(\tau)\} \geq 2. \quad (3.10)$$

Now, by (3.3) in Proposition 3.3, we have

$$\|\tau(X)\|_{HS} \geq m(\tau) \|X\|_{HS} = 2 \|X\|_{HS}, \quad (3.11)$$

as desired.

4. From the Hilbert-Schmidt Norm to the Operator Norm

Given an $n \times n$ matrix $X$, let $s_i(X)$ denote the $i$–th singular value of $X$ ($1 \leq i \leq n$). It is well known that if $A$ and $B$ are self-adjoint operators acting on a Hilbert space $\mathcal{H}$, then the elementary operator $L_A - R_B$ acting on $\mathcal{B}(\mathcal{H})$ is Hermitian, in the sense that

$$W(L_A - R_B) := \{s(L_A - R_B) : \text{s is a state on } \mathcal{B}(\mathcal{H})\} \subseteq \mathbb{R}$$

(cf. [1]). Observe now that a similar result holds for $\tau_{(A,B)}$ as defined in eq. (3.4), where $A$ and $B$ are self-adjoint and invertible; that is, $\tau_{(A,B)}$ is a Hermitian operator acting on $\mathcal{B}(\mathcal{H})$. We need this fact at the start of the proof of the next result.

Theorem 4.1. Assume that $d := \dim \mathcal{H} < \infty$ and let $A$ and $B$ be self-adjoint and invertible operators on $\mathcal{H}$. Then $\tau_{(A,B)}$ is bounded below by 2 in $\mathcal{B}(\mathcal{H})$.

Proof. Since $\tau$ is a Hermitian operator acting on the finite dimensional Banach space $\mathcal{B}(\mathcal{H})$, and since the spectrum of $\tau$ acting on $\mathcal{B}(\mathcal{H})$ is the same as the spectrum of $\tau$ acting on $C_2$ (by Theorem 7.4), it follows that the singular values of $\tau$ are the same irrespective of the space.

Let $s_1(\tau) \geq s_2(\tau) \geq \cdots \geq s_d(\tau) > 0$ be the singular values of $\tau$, and let $\|\cdot\|$ denote the operator norm (in $\mathcal{B}(\mathcal{H})$). We wish to prove that

$$\|\tau(X)\| \geq 2 \|X\|.$$ 

To show this, we use Ky Fan’s Dominance Theorem (Theorem 7.13). We know that

$$s_i(\tau^{-1}) \leq \|\tau^{-1}\|_{HS} \leq \frac{1}{2}$$
for all $i = 1, 2, \cdots, d$. Therefore,

$$\sum_{i=1}^{k} s_i \left( \tau^{-1} \right) \leq \frac{k}{2},$$  \hspace{1cm} (4.1)

On the other hand,

$$\sum_{i=1}^{k} s_i \left( \frac{1}{2} I \right) = \frac{k}{2}.$$  \hspace{1cm} (4.2)

Therefore,

$$\sum_{i=1}^{k} s_i \left( \tau^{-1} \right) \leq \sum_{i=1}^{k} s_i \left( \frac{1}{2} I \right),$$  \hspace{1cm} (4.3)

and by Lemma 7.13 (Ky Fan’s Dominance Theorem), we know $\|\tau^{-1}\| \leq \left\| \frac{1}{2} I \right\| = \frac{1}{2}$.

It follows that

$$2 \|X\| = 2 \|\tau^{-1}(\tau(X))\| \leq 2 \|\tau^{-1}\| \cdot \|\tau(X)\| \leq \|\tau(X)\|,$$

as desired.

**Remark 4.3.** Observe that the above reasoning similarly applies to any unitarily invariant norm $\|\cdot\|$ on $\mathcal{B}(\mathcal{H})$ when $\dim \mathcal{H} < \infty$. As a consequence, we have also established that $\|\tau(X)\| \leq 2 \|X\|$ for all $X \in \mathcal{B}(\mathcal{H})$ when $\dim \mathcal{H} < \infty$.

### 5. The Case of Infinite Dimensional $\mathcal{H}$

First, recall that if an increasing net $\{P_F\}_{F \in \mathcal{F}}$ (where $\mathcal{F}$ is the index set for the net) of finite dimensional orthogonal projections converges to the identity operator $I$ in the strong operator topology, and if $S \in \mathcal{B}(\mathcal{H})$, then the net $\{P_F SP_F\}_{F \in \mathcal{F}}$ converges to $S$ in the strong operator topology (cf. [10, Exercise 4.21]). Recall also that, as an application of the Spectral Theorem, any self-adjoint operator $S$ on $\mathcal{H}$ admits an increasing net of finite dimensional orthogonal projections $\{P_F\}_{F \in \mathcal{F}}$ such that (i) $P_F S = SP_F$ for all $F \in \mathcal{F}$, and (ii) $\{P_F\}_{F \in \mathcal{F}}$ converges to $I$ in the strong operator topology (cf. [6, page 155, lines 1–5]).

Now let $A$ and $B$ be self-adjoint and invertible operators on $\mathcal{H}$, let $X$ be a bounded operator on $\mathcal{H}$, and consider the operators $S, Z \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ given by

$$S := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad Z := \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}.$$
Observe that $S$ is self-adjoint and invertible, and $\|Z\| = \|X\|$ for all $X \in \mathcal{B}(\mathcal{H})$. Moreover,

$$S Z S^{-1} + S^{-1} Z S = \begin{pmatrix} 0 & A X B^{-1} + A^{-1} X B \\ B^{-1} X^* A + B X^* A^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tau(X) \\ (\tau(X))^* & 0 \end{pmatrix}.$$ 

It readily follows that $\|\tau(X)\| = \|S Z S^{-1} + S^{-1} Z S\|$; that is,

$$\|\tau(A,B)(X)\| = \|\tau(S,S)(Z)\|.$$ 

Let $\{P_F\}_{F \in \mathcal{F}}$ and $\{Q_G\}_{G \in \mathcal{G}}$ be two nets of finite dimensional orthogonal projections increasing to the identity operator $I$, and such that $P_F$ commutes with $A$ (for all $F \in \mathcal{F}$) and $Q_G$ commutes with $B$ (for all $G \in \mathcal{G}$). Then the net $\{P_F \oplus Q_G\}_{(F,G) \in \mathcal{F} \times \mathcal{G}}$ increases to the identity in the strong operator topology, provided we make $\mathcal{F} \times \mathcal{G}$ into a directed set by defining $(F,G) \leq (F',G')$ whenever $F \leq F'$ and $G \leq G'$. Moreover, $(P_F \oplus Q_G)S = S(P_F \oplus Q_G)$ for all $(F,G) \in \mathcal{F} \times \mathcal{G}$. It follows that

$$(P_F \oplus Q_G)S Z S^{-1}(P_F \oplus Q_G) \xrightarrow{SOT} S Z S^{-1}.$$ 

As a consequence,

$$\|(P_F \oplus Q_G)S Z S^{-1}(P_F \oplus Q_G)\| \to \|S Z S^{-1}\|.$$ 

(Recall that for a net of operators $A_\alpha$, $A_\alpha \xrightarrow{SOT} A \Rightarrow \|A_\alpha\| \to \|A\|$.)

