THE MEAN TRANSFORM AND THE MEAN LIMIT OF AN OPERATOR

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ABSTRACT. Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$, and let $T \equiv V|T|$ be the polar decomposition of T. The mean transform of T is defined by $\widehat{T} := \frac{1}{2}(V|T| + |T|V)$. In this paper we study the iterates of the mean transform and we define the mean limit of an operator as the limit (in the operator norm) of those iterates. We obtain new estimates for the numerical range and numerical radius of the mean transform in terms of the original operator. For the special class of unilateral weighted shifts we describe the precise relationship between the spectral radius and the mean limit, and obtain some sharp estimates.

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space equipped with its inner product $\langle .,. \rangle$. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and by $\mathcal{U}(\mathcal{H})$ the set of unitary operators on $\mathcal{H}$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* the range, the null subspace, and the operator adjoint of T, respectively. The numerical range of T is the set

$$W(T) := \Big\{\langle Tx, x\rangle \ : \ x \in \mathcal{H}, \ \ \|x\| = 1\Big\},$$

and the numerical radius of T is defined as

$$w(T) := \sup\{|\lambda| \ : \ \lambda \in W(T)\}.$$

We refer the reader to [14] for more information about the numerical range and numerical radius.

We also let $\sigma(T)$ denote the spectrum of T, and $r(T)$ denote its spectral radius. An operator T is said to be quasinormal if it commutes with T^*T. For $0 < p \leq 1$, we say that T is p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. In the case when $p = 1$ and $p = \frac{1}{2}$, the operator T is called hyponormal and semi-hyponormal, respectively.

As usual, for $T \in \mathcal{B}(\mathcal{H})$ we denote the modulus of T by $|T| := (T^*T)^{1/2}$ and we shall always write, without further mention, $T = V|T|$ to be the canonical polar decomposition of T, where V is the appropriate partial isometry satisfying $\mathcal{N}(V) = \mathcal{N}(T)$. The Aluthge transform of T was defined in [1] by

$$\Delta(T) := |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}.$$

The mean transform of T, recently introduced in [11], is given as

$$\widehat{T} := \frac{1}{2}(V|T| + |T|V). \tag{1.1}$$

We refer the reader to [1, 7, 8, 9, 11, 10] for such operator transforms.

It is well known that the quasinormal operators are exactly the fixed points of the Aluthge transform and of the mean transform (see [11]). The sequences of iterates of mean and Aluthge transforms of an operator T are denoted by $\widehat{T}^{(n)}$ and $\Delta^{(n)}(T)$ (respectively), with $\widehat{T}^{(0)} = \Delta^{(0)}(T) = T$, $\widehat{T}^{(n+1)} = \widehat{\widehat{T}^{(n)}}$ and $\Delta^{(n+1)}(T) := \Delta(\Delta^{(n)}(T))$. A list of detailed and informative articles on the subject includes [2, 9, 11, 10].

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In the next sections we establish some new properties of the mean transform. We also introduce the mean limit of an operator, and we study the mean transform iterates in some particular cases. Moreover, we obtain several new relations between the Aluthge and mean transforms for unilateral weighted shifts. For this class of operators, we obtain some sharp estimates for the spectral radius of the mean transform and the mean limit.

2. Some results about the mean transform

Contrary to what happens with the Aluthge transform, the mean transform does depend on the polar decomposition of the given operator. For example, consider $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $\mathbb{C}^2$. The canonical polar decomposition of T is $T = V|T|$, where $|T| = \sqrt{T^*T} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. On the other hand, we can also write $T = U_{max}|T|$, where $U_{max} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is unitary. This is the so-called *maximal* polar decomposition of T, since the partial isometry is unitary. In this case,

$$U_{max}|T| + |T|U_{max} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq V|T| + |T|V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which shows that the mean transform depends on the polar decomposition. In what follows, we will always use the canonical polar decomposition when dealing with the mean transform.

Proposition 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an arbitrary operator. Then we have*

$$\mathcal{N}(\widehat{T}) = \mathcal{N}(T).$$

In particular $\widehat{T} = 0$ if and only if $T = 0$.

Proof. Let $T = V|T|$ be the canonical polar decomposition of T, and let $x \in \mathcal{N}(T) = \mathcal{N}(V)$. Then $|T|x = Vx = 0$ and thus $\widehat{T}x = \frac{1}{2}(V|T|x + |T|Vx) = 0$. This show that $\mathcal{N}(T) \subseteq \mathcal{N}(\widehat{T})$.

Conversely, if $x \in \mathcal{N}(\widehat{T})$ then

$$V|T|x + |T|Vx = 0,$$

and hence

$$|T|x + V^*|T|Vx = V^*(V|T| + |T|V)x = 0.$$

It follows that

$$\langle |T|x, x\rangle + \langle V^*|T|Vx, x\rangle = \langle |T|x + V^*|T|Vx, x\rangle = 0.$$

Since $|T|$ and $V^*|T|V$ are both positive, we obtain

$$\||T|^{\frac{1}{2}}x\| = \langle |T|x, x\rangle = 0.$$

As a consequence,

$$\mathcal{N}(\widehat{T}) \subseteq \mathcal{N}(|T|) = \mathcal{N}(T),$$

as desired. ◻

Theorem 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

(i) T is invertible.
(ii) $\widehat{T}$ is invertible and $\mathcal{R}(T)$ is closed.

