ITERATES OF THE SPHERICAL ALUTHGE TRANSFORM OF 2-VARIABLE WEIGHTED SHIFTS

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To the memory of Professor Victor Lomonosov

Abstract. Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of Hilbert space operators, and let $P := \sqrt{T_1^* T_1 + T_2^* T_2}$ be the positive factor in the (joint) polar decomposition of $\mathbf{T}$; i.e., $T_i = V_i P$ $(i = 1, 2)$. The spherical Aluthge transform of $\mathbf{T}$ is the (necessarily commuting) pair $\Delta_{\rm sph}(\mathbf{T}) := (\sqrt{P} V_1 \sqrt{P}, \sqrt{P} V_2 \sqrt{P})$.

In this paper we focus on the asymptotic behavior of the sequence $\{\Delta_{\rm sph}^{(n)}(\mathbf{T})\}_{n \geq 1}$ as $n \to \infty$, where $\Delta_{\rm sph}^{(1)}(\mathbf{T}) := \Delta_{\rm sph}(\mathbf{T})$ and $\Delta_{\rm sph}^{(n+1)}(\mathbf{T}) := \Delta_{\rm sph}(\Delta_{\rm sph}^{(n)}(\mathbf{T}))$ $(n \geq 1)$. In those cases when the limit exists, the limit pair is a fixed point for the spherical Aluthge transform, that is, a spherically quasinormal pair. For a suitable class of 2-variable weighted shifts we establish the convergence of the sequence of iterates in the weak operator topology.

1. Introduction

The Aluthge transform of a bounded operator T acting on a Hilbert space $\mathcal{H}$ was introduced by A. Aluthge in ([1]). If $T \equiv V\,|T|$ is the canonical polar decomposition of T, the Aluthge transform $\Delta(T)$ is given as $\Delta(T) := \sqrt{|T|}V\sqrt{|T|}$. One of Aluthge's motivations was to use this transform in the study of p-hyponormal and log-hyponormal operators. Roughly speaking, the idea was to convert an operator, T, into another operator, $\Delta(T)$, which shares with the first one many structural and spectral properties, but which is closer to being a normal operator. Over the last two decades, substantial and significant results about $\Delta(T)$, and how it relates to T, have been obtained by a long list of mathematicians who devoted considerable attention to this topic (see, for instance, [2], [9], [20], [26–28], [29–31]). Aluthge transforms have been generalized to the case of powers of $|T|$ different from $\frac{1}{2}$ ([5]) and to the case of commuting pairs of operators ([17], [18]).

This generalization, called the *spherical Aluthge transform* of $\mathbf{T}$, is the (necessarily commuting) pair $\Delta_{\rm sph}(\mathbf{T}) := (\sqrt{P} V_1 \sqrt{P}, \sqrt{P} V_2 \sqrt{P})$, where $P := \sqrt{T_1^* T_1 + T_2^* T_2}$ is the positive factor in the (joint) polar decomposition of $\mathbf{T}$ and (V_1, V_2) is the joint partial isometry. In this paper, we study the asymptotic behavior of the iterates of the spherical Aluthge transform of $\mathbf{T}$; that is, the behavior as $n \to \infty$ of the sequence of commuting pairs given

2010 *Mathematics Subject Classification.* Primary: 47B37, 47B20, 47A13, 37E25; Secondary: 47A57, 44A60.

Key words and phrases. spherical Aluthge transform, iterates, polar decomposition, spherical isometry.

The first named author was partially supported by Labex CEMPI (ANR-11-LABX-0007-01). The second named author was partially supported by NSF Grant DMS-1302666.

by $\Delta^{(1)}_{\text{sph}}(\mathbf{T}) := \Delta_{\text{sph}}(\mathbf{T})$ and $\Delta^{(n+1)}_{\text{sph}}(\mathbf{T}) := \Delta_{\text{sph}}(\Delta^{(n)}_{\text{sph}}(\mathbf{T}))$ $(n \geq 1)$. We do this for a class of 2-variable weighted shifts obtained as finite-rank perturbations of spherical isometries. In those cases when the limit exists, the limit pair is a fixed point for the spherical Aluthge transform; that is, a spherically quasinormal pair. For this class of 2-variable weighted shifts we will establish the convergence of the sequence of iterates in the weak operator topology; see details in Section 4.

2. Notation and Preliminaries

2.1. **The classical Aluthge transform.** Let $\mathcal{H}$ denote a (complex, separable) Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, let $T \equiv V\,|T|$ be the canonical polar decomposition of T; that is, $|T| := (T^*T)^{\frac{1}{2}}$, V is a partial isometry, and $\ker V = \ker |T| = \ker T$. The Aluthge transform of T is the operator

$$\Delta(T) := |T|^{\frac{1}{2}}\, V\, |T|^{\frac{1}{2}}\,.$$

The Aluthge transform has been extensively studied, in terms of algebraic, structural and spectral properties. We list below a brief sample of the results obtained over the last several years.

(i) T is a fixed point of Δ (that is, $\Delta(T) = T$) if and only if T is quasinormal, i.e., T commutes with $|T|$.