On the other hand, a calculation reveals that

$$\begin{pmatrix} P_F & 0 \\ 0 & Q_G \end{pmatrix} \tau(S,S)(Z) \begin{pmatrix} P_F & 0 \\ 0 & Q_G \end{pmatrix} = \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix},$$

where $W := \tau(P_F A P_F, Q_G B Q_G)(P_F X Q_G)$. Thus, the compression to Ran $P_F \oplus$ Ran $Q_G$ establishes a bridge between dim $\mathcal{H} = \infty$ and dim $\mathcal{H} < \infty$. Now recall that in Section 4 we proved that $\|\tau(X)\| \geq 2 \cdot \|X\|$ for all operators $X$ acting on a finite dimensional Hilbert space. As a result, we have now established:

**Theorem 5.1.** Let $A$ and $B$ be self-adjoint and invertible operators on $\mathcal{H}$. Then $\tau(A,B)$ is bounded below by 2 in $\mathcal{B}(\mathcal{H})$; that is,

$$\|\tau(A,B)(X)\| \geq 2 \cdot \|X\|$$

holds for all $X \in \mathcal{B}(\mathcal{H})$. 

11
6. Proofs of the Main Results

We begin this section with a simple application of the results in Section 5 to obtain a new proof (for bounded operators) of a special case of a well-known inequality for complex matrices. In [4, Theorem 1], R. Bhatia and C. Davis established the arithmetic-geometric mean inequality for arbitrary matrices $A, B, X \in M_n(\mathbb{C})$:

$$2 \cdot \|AXB\| \leq \|A^*AX + XBB^*\|$$

holds for every unitarily invariant norm $\|\cdot\|$ on $M_n(\mathbb{C})$.

**Theorem 6.1.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint and invertible operators, and let $X \in \mathcal{B}(\mathcal{H})$. Then

$$2 \cdot \|AXB\| \leq \|A^2X + XB^2\| \quad (X \in \mathcal{B}(\mathcal{H})).$$

**Proof.** Let $Y := A^*XB = AXB$. Then, by Theorems 3.6, 4.1 and 5.1, we have

$$2 \cdot \|Y\|_{HS} \leq \|\tau(Y)\|_{HS} \quad \implies \quad 2 \cdot \|Y\| \leq \|\tau(Y)\| \quad \implies \quad 2 \cdot \|AXB\| \leq \|A^2X + XB^2\| \quad (X \in \mathcal{B}(\mathcal{H})),$$

as desired.

In [4, Theorem 2], R. Bhatia and C. Davis showed the convexity of the non-negative function

$$f(p) := \|A^{1+p}XB^{1-p} + A^{1-p}XB^{1+p}\|$$

for $p$ in the interval $[-1, 1]$, $A, B \in M_n(\mathbb{C})$ positive semidefinite, and $X \in M_n(\mathbb{C})$ arbitrary; they also proved that the minimum value of $f$ on $[-1, 1]$ is attained at $p = 0$. If we now replace $A$ and $B$ by $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, respectively, and if we let $r := \frac{1+p}{2}$, from the convexity of $f$ we can readily obtain the inequalities

$$2 \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| = f(0) \leq \left\|A^rXB^{1-r} + A^{1-r}XB^r\right\| \leq \max\{f(-1), f(1)\} = \|AX + XB\|. \quad (6.2)$$

In eq. (6.2), $A, B \in M_n(\mathbb{C})$ are positive semidefinite, $X \in M_n(\mathbb{C})$ arbitrary, and $r$ runs in the interval $[0, 1]$.

In what follows, we extend the inequalities (1) and (2) in eq. (6.2) to all bounded linear operators $X \in \mathcal{H}$, but using invertible and positive $A$ and $B$, and for the case when the norm $\|\cdot\|$ is the operator norm $\|.\|$. Instead of trying to establish the convexity of $f$, we resort to tools and techniques from multivariable spectral theory applied to a suitable elementary operator.
**Theorem 6.3.** With $A$, $B$, $X$ as in Theorem 6.1, and $r \in [0, 1]$, we have

\[ 2 \left\| A^{\frac{1}{2}} XB^{\frac{1}{2}} \right\| \leq \left\| A^r XB^{1-r} + A^{1-r} XB^r \right\| \leq \left\| AX + XB \right\|, \]

**Proof.** For the first inequality (1) in (6.2), we consider again the commuting 4–tuple $(L_A, L_{A^{-1}}, R_B, R_{B^{-1}})$, with Taylor spectrum given as

\[ \{ (\lambda, \lambda^{-1}, \rho, \rho^{-1}) : \lambda \in \sigma(A), \rho \in \sigma(B) \}. \]

For fixed $r \in [0, 1]$, consider now the continuous map $\psi_r : (0, \infty) \times (0, \infty) \to \mathbb{R}$ given by

\[ \psi_r(s, t) := s^{r-1} t^{r-1} + s^{\frac{1}{2}} t^{\frac{1}{2}} \quad (s, t > 0) \]

and let

\[ \psi(X) := A^{r-\frac{1}{2}} XB^{\frac{1}{2}} + A^{\frac{1}{2}} \rho^{-r} XB^r \quad (X \in B(H)). \]

Then, as in the Proof of Theorem 3.6, we have:

\[ \sigma(\psi) = \left\{ \lambda^{r-\frac{1}{2}} \rho^{\frac{1}{2}} + \lambda^{\frac{1}{2}} \rho^{-r} \right\} \quad \lambda \in \sigma(A), \rho \in \sigma(B) \}