Proof. Let $T = V|T|$ be the canonical polar decomposition of T. $(i) \Rightarrow (ii)$ We assume that T is invertible; then $\mathcal{R}(T) = \mathcal{H}$ is closed. Moreover, V is unitary and the operators $|T|$ and $V^*|T|V$ are positive and invertible. Hence $|T| > 0$, $V^*|T|V > 0$ and $V^*\widehat{T} = \frac{1}{2}(|T|+V^*|T|V) > 0$ is invertible. It follows that $\widehat{T} = \frac{1}{2}V(|T| + V^*|T|V)$ is also invertible.

$(ii) \Rightarrow (i)$ Assume now that $\widehat{T}$ is invertible. From Proposition 2.1, T is one-to-one, so V is isometry, i.e. $V^*V = I$. It follows that $V^*\widehat{T} = \frac{1}{2}(|T| + V^*|T|V)$ is also invertible, and its inverse is $(\widehat{T})^{-1}V$. Therefore, V is unitary, since it maps $(\mathcal{N}(T)^{\perp}$ isometrically onto $\overline{\mathcal{R}(T)}$. Therefore, $\mathcal{N}(T) = \mathcal{N}(T^*) = \{0\}$. Since $\mathcal{R}(T)$ is closed, $\mathcal{R}(T) = \overline{\mathcal{R}(T)} = (\mathcal{N}(T^*))^{\perp} = \mathcal{H}$. Hence T is invertible. This completes the proof. □

Remark 2.3. *In Theorem 2.2 (ii), the condition "$\mathcal{R}(T)$ is closed" is required; without it, the reverse implication is false, as shown by the following example.*

Example 2.4. *Let us denote by $(e_n)_{n\in\mathbb{Z}}$ the canonical basis of $\ell^2(\mathbb{Z})$, and by $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ the weighted bilateral shift defined by $Te_n = \alpha_n e_{n+1}$ for all $n \in \mathbb{Z}$, where*

$$\alpha_n = \begin{cases} 1 & \text{if } n \text{ even} \\ \frac{1}{n^2} & \text{if } n \text{ odd} . \end{cases}$$

The mean transform $\widehat{T}$ is also a weighted shift, and we have $\widehat{T}e_n = \widehat{\alpha}_n e_{n+1}$ for $n \in \mathbb{Z}$, where

$$\widehat{\alpha}_n = \frac{\alpha_n + \alpha_{n+1}}{2} = \begin{cases} \dfrac{1 + \frac{1}{(n+1)^2}}{2} & \text{if } n \text{ even} \\ \dfrac{1 + \frac{1}{n^2}}{2} & \text{if } n \text{ odd} . \end{cases}$$

Clearly, $Te_{2n+1} \underset{n\to\infty}{\longrightarrow} 0$, and therefore the operator T is not invertible. On the other hand, we have $1 \geq \widehat{\alpha}_n \geq \frac{1}{2}$ for all $n \in \mathbb{Z}$, and from this it follows that $\widehat{T}$ is invertible. □

Remark 2.5. *In general we have:*
(1) $\sigma(T) \neq \sigma(\widehat{T})$ (see [11]);
(2) $(\widehat{T})^{-1} \neq \widehat{T^{-1}}$ (see [10]).

Proposition 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following properties hold.*
(i) *For all $\alpha \in \mathbb{C}$, $\widehat{(\alpha T)} = \alpha\widehat{T}$.*
(ii) *For every unitary or anti-unitary operator $U : \mathcal{H} \to \mathcal{H}$, we have*

$$\widehat{UTU^*} = U\widehat{T}U^*.$$

(iii) $\widehat{T} = I \iff T = I$.

Proof. (i) Straightforward from (1.1).
(ii) Let $T = V|T|$ be the polar decomposition of T and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. First note that

$$|UTU^*| = U|T|U^*,$$

and we therefore have

$$UTU^* = UV|T|U^* = (UVU^*)(U|T|U^*) = \tilde{V}|UTU^*|,$$

where $\tilde{V} = UVU^*$. Observe that $\tilde{V}$ is a partial isometry and $\mathcal{N}(UTU^*) = \mathcal{N}(\tilde{V})$; it follows that $\tilde{V}|UTU^*|$ is the polar decomposition of UTU^*. This implies that

$$\begin{aligned}(\widehat{UTU^*}) &= \frac{1}{2}(\tilde{V}|UTU^*| + |UTU^*|\tilde{V}) \\ &= \frac{1}{2}(UV|T|U^* + U|T|VU^*) \\ &= U\widehat{T}U^*.\end{aligned}$$

When U is anti-unitary, the result is obtained in a similar fashion.
(iii) The implication $(\Leftarrow)$ is obvious, so we focus on $(\Rightarrow)$. Assume that $\widehat{T} = I$; hence $V^* = V^*\widehat{T} = \frac{1}{2}(|T| + V^*|T|V)$ is a positive partial isometry. In particular, $V = V^*$ is an orthogonal projection. On the other hand, still using $\widehat{T} = I$, we can use Proposition 2.1(i) and conclude that T is one-to-one. Then V is an isometry, so $V^*V = V^2 = I$. Therefore $V = I$ and $|T| = \widehat{T} = I$. □

Lemma 2.7. *(**Heinz inequality**, cf. [6]) Let $A, B, X \in \mathcal{B}(\mathcal{H})$ such that A and B are positive operators. Then*

$$\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \le \left\|\frac{AX + XB}{2}\right\|.$$

Corollary 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\left\|\Delta(T)\right\| \le \left\|\widehat{T}\right\| \le \left\|T\right\|.$$

In particular, $r(T) \le \left\|\widehat{T}\right\|$.

