JOINT SPECTRA OF SPHERICAL ALUTHGE TRANSFORMS OF COMMUTING n-TUPLES OF HILBERT SPACE OPERATORS

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ABSTRACT. Let $\mathbf{T} \equiv (T_1, \cdots, T_n)$ be a commuting n-tuple of operators on a Hilbert space $\mathcal{H}$, and let $T_i \equiv V_i P$ $(1 \le i \le n)$ be its canonical joint polar decomposition (i.e., $P := \sqrt{T_1^* T_1 + \cdots + T_n^* T_n}$, $(V_1, \cdots, V_n)$ a joint partial isometry, and $\bigcap_{i=1}^n \ker T_i = \bigcap_{i=1}^n \ker V_i = \ker P$). The spherical Aluthge transform of $\mathbf{T}$ is the (necessarily commuting) n-tuple $\hat{\mathbf{T}} := (\sqrt{P} V_1 \sqrt{P}, \cdots, \sqrt{P} V_n \sqrt{P})$. We prove that $\sigma_T(\hat{\mathbf{T}}) = \sigma_T(\mathbf{T})$, where σ_T denotes Taylor spectrum. We do this in two stages: away from the origin we use tools and techniques from criss-cross commutativity; at the origin we show that the left invertibility of $\mathbf{T}$ or $\hat{\mathbf{T}}$ implies the invertibility of P. As a consequence, we can readily extend our main result to other spectral systems that rely on the Koszul complex for their definition.

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Résumé. Soit $\mathbf{T} \equiv (T_1, \cdots, T_n)$ un n-uplet commutatif d'opérateurs sur un espace de Hilbert $\mathcal{H}$, et soit $T_i \equiv V_i P$ $(1 \le i \le n)$ sa décomposition polaire jointe canonique (i.e., $P := \sqrt{T_1^* T_1 + \cdots + T_n^* T_n}$, $(V_1, \cdots, V_n)$ est une isométrie partielle jointe, et $\bigcap_{i=1}^n \ker T_i = \bigcap_{i=1}^n \ker V_i = \ker P$). La transformée d'Aluthge sphérique de $\mathbf{T}$ est le n-uplet (nécessairement commutatif) $\hat{\mathbf{T}} := (\sqrt{P} V_1 \sqrt{P}, \cdots, \sqrt{P} V_n \sqrt{P})$. Nous démontrons que $\sigma_T(\hat{\mathbf{T}}) = \sigma_T(\mathbf{T})$, où σ_T désigne le spectre de Taylor. Nous procédons pour cela en deux étapes: En dehors de l'origine nous utilisons les outils et les techniques de la commutativité criss-cross; à l'origine nous prouvons que l'inversibilité à gauche de $\mathbf{T}$ ou de $\hat{\mathbf{T}}$ implique l'inversibilité de P. Comme conséquence, nous pouvons étendre notre résultat à d'autres systèmes spectraux définis à partir des complexes de Koszul.

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1. Introduction

Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space, let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$, and let $T \in \mathcal{B}(\mathcal{H})$. For $T \equiv V|T|$ the canonical polar decomposition of T, we let $\tilde{T} := |T|^{1/2} V |T|^{1/2}$ denote the Aluthge transform of T [1]. It is well known that T is invertible if and only if $\tilde{T}$ is invertible; moreover, the spectra of T and $\tilde{T}$ are equal. Over the last two decades, considerable attention has been given to the study of the Aluthge transform; cf. [2]–[5], [11], [16]–[29], [33], [39]–[42]). Moreover, the Aluthge transform has been generalized to the case of powers of $|T|$ different from $\frac{1}{2}$ ([6], [7], [10], [30]) and to the case of commuting pairs of operators ([16], [17]).