Since

\[ \lambda^{r-\frac{1}{2}} \rho^{\frac{1}{2}} + \lambda^{\frac{1}{2}} \rho^{-r} = \left( \frac{\lambda}{\rho} \right)^{r-\frac{1}{2}} + \left( \frac{\rho}{\lambda} \right)^{r-\frac{1}{2}} \geq 2, \]

it follows that

\[ m(\psi) := \inf \{ |\lambda| : \lambda \in \sigma(\tau) \} \geq 2. \]

Let $Z := A^{\frac{1}{2}} XB^{\frac{1}{2}}$. Then, by (3.3) in Proposition 3.3 and, as done in the Proofs of Theorem 3.6, Theorem 4.1 and Theorem 5.1, we have

\[ 2 \left\| Z \right\|_{HS} \leq \left\| \psi(Z) \right\|_{HS} \leq 2 \left\| Z \right\| \leq \left\| \psi(Z) \right\| \leq 2 \left\| A^{\frac{1}{2}} XB^{\frac{1}{2}} \right\| \leq \left\| A^r XB^{1-r} + A^{1-r} XB^r \right\|, \]

as desired. For the second inequality (2) in eq. (6.2), we recall from [20] that

\[ f(r) = \left\| A^r XB^{1-r} + A^{1-r} XB^r \right\| \]

is convex on $[0, 1]$ and attains its maximum at $r = 0, 1$. This immediately yields (2).
R. Kaur, M. Moslehian, M. Singh and C. Conde in ([19, Theorem 4.1]) showed a further refinement of (2) in eq. (6.2) using the Schur (resp. Hadamard) product: Let $A, B, X \in M_n(\mathbb{C})$ with $A, B$ positive semidefinite. Then, for any real numbers $\alpha, \beta$,

$$2 \left\| A^{\frac{\alpha+\beta}{2}} X B^{\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} X B^{\frac{\alpha+\beta}{2}} \right\| \leq \left\| A^\alpha X B^{1-\alpha} + A^{1-\alpha} X B^\alpha + A^\beta X B^{1-\beta} + A^{1-\beta} X B^\beta \right\|$$

(6.3)

In the following theorem, we extend (6.3) to the case of bounded linear operators on $\mathcal{H}$ and also obtain new inequalities.

**Theorem 6.5.** Let $A, B, X \in B(\mathcal{H})$ with $A, B$ invertible positive semidefinite. Then, for any real numbers $\alpha, \beta$,

$$2 \left\| A^{\frac{\alpha+\beta}{2}} X B^{\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} X B^{\frac{\alpha+\beta}{2}} \right\| \leq \left\| A^\alpha X B^{1-\alpha} + A^{1-\alpha} X B^\alpha + A^\beta X B^{1-\beta} + A^{1-\beta} X B^\beta \right\| \leq 2 \left\| AX + XB \right\|.$$ 

(6.4)

**Proof.** For the inequality (3) in (6.4), we consider the commuting pairs

$$\left( A^{\frac{\alpha}{2}}, A^{\frac{\beta}{2}} \right) \text{ and } \left( B^{\frac{\alpha}{2}}, B^{\frac{\beta}{2}} \right),$$

the 4-tuple $(L_A, L_{A^{-1}}, R_B, R_{B^{-1}})$, and self-adjoint and invertible operators $A, B \in B(\mathcal{H})$. For fixed $r \in [0, 1]$, consider now the continuous map $\phi_r : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ given by

$$\phi_{\alpha, \beta}(s, t) := s^{\frac{\beta}{2}} t^{\frac{\alpha}{2}} + s^{\frac{\alpha}{2}} t^{\frac{\beta}{2}} \quad (s, t > 0)$$

and let

$$\phi(X) := A^{\frac{\alpha}{2}} X B^{\frac{\beta}{2}} + A^{\frac{\beta}{2}} X B^{\frac{\alpha}{2}} \quad (X \in B(\mathcal{H})).$$

Then, as in the Proof of Theorem 3.6, we have:

$$\sigma(\phi) = \left\{ \lambda^{\frac{\alpha}{2}} \rho^{\frac{\beta}{2}} + \lambda^{\frac{\beta}{2}} \rho^{\frac{\alpha}{2}} : \lambda \in \sigma(A), \rho \in \sigma(B) \right\}.$$ 

Since

$$\lambda^{\frac{\alpha}{2}} \rho^{\frac{\beta}{2}} + \lambda^{\frac{\beta}{2}} \rho^{\frac{\alpha}{2}} = \left( \frac{\lambda}{\rho} \right)^{\alpha - \beta} + \left( \frac{\rho}{\lambda} \right)^{\alpha - \beta} \geq 2,$$

it follows that

$$m(\phi) := \inf \{|\lambda| : \lambda \in \sigma(\tau)| \geq 2.$$
Let \( W := A^{\frac{\alpha+\beta}{2}} XB^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} XB^{\frac{\alpha+\beta}{2}} \). Then, by (3.3) in Proposition 3.3, we have
\[
2 \|W\|_{HS} \leq \|\phi(W)\|_{HS} \implies 2 \|W\| \leq \|\phi(W)\|
\]

\[
\implies 2 \left\| A^{\frac{\alpha+\beta}{2}} XB^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} XB^{\frac{\alpha+\beta}{2}} \right\| 
\leq \left\| A^\alpha XB^{1-\alpha} + A^{1-\alpha} XB^\alpha + A^\beta XB^{1-\beta} + A^{1-\beta} XB^\beta \right\|. 
\] (6.5)

Next, we consider the inequality (4) in (6.4). By Theorem 7.10, if we let \( f(r) := \lambda \left( \frac{r}{\lambda} \right)^r + \rho \left( \frac{1}{\rho} \right)^r \), then
\[
\sup_{0 \leq r \leq 1} f(r) \leq f(0) = f(1) = \lambda + \rho.
\]

Next, still with same assumptions on \( A, B, X \), and for any real number \( \alpha \), we have
\[
\| A^\alpha XB^{1-\alpha} + A^{1-\alpha} XB^\alpha \| \leq \| AX + XB \|. 
\] (6.6)