For partial isometries, we have the following result.

Proposition 2.9. *Let $V \in \mathcal{B}(\mathcal{H})$ be a partial isometry. Then*

$$\widehat{V} = \frac{1}{2}(I + V^*V)V.$$

In particular,

$$\sigma(V) = \sigma(\widehat{V}).$$

Proof. The modulus of V is $|V| = V^*V$ and the polar decomposition of V is $V = V(V^*V)$. Hence

$$\widehat{V} = \frac{1}{2}(V + V^*VV) = \frac{1}{2}(I + V^*V)V. \tag{2.1}$$

Since $VV^*V = V$, it follows that

$$\sigma(\widehat{V})\backslash\{0\} = \sigma(\frac{1}{2}(I + V^*V)V)\backslash\{0\} = \sigma(\frac{1}{2}V(I + V^*V))\backslash\{0\} = \sigma(V)\backslash\{0\}. \tag{2.2}$$

Now observe that if V is invertible then $V^*V = I$ and therefore $\widehat{V} = V$; it follows that $\widehat{V}$ is also invertible. Conversely, if $\widehat{V}$ is invertible then (2.1) implies that V is left invertible, that is, V is an isometry. This means $V^*V = I$, and therefore $\widehat{V} = V$, and a fortiori V is also invertible. This argument together with (2.2) establishes the equality of the spectra. □

By Corollary 2.8, $\|\widehat{T}\| \le \|T\|$. As a consequence, the norm of the iterated mean transforms $(\|\widehat{T}^{(n)}\|)_{n\in\mathbb{N}}$ is a non-increasing sequence. Since it is bounded below by 0, it converges; we denote the limit by $\ell(T)$.

Definition 2.10. *Let $T \in \mathcal{B}(\mathcal{H})$. The mean limit $\ell(T)$ is the limit in norm of the sequence of mean transform iterates; that is,*

$$\ell(T) = \lim_{n\to\infty} \|\widehat{T}^{(n)}\| = \inf_{n\in\mathbb{N}} \|\widehat{T}^{(n)}\|.$$

Remark 2.11. *Let $U \in \mathcal{U}(\mathcal{H})$ and $\alpha \in \mathbb{C}$. Then*

(i) $\ell(UTU^*) = \ell(T)$ and $\ell(\alpha T) = |\alpha|\ell(T)$.
(ii) In the case when T is quasinormal, $\ell(T) = \|T\| = r(T)$.
(iii) If $T^2 = 0$ then $\widehat{T}^{(n)} = \frac{1}{2^n}T$; as a consequence, $\ell(T) = 0$.

For the reader's convenience we provide a proof of (iii). Consider the canonical polar decomposition $T = V|T|$, and recall that $\overline{\mathcal{R}(T)} = \mathcal{R}(V)$. Since $T^2 = 0$ we must have $\mathcal{R}(T) \subseteq \mathcal{N}(T) = \mathcal{N}(|T|)$. It then follows that $|T|V = 0$, which readily implies $\widehat{T} = \frac{1}{2}T$. The desired result is now clear.

It is now natural to formulate the following

Problem 2.12. *For a general bounded linear operator $T \in \mathcal{B}(\mathcal{H})$, describe what $\ell(T)$ says about T.*

For $x, y \in \mathcal{H}$ $(x \ne 0, y \ne 0)$, we denote by $x \otimes y$ the rank one operator defined by

$$(x \otimes y)u := \langle u, y\rangle x \ (u \in \mathcal{H}).$$

The λ-Aluthge transform [2] of a rank one operator is given in [3] as follows:

$$\Delta_\lambda(x\otimes y)=\frac{\langle x,y\rangle}{||y||^2}y\otimes y.$$

In the following lemma, we give the mean transform of this class of operators, and we show that the sequence of their mean iterates converges to the Aluthge transform.

Lemma 2.13. *Let $x,y\in\mathcal{H}$ be two nonzero vectors, let $T:=x\otimes y$ be the rank one operator with range generated by x, and let $n\in\mathbb{N}$. Then the n-th iterate of T is*

$$\widehat{T}^{(n)}=\frac{1}{2^n}(x+(2^n-1)\frac{\langle x,y\rangle}{||y||^2}y)\otimes y.$$

In particular, $\widehat{T}^{(n)}\underset{n\to+\infty}{\longrightarrow}\Delta_\lambda(T)$ and $\ell(T)=|\langle x,y\rangle|=r(T)$.