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In this note, we focus on the spherical Aluthge transform [17]. Although our results hold for arbitrary $n > 2$, for the reader's convenience we will focus on the case $n = 2$; that is, the case of commuting pairs of Hilbert space operators. Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$. We now consider the canonical polar decomposition of the column operator $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$; that is, $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \equiv \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P$, where $P := \sqrt{T_1^*T_1 + T_2^*T_2}$ and $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a (joint) partial isometry, and subject to the constraint $\bigcap_{i=1}^2 \ker T_i = \bigcap_{i=1}^2 \ker V_i = \ker P$.

The *spherical Aluthge transform* of $\mathbf{T}$ is the (necessarily commuting) n-tuple

$$\widehat{\mathbf{T}} := (\sqrt{P}V_1\sqrt{P}, \cdots, \sqrt{P}V_n\sqrt{P}) \quad ([16], [17]). \tag{1.1}$$

For a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ of operators on $\mathcal{H}$, the Koszul complex associated with $\mathbf{T}$ is given as

$$K(\mathbf{T},\mathcal{H}) : 0 \stackrel{0}{\to} \mathcal{H} \xrightarrow{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}} \mathcal{H} \oplus \mathcal{H} \xrightarrow{(-T_2\ T_1)} \mathcal{H} \xrightarrow{0} 0.$$

Definition 1.1. *A commuting pair $\mathbf{T}$ is said to be (Taylor) invertible if its associated Koszul complex $K(\mathbf{T},\mathcal{H})$ is exact. The Taylor spectrum of $\mathbf{T}$ is*

$$\sigma_T(\mathbf{T}) := \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : K\left((T_1 - \lambda_1, T_2 - \lambda_2), \mathcal{H}\right) \text{ is not invertible}\right\}.$$

The pair $\mathbf{T}$ is called Fredholm if each map in the Koszul complex $K(\mathbf{T}, \mathcal{H})$ has closed range and all the homology quotients are finite-dimensional. The Taylor essential spectrum is

$$\sigma_{Te}(\mathbf{T}) := \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : (T_1 - \lambda_1, T_2 - \lambda_2) \text{ is not Fredholm}\right\}.$$

J.L. Taylor showed in [36] and [37] that, if $\mathcal{H} \neq \{0\}$, then $\sigma_T(\mathbf{T})$ is a nonempty, compact subset of the polydisc of multiradius $r(\mathbf{T}) := (r(T_1), r(T_2))$, where $r(T_i)$ is the spectral radius of T_i $(i = 1, 2)$. (For additional facts about these joint spectra, the reader is referred to [12]–[15] and [38].)

As shown in [13] and [14], the Fredholmness of $\mathbf{T}$ can be detected in the Calkin algebra $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. (Here $\mathcal{K}$ denotes the closed two–sided ideal of compact operators; we also let $\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{Q}(\mathcal{H})$ denote the quotient map.) Concretely, $\mathbf{T}$ is Fredholm on $\mathcal{H}$ if and only if the pair of left multiplication operators $L_{\pi(\mathbf{T})} := (L_{\pi(T_1)}, L_{\pi(T_2)})$ is Taylor invertible when acting on $\mathcal{Q}(\mathcal{H})$. In particular, $\mathbf{T}$ is left Fredholm on $\mathcal{H}$ if and only if $L_{\pi(\mathbf{T})}$ is bounded below on $\mathcal{Q}(\mathcal{H})$.

Problem 1.2. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of operators.*
(i) Assume that $\mathbf{T}$ be (Taylor) invertible (resp. Fredholm). Is $\widehat{\mathbf{T}}$ also (Taylor) invertible (resp. Fredholm)?
(ii) Is the Taylor spectrum (resp. Taylor essential spectrum) of $\widehat{\mathbf{T}}$ equal to that of $\mathbf{T}$?