For, using the \( 2 \times 2 \) matrix trick employed in Theorem 6.17, we can reduce the proof to the case \( A = B \). Then, by the unitary invariance, we could assume that a positive matrix \( A \in M_n(\mathbb{C}) \) is diagonal with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Observe
\[
A^\alpha X A^{1-\alpha} + A^{1-\alpha} X A^\alpha = \left[ \frac{\lambda_i \left( \frac{\lambda_i}{\lambda_j} \right)^\alpha + \lambda_i \left( \frac{\lambda_j}{\lambda_i} \right)^\alpha}{\lambda_i + \lambda_j} \right] \circ (AX +XA),
\]
where \( \circ \) means the Schur (or Hadamard) product. By theorem 7.10, we can see that the all \((i,j)\) entries of the matrix \( \left[ \frac{\lambda_i \left( \frac{\lambda_i}{\lambda_j} \right)^\alpha + \lambda_i \left( \frac{\lambda_j}{\lambda_i} \right)^\alpha}{\lambda_i + \lambda_j} \right] \) are less than or equal to 1. It is known in ([3, 2.7.12 Exercise]) that if \( A \geq 0 \), then \( \| A \circ X \| \leq \max a_{ii} \| X \| \) for \( X \in M_n(\mathbb{C}) \), where \( a_{ii} \) is the diagonal entry of \( A \). Thus, we have eq. (6.6), as desired. Now, using the above result, and the triangle inequality, we can establish (4) in eq. (6.4). Finally, to transition from matrices to operators on an infinite dimensional Hilbert space \( \mathcal{H} \), we apply the approximation-of-the-identity technique employed in the proof of Theorem 5.1.

**Theorem 6.7.** Let \( A, B, X \in \mathcal{B}(\mathcal{H}) \) with \( A, B \) positive and invertible. Then, for real numbers \( \ell \) and \( r \), we have
\[
2 \left\| A^\ell XB^\ell \right\| \leq \left\| A^{r}XB^{2\ell-r} + A^{2\ell-r}XB^r \right\|. \] (6.7)

**Proof.** Once again, we consider the commuting 4–tuple \( (L_A, L_{A^{-1}}, R_B, R_{B^{-1}}) \). For fixed \( r \in [0,1] \), consider now the continuous map \( \psi_r : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R} \) given by
\[
\psi_{\ell,r}(s,t) := t^{r-\ell} s^{s-r} + s^{s-r} t^{-\ell} \quad (s,t > 0).
\]
Let

$$\Psi(X) := A^{r-\ell}XB^{\ell-r} + A^{\ell-r}XB^{r-\ell} \quad (X \in \mathcal{B}(\mathcal{H})).$$

Then, as in the Proof of Theorem 3.6, we have:

$$\sigma(\Psi) = \left\{ \lambda^{r-\ell} \rho^{\ell-r} + \lambda^{\ell-r} \rho^{r-\ell} : \lambda \in \sigma(A), \rho \in \sigma(B) \right\}.$$ 

Since

$$\lambda^{r-\ell} \rho^{\ell-r} + \lambda^{\ell-r} \rho^{r-\ell} = \left( \frac{\lambda}{\rho} \right)^{r-\ell} + \left( \frac{\rho}{\lambda} \right)^{r-\ell} = \left( \left( \frac{\lambda}{\rho} \right)^{\frac{r-\ell}{2}} \right)^{2} + \left( \left( \frac{\rho}{\lambda} \right)^{\frac{r-\ell}{2}} \right)^{2} \geq 2 \left( \frac{\lambda}{\rho} \right)^{\frac{r-\ell}{2}} \left( \frac{\rho}{\lambda} \right)^{-\frac{r-\ell}{2}} = 2,$$

it follows that

$$m(\Psi) := \inf \{ |\lambda| : \lambda \in \sigma(\tau) \} \geq 2.$$

Let $$Y := A^\ell XB^\ell.$$ We have

$$2 \|Y\|_{HS} \leq \|\Psi(Y)\|_{HS} \implies 2 \|Y\| \leq \|\Psi(Y)\| \implies 2 \|A^\ell XB^\ell\| \leq \|A^{r-\ell}XB^{2\ell-r} + A^{2\ell-r}XB^{r}\|,$$

as desired.

**Theorem 6.9.** Let $$A, B, X \in \mathcal{B}(\mathcal{H})$$ with $$A, B$$ positive and invertible. If $$A$$ and $$B$$ are contractive, then for real numbers $$s, t,$$ we have

$$\|A^sXB^{1+t} + A^{1+s}XB^t\| \leq \|AX + XB\|. \quad (6.8)$$

**Proof.** For the inequality (6) in (6.8), for real numbers $$s, t$$ we consider the commuting pair $$(A^{-s}, B^{-t}).$$ Let

$$\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

be given by $$\Phi(X) := A^{-s}XB^{-t}.$$ Since $$\|A\|, \|B\| \leq 1,$$ $$\Phi$$ is bounded below with bound 1 and invertible. Since $$\sigma(\Phi^{-1}) \subseteq \overline{D}$$ and $$\Phi^{-1}$$ is self-adjoint in $$\mathcal{C}_2(\mathcal{H}),$$ we know that

$$\|\Phi^{-1}\|_{HS} \leq 1.$$
For the following inequality,
\[ \|Y\| \leq \|\Phi(Y)\|, \quad Y \in \mathcal{B}(\mathcal{H}), \]
we need Ky Fan’s Dominance Theorem, used in the same way as in the proof of Theorem 4.1 with \( Y = A^vXB^u + A^wXB^t \), where \( v, w \) are real numbers. Thus, we have
\[ \|A^vXB^u + A^wXB^t\| \leq \|A^{w-s}X + XB^{u-t}\|. \tag{6.9} \]
If \( w = 1 + s \) and \( v = 1 + t \), then (6.9) becomes
\[ \|A^vXB^{1+t} + A^{1+s}XB^t\| \leq \|AX + XB\|, \tag{6.10} \]
as desired.

For the next result, we recall the notion of Kwong function. A continuous real-valued function \( f \) defined on an interval \((a, b)\) with \( a \geq 0 \) is called a Kwong function if the matrix
\[ K_f := \left( \frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i + \lambda_j} \right)_{i,j=1,2,...,n} \]
is positive semidefinite for any distinct real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) in \((a, b)\). It is easy to see that if \( f \) is a non-zero Kwong function then \( f \) is positive and \( \frac{1}{f} \) is Kwong \[21\]. Using a Kwong function, H. Najafi \[21\] proved the following result.