Proof. We first exhibit the mean transform of a rank one operator. A simple calculation yields

$$|T|=\sqrt{T^*T}=\frac{||x||}{||y||}(y\otimes y).$$

Let

$$V:=\frac{1}{||x||||y||}x\otimes y.$$

We then have $\mathcal{N}(V)=\mathcal{N}(T)$ and $V^*V=\dfrac{1}{||y||^2}y\otimes y$ is an orthogonal projection. Hence V is a partial isometry and

$$V|T|=\Big(\frac{1}{||x||||y||}(x\otimes y)\Big)\Big(\frac{||x||}{||y||}(y\otimes y)\Big)=T.$$

Therefore $T=V|T|$ is the canonical polar decomposition of T. It follows that

$$\begin{aligned}\widehat{T}&=\frac{1}{2}(V|T|+|T|V)\\&=\frac{1}{2}(x\otimes y+\frac{||x||}{||y||}(y\otimes y)\frac{1}{||x||||y||}(x\otimes y))\\&=\frac{1}{2}(x\otimes y+\frac{\langle x,y\rangle}{||y||^2}(y\otimes y))\\&=\frac{1}{2}(x+\frac{\langle x,y\rangle}{||y||^2}y)\otimes y.\end{aligned}$$

Now, by induction on n the equality

$$\widehat{T}^{(n)}=\frac{1}{2^n}(x+(2^n-1)\frac{\langle x,y\rangle}{||y||^2}y)\otimes y$$

holds immediately. Moreover,

$$\widehat{T}^{(n)}\underset{n\to+\infty}{\longrightarrow}\frac{\langle x,y\rangle}{||y||^2}y\otimes y=\Delta(T).$$

In particular,

$$\ell(T)=|\langle x,y\rangle|=r(T).$$

□

To study the mean iterates of a large class of Hilbert space operators we will first need the following result.

Lemma 2.14. *Let $T\in\mathcal{B}(\mathcal{H})$, with canonical polar decomposition $T=V|T|$. The following assertions are equivalent.*

(1) $\mathcal{N}(T^*)\subseteq\mathcal{N}(T)$.
(2) $VV^*|T|=|T|VV^*=|T|$.
(3) V^* is quasinormal (i.e., $VV^*V^*=V^*$).

In this case, we have

$$\widehat{T} = \frac{1}{2}(V|T| + |T|V) = \frac{1}{2}V(|T| + V^*|T|V).$$

Proof. (1) ⇒ (2) Suppose that (1) holds. Then

$$\mathcal{N}(VV^*) = \mathcal{N}(V^*) = \mathcal{N}(T^*) \subseteq \mathcal{N}(T) = \mathcal{N}(|T|).$$

Hence, $|T|(I - VV^*) = 0$. Thus $VV^*|T| = |T|VV^* = |T|$.

(2) ⇒ (3) Suppose that (2) holds. Then

$$\mathcal{N}(V^*) \subseteq \mathcal{N}(|T|) = \mathcal{N}(V).$$

It follows, $V(I - VV^*) = 0$. Hence $VVV^* = V$ and thus $V^* = VV^*V^*$.

(3) ⇒ (2) Since $V^* = VV^*V^*$, we have $V = VVV^*$. Hence $\mathcal{N}(V^*) \subseteq \mathcal{N}(V)$ and

$$\mathcal{N}(T^*) = \mathcal{N}(V^*) \subseteq \mathcal{N}(V) = \mathcal{N}(T).$$

This completes the proof. □

We now state and prove one of our main results.

Theorem 2.15. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\mathcal{N}(T^*) \subseteq \mathcal{N}(T)$. Let $T = V|T|$ be the canonical polar decomposition of T, and let $n \in \mathbb{N}$. Then*

$$\widehat{T}^{(n)} = \frac{1}{2^n}V\Big(\sum_{j=0}^{j=n}\tbinom{n}{j}(V^*)^j|T|V^j\Big). \tag{2.3}$$

Proof. We will use induction on n. For $n = 0$, the equality (2.3) holds immediately. Since $\mathcal{N}(T^*) \subseteq \mathcal{N}(T)$, we can use Lemma 2.14 to conclude that

$$\widehat{T} = \frac{1}{2}(V|T| + |T|V) = \frac{1}{2}V(|T| + V^*|T|V).$$

In particular, (2.3) holds also for $n = 1$.

We now assume that (2.3) holds for $n \in \mathbb{N}$, $n \geq 1$. From Proposition 2.1 we have

$$\mathcal{N}(V) = \mathcal{N}(T) = \mathcal{N}(\widehat{T}^{(n)}).$$

Since $V^*VV^* = V^*$, it follows that

$$|\widehat{T}^{(n)}| = \frac{1}{2^n}\Big(\sum_{j=0}^{j=n}\tbinom{n}{j}(V^*)^j|T|V^j\Big).$$

Hence $\widehat{T}^{(n)} = V|\widehat{T}^{(n)}|$ is the canonical polar decomposition of $\widehat{T}^{(n)}$. Thus

$$\begin{aligned}
\widehat{T}^{(n+1)} &= \frac{\widehat{T}^{(n)} + |\widehat{T}^{(n)}|V}{2} \\
&= \frac{1}{2^{n+1}}\Big(V\sum_{j=0}^{j=n}\tbinom{n}{j}(V^*)^j|T|V^j + \Big(\sum_{j=0}^{j=n}\tbinom{n}{j}(V^*)^j|T|V^j\Big)V\Big) \\
&= \frac{1}{2^{n+1}}V\sum_{j=0}^{j=n}\tbinom{n}{j}\Big((V^*)^j|T|V^j + (V^*)^{j+1}|T|V^{j+1}\Big) \\
&= \frac{1}{2^{n+1}}V\sum_{j=0}^{j=n+1}\tbinom{n+1}{j}(V^*)^j|T|V^j.
\end{aligned}$$