We first prove that $\sigma_T(\widehat{\mathbf{T}}) = \sigma_T(\mathbf{T})$. We do this in two stages: away from the origin we use tools and techniques from criss-cross commutativity; at the origin we show that the left invertibility of $\mathbf{T}$ or $\widehat{\mathbf{T}}$ implies the invertibility of P; P then helps establish an isomorphism between the relevant Koszul complexes. As a consequence, we can readily extend the above result to other spectral systems that rely on the Koszul complex for their definition, including spectral systems on $\mathcal{Q}(\mathcal{H})$.

2. Main Results

Recall the joint polar decomposition of $\mathbf{T}$ and the spherical Aluthge transform of $\mathbf{T}$; cf. (1.1). We now state our first main result.

Theorem 2.1. *Assume that* $\mathbf{T}$ *or* $\widehat{\mathbf{T}}$ *is left invertible; that is, the associated Koszul complex is exact at the left stage, and the range of the corresponding boundary map is closed. Then the operator* P *is invertible.*

Proof. **Case 1**. If $\mathbf{T}$ is left invertible, then $T_1^*T_1 + T_2^*T_2$ is invertible, and therefore P is invertible.
Case 2. If $\widehat{\mathbf{T}}$ is left invertible, then it is bounded below; that is, there exists a constant $c > 0$ such that

$$\left\|\sqrt{P}V_1\sqrt{P}x\right\|^2 + \left\|\sqrt{P}V_2\sqrt{P}x\right\|^2 \geq c^2 \left\|x\right\|^2.$$

Since (V_1, V_2) is a joint partial isometry, it readily follows that

$$\left\|\sqrt{P}x\right\|^2 + \left\|\sqrt{P}x\right\|^2 \geq \frac{c^2}{\|P\|} \left\|x\right\|^2.$$

As a result, $\sqrt{P}$ is bounded below, so P is invertible. □

We are now ready to state our second main result.

Theorem 2.2. *Let* $\mathbf{T} = (T_1, T_2)$ *be a commuting pair of operators on* $\mathcal{H}$. *Then*

$$\mathbf{T} \text{ is (Taylor) invertible } \Longleftrightarrow \widehat{\mathbf{T}} \text{ is (Taylor) invertible.}$$

We now recall the notion of criss-cross commutativity.

Definition 2.3. *Let* $\mathbf{A} \equiv (A_1, \cdots, A_n)$ *and* $\mathbf{B} \equiv (B_1, \cdots, B_n)$ *be two* n*-tuples of operators on* $\mathcal{H}$. *We say that* $\mathbf{A}$ *and* $\mathbf{B}$ *criss-cross commute (or that* $\mathbf{A}$ *criss-cross commutes with* $\mathbf{B}$*) if* $A_iB_jA_k = A_kB_jA_i$ *and* $B_iA_jB_k = B_kA_jB_i$ *for all* $i, j, k = 1, \cdots, n$. *Observe that we do not assume that* $\mathbf{A}$ *or* $\mathbf{B}$ *is commuting.*

Definition 2.4. *Given two* n*-tuples* $\mathbf{A}$ *and* $\mathbf{B}$ *we define* $\mathbf{AB} := (A_1B_1, \cdots, A_nB_n)$ *and* $\mathbf{BA} := (B_1A_1, \cdots, B_nA_n)$.

Remark 2.5. It is an easy consequence of Definition 2.3 that, if $\mathbf{A}$ and $\mathbf{B}$ criss-cross commute and $\mathbf{AB}$ is commuting, then $\mathbf{BA}$ is also commuting.