**Lemma 6.11.** \[21\] Let \( A \) and \( B \) be positive \( n \times n \) complex matrices, let \( X \) be any \( n \times n \) matrix, and let \( f \) and \( g \) be two continuous functions on \((0, \infty)\) such that \( \frac{f(x)}{g(x)} \) is a Kwong function and \( f(x)g(x) \leq x \). Then
\[ \|f(A)Xg(B) + g(A)Xf(B)\| \leq \|AX + XB\|. \]

**Theorem 6.12.** Let \( \ell \) and \( r \) be real numbers. If \( -\frac{1}{2} \leq r - \ell \leq \frac{1}{2} \) and \( A, B, X \in \mathcal{M}_n(\mathbb{C}) \) with \( A, B \) positive and invertible, then
\[ \|A^{-\ell+r+\frac{1}{2}}XB^\ell-r+\frac{1}{2} + A^{\ell-r+\frac{1}{2}}XB^{-\ell+r+\frac{1}{2}}\| \leq \|AX + XB\|. \tag{6.11} \]

**Proof.** To establish eq. (6.11), we need the following variant of theorem 7.12: For real numbers \( \ell, r \) with \( -\frac{1}{2} \leq r - \ell \leq \frac{1}{2} \) and for \( i, j = 1, 2, \ldots, n \) nonzero positive real numbers \( \lambda_i, \lambda_j \), consider the matrix \( Y \) whose entries are
\[ \frac{\lambda_i^{2r-2\ell} + \lambda_j^{2r-2\ell}}{\lambda_i + \lambda_j}. \]
Then $Y$ is positive semidefinite.

Let $h(x) = x^{2r-2\ell}$, and let

$$K_h := \left( \frac{h(\lambda_i) + h(\lambda_j)}{\lambda_i + \lambda_j} \right)_{i,j=1,2,...,n} = \left( \frac{\lambda_i^{2r-2\ell} + \lambda_j^{2r-2\ell}}{\lambda_i + \lambda_j} \right)_{i,j=1,2,...,n}.$$

By theorem 7.8 applied to real numbers $\ell, r$ with $-\frac{1}{2} \leq r - \ell \leq \frac{1}{2}$, we know that $h(x) = \frac{f(x)}{g(x)}$ is a Kwong function, using $f(x) = x^{-\ell+r+\frac{1}{2}}$ and $g(x) = x^{\ell-r+\frac{1}{2}}$. Moreover, $f(x)g(x) = x$. Thus, by Theorem 6.11, we obtain the desired inequality eq. (6.11).

By Theorem 6.12, we have the following:

**Remark 6.14.** (i) With the same conditions as in Theorem 6.12, and with $\|A\|, \|B\| \leq 1$, consider the commuting pair $\left( A^{\ell-r+\frac{1}{2}}, B^{\ell-r+\frac{1}{2}} \right)$ and the map

$$\Psi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}),$$

given by $\Psi(X) := A^{\ell-r+\frac{1}{2}}XB^{\ell-r+\frac{1}{2}}$. Since $\|A\|, \|B\| \leq 1$, $\Psi$ is bounded below with bound 1 and therefore invertible. Since $\sigma(\Psi^{-1}) \subseteq \overline{D}$ and $\Psi^{-1}$ is self-adjoint in $C_2(\mathcal{H})$, we know that

$$\|\Psi^{-1}\|_{HS} \leq 1.$$

By Ky Fan’s Dominance Theorem, and employing the technique used in the Proof of theorem 3.6 to establish eq. (6.7), we obtain

$$\|AXB^{2\ell-2r+1} + A^{2\ell-2r+1}XB\| \leq \|AX + XB\|.$$

(ii) Using the convexity of the nonnegative function $f$ in (eq. (6.1)), we can readily extend theorem 6.12 to the case $|\ell - r| \leq 1$.

The following result appears in [5, Theorem 1]); we give a new proof using multivariable techniques similar to those employed in Section 3. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be a complex infinite dimensional Hilbert spaces and Let $A, B$ be operators on $\mathcal{H}_1, \mathcal{H}_2$, respectively, such that $A$ and $B^*$ are subnormal and $\text{dist}(\sigma(A), \sigma(B)) = \delta > 0$. Let $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. If $AX - XB$ is a Hilbert-Schmidt operator, then $X$ is also Hilbert-Schmidt and

$$\delta \|X\|_{HS} \leq \|AX - XB\|_{HS}. \quad (6.12)$$
If $AX - XB$ lies in a subspace $\mathcal{L}$ of $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ which is the natural domain of a symmetric norm $\| \cdot \|_s$, then $X \in \mathcal{L}$ and
\[ \delta \|X\|_s \leq c \|AX - XB\|_s. \tag{6.13} \]
where $c$ is a universal constant. By the technique used in the proof of Theorem 3.6, we have the following:

**Theorem 6.15.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be invertible self-adjoint operators and $X \in \mathcal{B}(\mathcal{H})$. We also let
\[ \varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \]
be given by
\[ \varphi(X) := AXB^{-1} - A^{-1}XB. \]
Assume $0 \leq A, B \leq I$. Then we have
\[ 2\delta \cdot \|X\| \leq \|\varphi(X)\|, \tag{6.14} \]
where $\delta = \text{dist} (\sigma(A), \sigma(B))$. (Note that if $\delta = 0$, then eq. (6.14) is trivial.)

**Proof.** For $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$, note
\[ \left| \frac{\alpha}{\beta} - \frac{\beta}{\alpha} \right| = \left| \frac{\alpha^2 - \beta^2}{\alpha\beta} \right| = \left| \frac{(|\alpha| - \beta)(\alpha + \beta)}{\alpha^2} \right| \geq \frac{\delta (\alpha + \beta)}{\alpha^2} \frac{\alpha + \beta}{\alpha\beta} \geq \delta \left( \frac{\alpha + \beta}{\alpha\beta} \right) = \delta \left( \frac{\alpha\beta + 1}{\alpha\beta} \right) \tag{6.15} \]
Thus, by eq. (3.9) and eq. (6.15), we have
\[ \left| \frac{\alpha}{\beta} - \frac{\beta}{\alpha} \right| \geq 2\delta. \tag{6.16} \]
By Ky Fan’s Dominance Theorem in [17] used in the proof of Theorem 3.6, we obtain
\[ 2\delta \|X\|_{HS} \leq \|\varphi(X)\|_{HS}. \]
Therefore, by the same technique used in the proof of Theorem 3.6, we have
\[ 2\delta \|X\| \leq \|\varphi(X)\|, \]
as desired. Finally, to transition from matrices to operators on an infinite dimensional Hilbert space $\mathcal{H}$, once again we apply the approximation-of-the-identity technique employed in the proof of Theorem 5.1.