Hence (2.3) holds for $n + 1$. This completes the proof. □

Corollary 2.16. *Let $T \in \mathcal{B}(\mathcal{H})$ be such that T and T^* are one-to-one. Then $\|\widehat{T}^{(n)}\| = \|\widehat{T^*}^{(n)}\|$ for all $n \in \mathbb{N}$. In particular, T and T^* have the same mean limit*

$$\ell(T) = \ell(T^*).$$

Theorem 2.17. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\mathcal{N}(T^*) \subseteq \mathcal{N}(T)$. If T is a semi-hyponormal operator then $\widehat{T}$ is also semi-hyponormal. Moreover, the sequence of mean iterates converges in the strong operator topology to a normal operator $L \in \mathcal{B}(\mathcal{H})$, and we have*

$$\ell(T) = \|L\| = \|\widehat{T}^{(n)}\| \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. Let $T = V|T|$ be the polar decomposition of T. It is easy to get that $|T^*| = V|T|V^*$. Suppose that T is semi-hyponormal. Then $|T^*| = V|T|V^* \leq |T|$. Multiplying this inequality by V^* on the left and by V on the right, we get that

$$V|T|V^* \leq |T| \leq V^*|T|V. \tag{2.4}$$

On the other hand, since $\mathcal{N}(T^*) \subseteq \mathcal{N}(T)$, Lemma 2.14 and a simple calculation yield

$$|\widehat{T}| = \frac{1}{2}(|T| + V^*|T|V).$$

Hence, it follows from Lemma 2.14 again that $\widehat{T} = V|\widehat{T}|$ is the canonical polar decomposition of $\widehat{T}$, and

$$\begin{aligned} |(\widehat{T})^*| &= V|\widehat{T}|V^* \\ &= \frac{1}{2}(V|T|V^* + VV^*|T|VV^*) \\ &= \frac{1}{2}(V|T|V^* + |T|), \quad \text{(since } VV^*|T| = |T|) \\ &\leq \frac{1}{2}(|T| + V^*|T|V) = |\widehat{T}| \quad \text{(see (2.4))}. \end{aligned}$$

This shows that $\widehat{T}$ is semi-hyponormal.

Since $\mathcal{N}(T^*) \subseteq \mathcal{N}(T)$, we have

$$\mathcal{R}(\widehat{T}) \subseteq \overline{\mathcal{R}(T)} \quad \text{and} \quad \mathcal{N}(\widehat{T}^*) \subseteq \mathcal{N}(\widehat{T}).$$

For the first inclusion, note that the condition on the kernels implies that $\mathcal{R}(T^*) \subseteq \overline{\mathcal{R}(T)}$. It follows, by definition of $\widehat{T}$, that $\mathcal{R}(\widehat{T}) \subseteq \overline{\mathcal{R}(T)}$. The second inclusion is obtained as follows:

$$\begin{aligned} \widehat{T}^*x = 0 \quad &\Rightarrow \quad T^*x + V^*|T|x = 0 \\ &\Rightarrow \quad V|T|V^*x + VV^*|T|x = 0 \\ &\Rightarrow \quad V|T|V^*x + |T|x = 0, \quad \text{(since } VV^*|T| = |T|) \\ &\Rightarrow \quad x \in \mathcal{N}(|T|) = \mathcal{N}(T) = \mathcal{N}(\widehat{T}). \end{aligned}$$

Now, by the induction we obtain, for all $n \in \mathbb{N}$,

$$\mathcal{R}(\widehat{T}^{(n)}) \subseteq \overline{\mathcal{R}(T)}, \quad \mathcal{N}((\widehat{T}^{(n)})^*) \subseteq \mathcal{N}(\widehat{T}^{(n)}) \quad \text{and} \quad \widehat{T}^{(n)} \ \text{ is semi-hyponormal.}$$

We also know that $\widehat{T}^{(n)} = V|\widehat{T}^{(n)}|$ is the canonical polar decomposition of $\widehat{T}^{(n)}$, with

$$|\widehat{T}^{(n+1)}| = \frac{1}{2}(|\widehat{T}^{(n)}| + V^*|\widehat{T}^{(n)}|V) \geq |\widehat{T}^{(n)}| \quad \text{(by (2.4))}. \tag{2.5}$$

In particular, $(|\widehat{T}^{(n)}|)_{n\in\mathbb{N}}$ is an increasing sequence, so it converges in the strong operator topology to a positive operator $A \in \mathcal{B}(\mathcal{H})$ with $\mathcal{R}(A) \subseteq \overline{\mathcal{R}(T)}$. It follows that $VV^*A = A$. From (2.5), A satisfies $A = \frac{1}{2}(A + V^*AV)$, and therefore $VA = AV$. It follows that $\widehat{T}^{(n)} = V|\widehat{T}^{(n)}|$ strongly converges to the normal operator $L := VA = AV$. Again, from (2.5) we obtain $\|\widehat{T}^{(n+1)}\| = \|\widehat{T}^{(n)}\| = \|T\| = \|L\|$, for all $n \in \mathbb{N}$, as desired. □

Remark 2.18. *Under the assumption of Theorem 3.1, if the sequence* $((V^*)^k|T|V^k)_{k\in\mathbb{N}}$ *converges in the strong (resp. weak) operator topology to an operator* $A \in \mathcal{B}(\mathcal{H})$*, then so does the sequence of operator mean iterates* $(\widehat{T}^{(n)})_{n\in\mathbb{N}}$*; the limit is the normal operator* $L = VA = AV$.