(ii) (A. Aluthge [1]) Let $0 < p < \frac{1}{2}$ and assume that T is p-hyponormal. Then $\Delta(T)$ is $(p + \frac{1}{2})$-hyponormal.

(iii) In [26], I.B. Jung, E. Ko and C. Pearcy showed that T and $\Delta(T)$ share many spectral properties; in particular, $\sigma(\Delta(T)) = \sigma(T)$.

(iv) In [26, Corollary 1.16], I.B. Jung, E. Ko and C. Pearcy proved that if $\Delta(T)$ has a nontrivial invariant subspace, then so does T; and if T has dense range, then the above implication becomes an equivalence [26, Theorem 1.15].

(v) M.H. Kim and E. Ko ([29]), and F. Kimura ([30]) proved that T has property (β) if and only if $\Delta(T)$ has property (β).

(vi) In [2], T. Ando established that for all $\lambda \notin \sigma(T)$, one has $\|(T-\lambda)^{-1}\| \geq \|(\Delta(T)-\lambda)^{-1}\|$.

(vii) G. Exner proved in [21, Example 2.11] that the subnormality of T is not preserved under the Aluthge transform.

(viii) Subsequently, S.H. Lee, W.Y. Lee and J. Yoon ([31]) showed that for $k \geq 2$, the Aluthge transform, when acting on weighted shifts, does not preserve k-hyponormality.

2.2. **Iterates of the Aluthge Transform.** The iterates of the Aluthge transform are given by

$$\Delta^{(1)}(T) := \Delta(T)$$

and

$$\Delta^{(n+1)}(T) := \Delta(\Delta^{(n)}(T)) \; (n \geq 1).$$

It is easy to verify that the Aluthge transform of a weighted shift W_ω is again a weighted shift; see Subsection 2.5. Concretely, the weights of $\Delta(W_\omega)$ are

$$\sqrt{\omega_0\omega_1}, \sqrt{\omega_1\omega_2}, \sqrt{\omega_2\omega_3}, \sqrt{\omega_3\omega_4}, \cdots .$$

If we let

$$W_{\sqrt{\omega}} := \text{shift } \left(\sqrt{\omega_0}, \sqrt{\omega_1}, \sqrt{\omega_2}, \cdots\right),$$

then $\Delta(W_\omega)$ is the Schur product of $W_{\sqrt{\omega}}$ and its restriction to the closed linear span $\bigvee\{e_1, e_2, \cdots\}$. Thus, a sufficient condition for the subnormality of $\Delta(W_\omega)$ is the subnormality of $W_{\sqrt{\omega}}$. (For more on this connection, see [15].)

Next, observe that

$$\Delta^{(2)}(W_\omega) = \text{shift}(\sqrt{\sqrt{\omega_0\omega_1}\sqrt{\omega_1\omega_2}}, \sqrt{\sqrt{\omega_1\omega_2}\sqrt{\omega_2\omega_3}}, \cdots),$$

$$\Delta^{(3)}(W_\omega) = \text{shift}((\omega_0\omega_1^3\omega_2^3\omega_3)^{\frac{1}{8}}, (\omega_1\omega_2^3\omega_3^3\omega_4)^{\frac{1}{8}}, \cdots),$$

and

$$\Delta^{(4)}(W_\omega) = \text{shift}((\omega_0\omega_1^4\omega_2^6\omega_3^4\omega_4)^{\frac{1}{16}}, (\omega_1\omega_2^4\omega_3^6\omega_4^4\omega_5)^{\frac{1}{16}}, \cdots).$$

Thus, if we let $\omega^{(n)}$ denote the weight sequence of $\Delta^{(n)}(W_\omega)$, we have

$$\omega_k^{(n+1)} = \sqrt{\omega_k^{(n)}\omega_{k+1}^{(n)}},$$

and an induction argument shows that

$$\omega_k^{(n)} = \left(\prod_{j=0}^{n} \omega_{k+j}^{\binom{n}{j}}\right)^{\frac{1}{2^n}} . \tag{2.1}$$

The study of the limiting behavior of the iterates of the Aluthge transform has received considerable attention. Below is a list of some major results in this direction.

(i) In [26], I.B. Jung, E. Ko and C. Pearcy conjectured that for every bounded operator T the sequence $\{\Delta^{(n)}(T)\}$ converges in norm to a quasinormal operator.

(ii) In [2, Theorem], T. Ando proved that the conjecture is true for 2×2 matrices.

(iii) In [4], J. Antezana, E. Pujals and D. Stojanoff proved the conjecture to be true for $\dim \mathcal{H} < \infty$; see also [5].

(iv) In 2003, J. Thompson (as communicated in [27, Example 5.5]) found an example of an operator for which the sequence converges to 0 in the strong operator topology (SOT), but it does not converge in norm.

(v) In 2001, M. Yanagida found an example of a unilateral weighted shift for which the sequence of iterates does not converge in the weak operator topology (WOT) (cf. [32, p.

2, lines 15 and 16]).