As an application of Theorems 6.5 and 6.7, we have:
Corollary 6.17. Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint and invertible operators and $X \in \mathcal{B}(\mathcal{H})$. Then we have:

(i) for any real numbers $r$ and $\ell$, $w(A^\ell XB^\ell) \leq w\left(\frac{A^rXB^{2\ell-r}+A^{2\ell-r}XB^r}{2}\right)$;
(ii) for any real numbers $\alpha$ and $\beta$, $w\left(A^{\frac{\alpha+\beta}{2}}XB^{\frac{2-\alpha-\beta}{2}}+A^{\frac{2-\alpha-\beta}{2}}XB^{\frac{\alpha+\beta}{2}}\right) \leq w\left(A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^{\alpha}+A^\beta XB^{1-\beta}+A^{1-\beta}XB^{\beta}\right)$;
(iii) $\delta \cdot w(X) \leq w\left(\frac{AXB^{-1}A^{-1}XB}{2}\right)$, where $\delta := \text{dist}(\sigma(A), \sigma(B))$.

Proof. For (i), by taking into consideration Lemma 7.6, it suffices to prove that for all $\mu \in \mathbb{C}$ and any real numbers $r$ and $\ell$,

$$\|X - \mu I\| \leq \left|\frac{A^{r-\ell}XA^{\ell-r} + A^{-r}XA^{r-\ell}}{2} - \mu I\right|.$$

By (6.7) in Theorem 6.7, we have

$$2\|X - \mu I\| \leq \|A^{r-\ell} (X - \mu I) A^{\ell-r} + A^{\ell-r} (X - \mu I) A^{r-\ell}\|
\Rightarrow \|X - \mu I\| \leq \left|\frac{A^{r-\ell}XA^{\ell-r} + A^{\ell-r}XA^{r-\ell}}{2} - \mu I\right|
\Rightarrow W(X) \leq W\left(\frac{A^{r-\ell}XA^{\ell-r} + A^{\ell-r}XA^{r-\ell}}{2}\right) \quad (6.17)
\Rightarrow w(X) \leq w\left(\frac{A^{r-\ell}XA^{\ell-r} + A^{\ell-r}XA^{r-\ell}}{2}\right).$$

Next, we let $Z := \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ and $S := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then, $Z$ and $S$ are self-adjoint. Moreover, $S$ is invertible and $w(Z) = w(X)$. Hence, by eq. (6.17), we have

$$w(X) = w(Z) \leq w\left(\frac{S^{r-\ell}ZS^{\ell-r} + S^{\ell-r}ZS^{r-\ell}}{2}\right) = w\left(\frac{A^{r-\ell}XB^{\ell-r} + A^{\ell-r}XB^{r-\ell}}{2}\right). \quad (6.18)$$

Define $Y := A^{-\ell}XB^{-\ell} \in \mathcal{B}(\mathcal{H})$, so that $X = A^\ell Y B^\ell$. Thus, eq. (6.18) becomes

$$w(A^\ell Y B^\ell) \leq w\left(\frac{A^{r-\ell}A^\ell Y B^\ell B^{\ell-r} + A^{\ell-r}A^{\ell+r} B^\ell B^{r-\ell}}{2}\right) = w\left(\frac{A^{\ell+r}Y B^{\ell-r} + A^{\ell-r}Y B^{\ell+r}}{2}\right),$$

as desired.
For (ii), we similarly prove that for all \( \mu \in \mathbb{C} \) and for any real numbers \( \alpha \) and \( \beta \),
\[
\|X - \mu I\| \leq \left| \frac{A^{\alpha-\beta} X \frac{\alpha-\beta}{2} + A^{\beta-\alpha} X \frac{\alpha-\beta}{2}}{2} - \mu I \right|.
\]
Then
\[
2 \|X - \mu I\| \leq \left| A^{\alpha-\beta} (X - \mu I) A^{\frac{\beta-\alpha}{2}} + A^{\frac{\beta-\alpha}{2}} (X - \mu I) A^{\frac{\alpha-\beta}{2}} \right|
\]
\[
\implies w(X) \leq w \left( A^{\alpha-\beta} X \frac{\alpha-\beta}{2} + A^{\beta-\alpha} \frac{\alpha-\beta}{2} X A^{\alpha-\beta} \right)
\]
and
\[
w(X) = w(Z) \leq w \left( \frac{2^\alpha - Z \frac{\alpha-\beta}{2} + 2^\beta - Z \frac{\alpha-\beta}{2}}{2} \right).
\]
(6.19)
In (6.19), replace \( X \) by \( A^{\frac{\alpha+\beta}{2}} Y B^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} Y B^{\frac{\alpha+\beta}{2}} \) to obtain
\[
w \left( A^{\alpha+\beta} Y B^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} Y B^{\frac{\alpha+\beta}{2}} \right)
\]
\[
\leq w \left( A^{\alpha} Y B^{1-\alpha} + A^{1-\alpha} Y B^{\alpha} + A^{\beta} Y B^{1-\beta} + A^{1-\beta} Y B^{\beta} \right)
\]
\[
\leq w \left( \frac{A^{\alpha} Y B^{1-\alpha} + A^{1-\alpha} Y B^{\alpha} + A^{\beta} Y B^{1-\beta} + A^{1-\beta} Y B^{\beta}}{2} \right),
\]
as desired.

For (iii), we similarly prove that for all \( \mu \in \mathbb{C} \),
\[
\delta \cdot \|X - \mu I\| \leq \left| \frac{AXB^{-1} - A^{-1}XB}{2} - \mu I \right|.
\]
Since
\[
2\delta \cdot \|X - \mu I\| \leq \left| A \left( X - \mu I \right) A^{-1} + A^{-1} \left( X - \mu I \right) A \right|
\]
\[
\implies \delta w(X) \leq w \left( \frac{AXA^{-1} - A^{-1}XA}{2} \right),
\]
we have
\[
\delta w(X) = \delta w(\hat{X}) \leq w \left( \frac{\hat{A}\hat{X}A^{-1} + A^{-1}\hat{X}\hat{A}}{2} \right)
\]
\[
= w \left( \frac{AXB^{-1} - A^{-1}XB}{2} \right),
\]
as desired.