3. Numerical range and numerical radius

Theorem 3.1. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then*

$$\overline{W(\widehat{T})} \subseteq \overline{W(T)},$$

where $\overline{W(T)}$ *denotes the closure of the numerical range* $W(T)$ *of* T*. In particular,*

$$w(\widehat{T}) \leq w(T).$$

Proof. Recall first the well known formula for the numerical range, (see [14, Theorem 4 and Corollary])

$$\overline{W(T)} = \bigcap_{\lambda\in\mathbb{C}} \Big\{ \ \mu \ : \ |\mu-\lambda| \leq \|T-\lambda I\| \ \Big\}.$$

Let $T = V|T|$ be the canonical polar decomposition of T. From [4, Lemma 2.3] we have

$$\|\, |T|V - \lambda I \,\| \leq \|T-\lambda I\|, \quad (\text{all} \quad \lambda \in \mathbb{C}).$$

Therefore,

$$\begin{aligned}
\overline{W(\widehat{T})} &= \bigcap_{\lambda\in\mathbb{C}} \Big\{ \ \mu \ : \ |\mu-\lambda| \leq \|\widehat{T}-\lambda I\| \ \Big\} \\
&= \bigcap_{\lambda\in\mathbb{C}} \Big\{ \ \mu \ : \ |\mu-\lambda| \leq \frac{1}{2}\|T-\lambda I + |T|V-\lambda I\| \ \Big\} \\
&\subseteq \bigcap_{\lambda\in\mathbb{C}} \Big\{ \ \mu \ : \ |\mu-\lambda| \leq \frac{1}{2}(\|T-\lambda I\| + \||T|V-\lambda I\|) \ \Big\} \\
&\subseteq \bigcap_{\lambda\in\mathbb{C}} \Big\{ \ \mu \ : \ |\mu-\lambda| \leq \|T-\lambda I\| \ \Big\} = \overline{W(T)}.
\end{aligned}$$

□

Lemma 3.2. *(Cf.* [12]*) Let* $A, B, X \in \mathcal{B}(\mathcal{H})$ *such that* A, B *are positive. Then*

$$w(A^{\frac{1}{2}}XB^{\frac{1}{2}}) \leq w\Big(\frac{A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha}{2}\Big).$$

for all $0 \leq \alpha \leq 1$.

As a direct consequence of Theorem 3.1 and Lemma 3.2 we get the following result.

Corollary 3.3. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be an arbitrary operator, and recall that* $\Delta(T)$ *denotes the Aluthge transform of* T*. Then*

$$w(\Delta(T)) \leq w(\widehat{T}) \leq w(T).$$

4. The mean limit for unilateral weighted shifts

Let $\ell^2(\mathbb{N})$ be the Hilbert space of complex square-summable sequences $x = (x_i)_{i\in\mathbb{N}}$, with the norm $\|x\| := (\sum_{i\in\mathbb{N}} |x_i|^2)^{\frac{1}{2}}$. Given any bounded sequence of strictly positive numbers $\alpha \equiv (\alpha_i)_{i\in\mathbb{N}}$, the associated unilateral weighted shift $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1 ...) : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ is defined by

$$W_\alpha(x_0, x_1, ...) := (0, \alpha_0 x_0, \alpha_1 x_1, ...),$$

where $x = (x_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{N})$. When $\alpha_i = 1$ for all i, $W_\alpha = W$ is simply the standard (unweighted) unilateral shift on $\ell^2(\mathbb{N})$.

Clearly, W_α is a bounded linear operator on $\ell^2(\mathbb{N})$, with operator norm $\|W_\alpha\| = \sup_{i\in\mathbb{N}} \alpha_i$. The spectral radius of W_α is well known (see, for example, [5, Problem 91]):

$$r(W_\alpha) = \lim_{n\to\infty}\Big(\sup_{k\in\mathbb{N}}(\alpha_k...\alpha_{k+n-1})\Big)^{\frac{1}{n}}. \tag{4.1}$$

The spectrum of W_α is given in [13, p.66, Theorem 4] by

$$\sigma(W_\alpha) = \{\lambda \in \mathbb{C} : |\lambda| \le r(W_\alpha)\}.$$

The Aluthge transform $\Delta(W_\alpha)$ of W_α is also a unilateral weighted shift:

$$\Delta(W_\alpha) = W_{\alpha^\Delta} = \text{shift}\Big(\sqrt{\alpha_0\alpha_1},\ \sqrt{\alpha_1\alpha_2}, ...\Big).$$

By induction, the iterates of the Aluthge transform are given in [9] by

$$\Delta^{(n)}(W_\alpha) = W_{\alpha^{\Delta^{(n)}}},$$

where

$$\alpha^{\Delta^{(n)}} = \Big\{\alpha_i^{\Delta^{(n)}}\Big\}_{i\in\mathbb{N}} = \Big\{\Big(\prod_{j=0}^{n}\alpha_{i+j}^{\binom{n}{j}}\Big)^{\frac{1}{2^n}}\Big\}_{i\in\mathbb{N}}$$

and $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$.