Lemma 2.6. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a commuting pair of operators on* $\mathcal{H}$, *let* $P := \sqrt{T_1^*T_1 + T_2^*T_2}$, *and let* $\widehat{\mathbf{T}}$ *be its spherical Aluthge transform. Then* $\mathbf{A} \equiv (A_1, A_2) := (\sqrt{P}, \sqrt{P})$ *and* $\mathbf{B} \equiv (B_1, B_2) := (V_1\sqrt{P}, V_2\sqrt{P})$ *criss-cross commute. As a consequence,* $\widehat{\mathbf{T}}(= \mathbf{BA})$ *is commuting.*

Lemma 2.7. *(cf.* [8] *and* [9]*) Let* $\mathbf{A}$ *criss-cross commute with* $\mathbf{B}$ *on* $\mathcal{H}$, *and assume that* $\mathbf{AB}$ *is commuting. Then* $\sigma_T(\mathbf{BA}) \setminus \{\mathbf{0}\} = \sigma_T(\mathbf{AB}) \setminus \{\mathbf{0}\}$.

We now prove our third main result.

Theorem 2.8. *Let* $\mathbf{T} = (T_1, T_2)$ *be a commuting pair of operators on* $\mathcal{H}$. *Then*

$$\sigma_T(\mathbf{T}) = \sigma_T(\widehat{\mathbf{T}}).$$

Proof. Let $\lambda \in \mathbb{C}^2$. If $\lambda = (0, 0)$, use Theorem 2.2; if $\lambda \neq (0, 0)$, use Lemma 2.7. □

Remark 2.9. (i) Theorems 2.1, 2.2 and 2.8 can be easily extended to other spectral systems whose definition is given in terms of the Koszul complex; e.g., the left k-spectral systems $\sigma_{\pi,k}$ defined by W. Słodkowski and W. Żelazko ([34], [35]). For, the Proof of Theorem 2.1 (which uses only left invertibility of the relevant Koszul complex) works well in case $\mathbf{T}$ or $\widehat{\mathbf{T}}$. Once we know that $\sqrt{P}$ is invertible, the Koszul complexes of $\mathbf{T}$ and $\widehat{\mathbf{T}}$ are isomorphic, so $0 \notin \sigma_{\pi,k}(\mathbf{T})$ if and only if $0 \notin \sigma_{\pi,k}(\widehat{\mathbf{T}})$.
(ii) Similarly, Theorem 2.7 admits an easy extension to Słodkowski's left k-spectra (cf. [8], [9]),

since the Proof of Theorem 2.7 relies on the isomorphism of the Koszul complexes for $\mathbf{T}$ and $\widehat{\mathbf{T}}$, implemented by $\sqrt{P}$.
(iii) On the other hand, the above results cannot be extended to Słodkowski's right k-spectra; for, consider the adjoint U_+^* of the (unweighted) unilateral shift U_+. It is easy to see that U_+^* is onto while $\widehat{U_+^*}$ is not.

Our final main result deals with Fredholmness.

Theorem 2.10. *Let* $\mathbf{T} = (T_1, T_2)$ *be a commuting pair of operators on* $\mathcal{H}$. *Then*

$$\sigma_{Te}(\mathbf{T}) = \sigma_{Te}(\widehat{\mathbf{T}}).$$

Moreover, for each $\lambda \notin \sigma_{Te}(\mathbf{T})$ *we have*

$$ind\ (\mathbf{T} - \lambda) = ind\ (\widehat{\mathbf{T}} - \lambda),$$

where ind denotes the Fredholm index.

Sketch of Proof. In Theorem 2.1 one can replace "left invertible" for the Koszul complex with "left Fredholm" and "invertible" for P with "Fredholm." A similar adjustment works for Theorems 2.2 and 2.8. In the analog of Theorem 2.2 one first proves that $\sqrt{P}$ is bounded below in the orthogonal complement of $\ker T_1 \cap \ker T_2$; since this kernel is finite dimensional, it follows that $\sqrt{P}$ is Fredholm. In Theorem 2.8 one needs to replace Lemma 2.7 with the results for Fredholmness proved in [8], [9], [31] and [32]. While Li's results only guarantee that ind $(\mathbf{T} - \lambda) =$ ind $(\widehat{\mathbf{T}} - \lambda)$ whenever $\lambda \neq (0, 0)$, the continuity of the Fredholm index (cf. [13]) does the rest. □

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