**Remark 6.19.** Observe that Theorem 6.17 readily implies Theorems 2.1(ii), 2.3, and 2.4.
7. Appendix

For the reader’s convenience, in this section, we gather several well-known auxiliary results which are needed for the proofs of the main results in this article.

7.1. The class \( \mathcal{C}_2 \)

Let \( \mathcal{H} \) be a complex separable infinite dimensional Hilbert space, let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of bounded linear operators on \( \mathcal{H} \), and let \( \mathcal{C}_2 \) denote the two-sided Schatten ideal associated with the 2–norm defined by the trace:

\[
\|X\|_{HS}^2 = \langle X, X \rangle_{HS} = \text{Tr}(X^*X) = \sum_{i=1}^{\infty} \left( s_i((X^*)^{\frac{1}{2}}) \right)^2
\]

where \( s_i((X^*)^{\frac{1}{2}}) \) denotes the \( i \)-th singular value of \( X \) \((i = 1, 2, \ldots)\). It is well known that \( (\mathcal{C}_2, \| \cdot \|_{HS}) \) is a separable infinite dimensional Hilbert space. Moreover, every bounded operator on \( \mathcal{H} \) gives rise to a bounded operator on \( \mathcal{C}_2 \) by left or right multiplication; that is, if \( A, B \in \mathcal{B}(\mathcal{H}) \) then \( L_A|_{\mathcal{C}_2}(X) := AX \) and \( R_B|_{\mathcal{C}_2}(X) := XB \) for all \( X \in \mathcal{C}_2 \); moreover, \( \|L_A|_{\mathcal{C}_2}\| = \|A\| \) and \( \|R_B|_{\mathcal{C}_2}\| = \|B\| \).

**Lemma 7.1.** Let \( A \) and \( B \) be bounded operators on \( \mathcal{H} \), and let \( L_A|_{\mathcal{C}_2} \) and \( R_B|_{\mathcal{C}_2} \) be the restrictions of the left and right multiplication operators to the Hilbert-Schmidt class \( \mathcal{C}_2 \). Then

\[
(L_A|_{\mathcal{C}_2})^* = (L_A^*)|_{\mathcal{C}_2} \quad \text{and} \quad (R_B|_{\mathcal{C}_2})^* = (R_B^*)|_{\mathcal{C}_2}.
\]

**Proof.** For \( X \) and \( Y \) in \( \mathcal{C}_2 \), we have

\[
\langle (L_A|_{\mathcal{C}_2})^*(X), Y \rangle_{HS} = \langle X, L_A|_{\mathcal{C}_2}(Y) \rangle_{HS} = \langle X, AY \rangle_{HS} = \text{Tr}((AY)^*X) = \text{Tr}(Y^*A^*X) = \langle (A^*X), Y \rangle_{HS} = \langle (L_{A^*}|_{\mathcal{C}_2})(X), Y \rangle_{HS}.
\]

It follows that, when restricted to \( \mathcal{C}_2 \), \( (L_A)^* = L_{A^*} \), as desired. Similarly,

\[
\langle (R_B|_{\mathcal{C}_2})^*X, Y \rangle_{HS} = \langle X, R_B|_{\mathcal{C}_2}(Y) \rangle_{HS} = \langle X, YB \rangle_{HS} = \text{Tr}((YB)^*X) = \text{Tr}(B^*Y^*X) = \text{Tr}(Y^*XB^*) = \langle XB^*, Y \rangle_{HS} = \langle (R_{B^*}|_{\mathcal{C}_2})(X), Y \rangle_{HS}.
\]

and therefore \( (R_B|_{\mathcal{C}_2})^* = (R_{B^*})|_{\mathcal{C}_2} \).
7.2. Taylor spectrum

Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ be two commuting operators, and consider the Koszul complex $K(T, \mathcal{H})$ associated to $T = (T_1, T_2)$ on $\mathcal{H}$:

$$K(T, \mathcal{H}): 0 \longrightarrow \mathcal{H} \overset{\delta_1=(-T_2,T_1)}{\longrightarrow} \mathcal{H} \oplus \mathcal{H} \overset{\delta_0=(T_1,T_2)}{\longrightarrow} \mathcal{H} \longrightarrow 0,$$

where $T = (T_1, T_2)$, $T = \left( \begin{array}{c} T_1 \\ T_2 \end{array} \right)$, and $T_1, T_2 \in \mathcal{B}(\mathcal{H})$.

**Definition 7.3.** A commuting pair $T$ is said to be (Taylor) invertible if its associated Koszul complex $K(T, \mathcal{H})$ is exact. The Taylor spectrum of $T$ is

$$\sigma_T(T) := \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : K((T_1 - \lambda_1, T_2 - \lambda_2), \mathcal{H}) \text{ is not invertible} \}.$$

The pair $T$ is called Fredholm if each map in the Koszul complex $K(T, \mathcal{H})$ has closed range and all the homology quotients are finite-dimensional. The Taylor essential spectrum is

$$\sigma_{Te}(T) := \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : (T_1 - \lambda_1, T_2 - \lambda_2) \text{ is not Fredholm} \}.$$

J.L. Taylor showed in [23] that, if $\mathcal{H} \neq \{0\}$, then $\sigma_T(T)$ is a nonempty, compact subset of the polydisc of multiradius $r(T) := (r(T_1), r(T_2))$, where $r(T_i)$ is the spectral radius of $T_i$ ($i = 1, 2$). He also showed that the Spectral Mapping Theorem for bivariate polynomials holds for the Taylor spectrum; that is, if $p : \mathbb{C}^2 \rightarrow \mathbb{C}$ is a polynomial, then

$$\sigma(p(T_1, T_2)) = p(\sigma_T(T_1, T_2))$$

(7.1)

for all commuting pairs of operators $T_1$ and $T_2$.

R.E. Curto proved in [7] that $\sigma_{Te}(T) \subseteq \sigma_T(T)$ is also nonempty and compact. (For additional facts about these joint spectra, the reader is referred to [8] and [24].)

For elementary operators, R.E. Curto and L.A. Fialkow proved the following result.