As explained in [11], the mean transform $\widehat{W_\alpha}$ of W_α is

$$\widehat{W_\alpha} = W_{\widehat{\alpha}} = \text{shift}\Big(\frac{\alpha_0+\alpha_1}{2}, \frac{\alpha_1+\alpha_2}{2}, ...\Big).$$

The mean iterates of the weighted shift $\widehat{W_\alpha}^{(n)}$ are also weighted shifts with weight sequences

$$\alpha^{(n)} = \{\widehat{\alpha}_i^{(n)}\}_{i\in\mathbb{N}} = \Big\{\frac{\sum_{i=0}^{n}\binom{n}{j}\alpha_{i+j}}{2^n}\Big\}_{i\in\mathbb{N}}. \tag{4.2}$$

We remark that, for a sequence of strictly positive numbers $\alpha = (\alpha_i)_{i\in\mathbb{N}}$, we have the following relation between the iterates of Aluthge and mean transforms,

$$W_{\exp(\widehat{\alpha}^{(n)})} = \Delta^{(n)}(W_{\exp(\alpha)}).$$

where, $\exp(\beta) = (\exp(\beta_i))_{i\in\mathbb{N}}$ for any sequence $\beta = (\beta_i)_{i\in\mathbb{N}}$.

In contrast to what happens with the iterates of the Aluthge transform, the spectrum of the mean transform is not the same as the spectrum of the original operator. Moreover, in general the sequence of norms of the mean iterates does not converge to the spectral radius, as shown by the following example.

Example 4.1. *Let W_α be the unilateral weighted shift defined by $\alpha \equiv (\alpha_i)_{i\in\mathbb{N}}$, where $\alpha_i := 2 + (-1)^i$. As proven in* [11], *$\widehat{W_\alpha} = 2U_+$ (the unweighted unilateral shift), and is therefore quasi-normal. However, W_α is not quasinormal. This proves that the inverse mean transform does not preserve the set of quasinormal operators.*

On the other hand, by the formula for the spectral radius of a unilateral weighted shift W_α (given in [5, Problem 91]*), we get the following*

$$\begin{aligned} r(W_\alpha) &= \lim_{n\to\infty}\Big(\sup_{k\in\mathbb{N}}(\alpha_k...\alpha_{k+n-1})\Big)^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\Big(\sup_{k\in\mathbb{N}}\prod_{k\le j\le k+n-1}(2+(-1)^j)\Big)^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\Big(\sup_{k\in\mathbb{N}}\prod_{\substack{k\le j\le k+n-1 \\ and\ j\ \ peer}} 3\Big)^{\frac{1}{n}} \\ &= \lim_{n\to\infty}(3^{\frac{n}{2}+\delta})^{\frac{1}{n}}, \quad (\ where\ \delta \in \{-1,0,1\}) \\ &= \lim_{n\to\infty} 3^{\frac{1}{2}+\frac{\delta}{n}} = \sqrt{3}. \end{aligned}$$

Hence, from [13, Section 4], *we conclude that*

$$\sigma(W_\alpha) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{3}\} \neq \sigma(\widehat{W_\alpha}) = 2\sigma(U_+) = \{\lambda \in \mathbb{C} : |\lambda| \leq 2\}.$$

On the other hand, the mean iterates of W_α are

$$\widehat{W_\alpha}^{(n)} = 2U_+ \ \ \text{for all} \quad n \in \mathbb{N}.$$

Therefore $\ell(W_\alpha) = 2 \neq r(W_\alpha)$. Thus, in general the sequence of operator mean iterates does not converge to the spectral radius. This is in sharp contrast to what happens for the Aluthge transform (see [15]*).* □

Theorem 4.2. *Let $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ be a sequence of strictly positive numbers, and let W_α be the associate weighted shift. Then*

$$r(W_\alpha) \leq r(\widehat{W_\alpha}).$$

Proof. From the spectral radius formula we obtain

$$\begin{aligned} r(W_\alpha) &= r(\Delta(W_\alpha)) \\ &= \lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}} \prod_{j=0}^{n-1} \alpha^{\Delta}_{i+j}\Big)^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}} \prod_{j=0}^{n-1} \sqrt{\alpha_{i+j}\alpha_{i+j+1}}\Big)^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}} \prod_{j=0}^{n-1} \frac{1}{2}(\alpha_{i+j} + \alpha_{i+j+1})\Big)^{\frac{1}{n}} \quad (\sqrt{ab} \leq \frac{1}{2}(a+b)) \\ &= \lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}} \prod_{j=0}^{n-1} \widehat{\alpha_{i+j}}\Big)^{\frac{1}{n}} \\ &= r(\widehat{W_\alpha}). \end{aligned}$$

□

As a direct consequence, we have the following result.

Corollary 4.3. *For a unilateral weighted shift W_α, we have*

$$\sigma(W_\alpha) \subseteq \sigma(\widehat{W_\alpha}).$$

Theorem 4.4. *Let $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ be a sequence of strictly positive numbers, and let $\beta :=$ $\exp(\alpha)$. The following estimate for the mean limit of W_α holds:*

$$r(W_\alpha) \leq \log(r(W_\beta)) = \ell(W_\alpha).$$

Proof. Using the iterates of Aluthge and mean transforms for the weighted shift W_α and Yamazaki's formula for the spectral radius (via the iterates of the Aluthge transform), we get

$$r(W_\beta) \quad = \quad \lim_{n\to\infty} \|\Delta^{(n)}(W_\beta)\| = \lim_{n\to\infty}\sup_{i\in\mathbb{N}}\Big(\prod_{j=0}^{n} \beta_{i+j}^{\binom{n}{j}}\Big)^{\frac{1}{2^n}} \quad \text{(Yamazaki's formula)}. \tag{4.3}$$