**Lemma 7.4.** [9] Let $A$ and $B$ be commuting $n$-tuples of operators on a Hilbert space $\mathcal{H}$, and let $(L_A, R_B)$ be the $(2n)$-tuple of left and right multiplications induced by $A$ and $B$ on $\mathcal{B}(\mathcal{H})$. Then

$$\sigma_T((L_A, R_B), \mathcal{B}(\mathcal{H})) = \sigma_T((L_A|_{C_2}, R_B|_{C_2}), C_2) = \sigma_T(A, \mathcal{H}) \times \sigma_T(B, \mathcal{H}).$$
7.3. Numerical range and radius

Given a bounded operator \( T \) on \( \mathcal{H} \), the numerical range and the numerical radius of \( T \) are defined by

\[
W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H} \text{ and } \|x\| = 1 \}
\]

and

\[
w(T) = \sup \{ |\xi| : \xi \in W(T) \},
\]

respectively.

**Lemma 7.5.** (cf. [13, Problem 218]) Let \( T \in \mathcal{B}(\mathcal{H}) \). Then \( w(T) \leq \|T\| \leq 2 \cdot w(T) \) and \( w(T^n) \leq (w(T))^n \).

**Lemma 7.6.** [15, 22] Let \( T \in \mathcal{B}(\mathcal{H}) \). Then

\[
\overline{W(T)} = \bigcap_{\mu \in \mathbb{C}} \{ \lambda \in \mathbb{C} : |\lambda - \mu| \leq \|T - \mu I\| \},
\]

where \( \overline{W(T)} \) denotes the closure of the numerical range of \( T \).

**Lemma 7.7.** [2] Let \( T \in \mathcal{B}(\mathcal{H}) \). Then \( w(T) \leq 1 \) is equivalent to

\[
\|T - zI\| \leq 1 + \left( 1 + |z|^2 \right)^{1/2} \text{ for all } z \in \mathbb{C}.
\]

7.4. Classical inequalities of Heinz type

**Lemma 7.8.** [14] Let \( \mathcal{H} \) be a complex Hilbert space and let \( A, B, X \in \mathcal{B}(\mathcal{H}) \). Then, for \( t \in [0,1] \) we have

\[
\|A^tXB^t\| \leq \|AXB\|^t \|X\|^{1-t} \tag{7.2}
\]

and

\[
\|A^tXB^{1-t}\| \leq \|AXB\|^t \|X\|^{1-t} \tag{7.3}
\]

**Lemma 7.9.** [11] For \( a,b \geq 0, 0 \leq t \leq 1 \leq r \) and \( p,q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

(a) \( a^t b^{1-t} \leq ta + (1 - t)b \leq (ta^r + (1-t)b^r)^{\frac{1}{r}} \).

(b) \( ab \leq \frac{a^p}{p} + \frac{a^q}{q} \leq \left( \frac{a^{rp}}{p} + \frac{a^{rq}}{q} \right)^{\frac{1}{r}} \).
Lemma 7.10. For $0 \leq r \leq 1$ and nonzero real numbers $\lambda, \rho$ with $\lambda \neq \rho$, consider the function

$$f (r) := \lambda \left( \frac{\rho}{\lambda} \right)^r + \rho \left( \frac{\lambda}{\rho} \right)^r.$$ 

Then

$$\sup_{0 \leq r \leq 1} f (r) \leq f (0) = f (1) = \lambda + \rho.$$ 

Proof. Since $f' (r) = \left[ \lambda \left( \frac{\rho}{\lambda} \right)^r - \rho \left( \frac{\lambda}{\rho} \right)^r \right] \ln \left( \frac{\rho}{\lambda} \right)$, we have that $f' (r) = 0 \implies \frac{\rho}{\lambda} = \left( \frac{\rho}{\lambda} \right)^r \implies r = \frac{1}{2}$. Thus $r = \frac{1}{2}$ is the critical point of $f (r)$ on $0 \leq r \leq 1$. Since

$$f'' (r) = \left[ \lambda \left( \frac{\rho}{\lambda} \right)^r + \rho \left( \frac{\lambda}{\rho} \right)^r \right] \left( \ln \left( \frac{\rho}{\lambda} \right) \right)^2 > 0,$$

it follows that

$$\sup_{0 \leq r \leq 1} f (r) = \max \{ f (0), f (1) \} = f (0) = f (1) = \lambda + \rho,$$

as desired.

Lemma 7.12. (cf. [3, Proof of Theorem 5.4.1]) For $-1 \leq \alpha \leq 1$ and $\lambda_i > 0$ ($i = 1, 2, \ldots, n$), the matrix $Y$ whose entries are

$$y_{ij} = \frac{\lambda_i^\alpha + \lambda_j^\alpha}{\lambda_i + \lambda_j}$$

is positive semidefinite.

7.5. Ky Fan’s Dominance Theorem

Recall now the notion of symmetric gauge function. A real-valued function $\Phi$ defined on $\mathbb{R}^n$, is called a symmetric gauge function if it satisfies the following conditions:

(i) $\Phi (x) > 0$ ($x \neq 0$);
(ii) $\Phi (\rho x) = |\rho| \Phi (x)$;
(iii) $\Phi (x + y) \leq \Phi (x) + \Phi (y)$;
(iv) $\Phi (\pi x) = \Phi (x)$; and
(v) $\Phi (J x) = \Phi (x)$,

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\rho \in \mathbb{C}$, $\pi \in M_n (\mathbb{C})$ is any permutation, and $J$ is any diagonal matrix whose diagonal elements are $\pm 1$. Moreover,
given two vectors \( x, y \in \mathbb{R}^n \), with coordinates listed in descending order, we say that \( x \) is weakly majorized by \( y \) (in symbols, \( x <_w y \)) if

\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad (k = 1, \ldots, n).
\]

For additional facts about symmetric gauge functions, the reader is referred to [18, Chapter 3, Definition 3.5.17 ff.]. It is known (see [12]) that a unitarily invariant norm stands in one-to-one correspondence to a symmetric gauge function \( \Phi \) on \( \mathbb{R}_+^n \) by the relation \( \|A\| = \Phi (s_1 (A), \ldots, s_n (A)) \), where \( (s_1 (A), \ldots, s_n (A)) \) is the \( n \)-tuple of the singular values of \( A \in M_n (\mathbb{C}) \).

**Lemma 7.13.** [17] (Ky Fan’s Dominance Theorem) Let \( x, y \in \mathbb{R}^n \). Then \( x <_w y \) if and only if \( \|x\| \leq \|y\| \) for all unitarily invariant norms on \( \mathbb{R}^n \), that is, \( \Phi (x) \leq \Phi (y) \) for every symmetric gauge functions (or symmetric norms) \( \Phi \) on \( \mathbb{R}^n \).

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**References**