Using (4.3) and the particular form of the mean iterates of a unilateral weighted shift (cf. (4.2)), we obtain

$$\begin{aligned}
\ell(W_\alpha) &= \lim_{n\to\infty}\left\|\widehat{W_\alpha}^{(n)}\right\| \\
&= \lim_{n\to\infty}\sup_{i\in\mathbb{N}}\frac{\sum_{i=0}^{n}\binom{n}{j}\alpha_{i+j}}{2^n} \\
&= \lim_{n\to\infty}\sup_{i\in\mathbb{N}}\frac{\sum_{i=0}^{n}\binom{n}{j}\log(\beta_{i+j})}{2^n} \quad \Big(\text{recall that} \quad \log(\beta_k)=\alpha_k,\ \text{for all } k\in\mathbb{N}\Big) \\
&= \log\Big(\lim_{n\to\infty}\sup_{i\in\mathbb{N}}\Big(\prod_{j=0}^{n}\beta_{i+j}^{\binom{n}{j}}\Big)^{\frac{1}{2^n}}\Big) \\
&= \log\Big(\lim_{n\to\infty}\left\|\Delta^{(n)}(W_\beta)\right\|\Big) \\
&= \log\left(r(W_\beta)\right) \qquad \text{(Yamazaki's formula for the spectral radius [15])}
\end{aligned}$$

$$\begin{aligned}
&= \log\Big(\lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}}\prod_{j=0}^{n-1}\beta_{i+j}\Big)^{\frac{1}{n}}\Big) \quad \text{(spectral radius formula for weighted shifts [13])} \\
&= \lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}}\frac{1}{n}\log\Big(\prod_{j=0}^{n-1}\beta_{i+j}\Big)\Big) = \lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}}\frac{1}{n}\sum_{j=0}^{n-1}\log(\beta_{i+j})\Big) \\
&= \lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}}\frac{1}{n}\sum_{j=0}^{n-1}\alpha_{i+j}\Big) \\
&\geq \lim_{n\to\infty}\Big(\sup_{i\in\mathbb{N}}\prod_{j=0}^{n-1}\alpha_{i+j}\Big)^{\frac{1}{n}} \quad \text{(using the arithmetic-geometric mean inequality)} \\
&= r(W_\alpha).
\end{aligned}$$

□

Remark 4.5. *In general the inequality in Theorem 4.4 can be strict, as shown in Example 4.1.*

On the other hand, when the sequence $(\alpha_i)_{i\in\mathbb{N}}$ converges we have the following.

Proposition 4.6. *Let $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ be a sequence of positive numbers ($\alpha_i > 0$) and assume that α converges. Then*

$$\ell(W_\alpha) = r(W_\alpha) = \lim_{i\to\infty} \alpha_i.$$

Proof. We let $r_0 := r(W_\alpha) = \lim_{i\to\infty} \alpha_i$. Then, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|\alpha_i - r_0| \leq \epsilon$ for all $i \geq n_0$.

On the other hand,

$$\frac{\binom{n}{j}}{2^n} \leq K\frac{n^{j+1}}{2^n} \underset{n\to\infty}{\longrightarrow} 0 \ \text{ for any } \ 0 \leq i \leq n_0,\ \text{where } K \text{ is a fixed constant.}$$

Then

$$\frac{\sum_{j=0}^{n_0}\binom{n}{j}}{2^n} \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, there exists $n_1 > n_0$, such that

$$\frac{\sum_{j=0}^{n_0}\binom{n}{j}}{2^n} \leq \epsilon, \quad \text{for all} \quad n \geq n_1.$$

On the other hand, for $n > n_1$ we have

$$\begin{aligned}
\left|\left\|\widehat{W_\alpha}^{(n)}\right\| - r_0\right| &= \left|\sup_{i\in\mathbb{N}} \ \alpha_i^{(n)} - r_0\right| \\
&\leq \sup_{i\in\mathbb{N}} \left|\alpha_i^{(n)} - r_0\right| \\
&= \sup_{i\in\mathbb{N}} \left|\frac{\sum_{j=0}^{n}\binom{n}{j}\alpha_{i+j}}{2^n} - r_0\right| \\
&= \sup_{i\in\mathbb{N}} \left|\frac{\sum_{j=0}^{n}\binom{n}{j}(\alpha_{i+j} - r_0)}{2^n}\right| \\
&\leq \sup_{i\in\mathbb{N}} \frac{\sum_{j=0}^{n}\binom{n}{j}|\alpha_{i+j} - r_0|}{2^n} \\
&= \sup_{i\in\mathbb{N}} \underbrace{\frac{\sum_{j=0}^{n_0}\binom{n}{j}|\alpha_{i+j} - r_0|}{2^n}}_{\leq M\epsilon} + \underbrace{\frac{\sum_{j=n_0+1}^{n}\binom{n}{j}|\alpha_{i+j} - r_0|}{2^n}}_{\leq \frac{\sum_{j=n_0+1}^{n}\binom{n}{j}\ \epsilon}{2^{n+1}} \leq \epsilon} \\
&\leq \epsilon(M+1),
\end{aligned}$$

where $M := \sup_{i\in\mathbb{N}}|\alpha_i - r_0|$. This completes the proof. □

Remark 4.7. *Any semi-hyponormal unilateral weighted shift has a weight sequence satisfying the hypothesis of Proposition 4.6.*

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