(vi) In [9], M. Chō and W.Y. Lee proved that for any $0 < a < b$ there exists a unilateral weighted shift W_ω such that the sequence $\{\omega_0^{(n)}\}_{n\geq 0}$ clusters at both a and b.

Possibly the most definitive results about the convergence of the iterates of the classical Aluthge transform were obtained by K. Rion in [32].

Proposition 2.1. *([32, Proposition 1]) The WOT and SOT convergences of $\{T_{\omega^{(n)}}\}$ are equivalent to the pointwise convergence of the sequence $\{\omega^{(n)}\}_n$, given by (2.1).*

Theorem 2.2. *([32, Theorem 7]) Assume ω is bounded below. Then the set $\mathcal{S}$ of SOT subsequential limits of $\{\Delta^{(n)}(T_\omega)\}$ is nonempty. Moreover, $\mathcal{S}$ is a closed interval of quasinormal shifts; that is, $\mathcal{S} = [a,b]U_+$ for some $a, b > 0$.*

2.3. **The Spherical Aluthge Transform.** We first recall the definition of the spherical Aluthge transform (introduced in [17] and [18]). Given a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ of operators acting on $\mathcal{H}$, let $P := (T_1^* T_1 + T_2^* T_2)^{\frac{1}{2}}$. Clearly, $\ker P = \ker T_1 \bigcap \ker T_2$. For $x \in \ker P$, let $V_i x := 0$ $(i = 1, 2)$; for $y \in \operatorname{Ran} P$, say $y = Px$, let $V_i y := T_i x$ $(i = 1, 2)$. It is easy to see that V_1 and V_2 are well defined, and extend continuously to $\overline{\operatorname{Ran} P}$. We then have

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 P \\ V_2 P \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P, \tag{2.2}$$

as operators from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$. Moreover, this is the canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. It follows that $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $(\ker P)^\perp$ onto $\overline{\operatorname{Ran} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}}$.

The spherical Aluthge transform of $\mathbf{T}$ is $\Delta_{\text{sph}}(\mathbf{T}) \equiv (\widehat{T}_1, \widehat{T}_2)$, where

$$\widehat{T}_i := P^{\frac{1}{2}} V_i P^{\frac{1}{2}} \ (i = 1, 2).$$

The spherical Aluthge transform was introduced in [17]; its general theory was developed in [18]. One of the basic results follows.

Lemma 2.3. *(cf. [18]) $\Delta_{sph}(\mathbf{T})$ is commutative.*

The equality of the spectra of an operator and its Aluthge transform (mentioned in Subsection 2.1) can be extended to commuting pairs $\mathbf{T}$ (cf. [8]). That is, one can use a bit of homological algebra applied to the appropriate Koszul complexes to prove directly that for a commuting pair $\mathbf{T} \equiv (T_1, T_2)$

$$\sigma_T(\Delta_{\text{sph}}(\mathbf{T})) = \sigma_T(\mathbf{T}), \tag{2.3}$$

where $\sigma_T(\mathbf{T})$ is the Taylor spectrum of $\mathbf{T}$. (For more information on the notion of Taylor spectrum and related results, the reader is referred to [11], [12], [34].)

Moreover, if $\mathbf{T} \equiv (T_1, T_2)$ is Taylor invertible and we represent it as a column matrix, then one can see that P is also invertible, and in this case,

$$\Delta_{\text{sph}}(\mathbf{T}) = \left(P^{\frac{1}{2}} \oplus P^{\frac{1}{2}}\right) \mathbf{T} P^{-\frac{1}{2}}.$$

2.4. **Spherically quasinormal pairs.** It is well known that the fixed points of the classical Aluthge transform are the quasinormal operators, that is, those operators $T = V\,|T|$ such that V and $|T|$ commute (equivalently, T and $|T|$ commute). For the spherical Aluthge transform, the fixed commuting pairs are the so-called spherically quasinormal pairs, which we now define. First, we need some terminology.

Following A. Athavale-S. Podder ([7]) and J. Gleason ([23]), we say that
(i) $\mathbf{T}$ is *matricially quasinormal* if T_i commutes with $T_j^*T_k$ for all $i,j,k = 1,2$;
(ii) $\mathbf{T}$ is *(jointly) quasinormal* if T_i commutes with $T_j^*T_j$ for all $i,j = 1,2$; and
(iii) $\mathbf{T}$ is *spherically quasinormal* if T_i commutes with

$$P := T_1^*T_1 + T_2^*T_2,$$

for $i = 1,2$. Also, recall that $\mathbf{T}$ is said to be *normal* if $T_1T_2 = T_2T_1$ and T_i is normal ($i = 1,2$).

It follows that

$$\begin{aligned} \text{normal} \quad &\Longrightarrow \quad \text{matricially quasinormal} \;\Longrightarrow\; \text{(jointly) quasinormal} \\ &\Longrightarrow \quad \text{spherically quasinormal} \;\Longrightarrow\; \text{subnormal ([7, Proposition 2.1])} \\ &\Longrightarrow \quad k\text{-hyponormal} \;\Longrightarrow\; \text{hyponormal.} \end{aligned} \tag{2.4}$$

On the other hand, results of R.E. Curto, S.H. Lee and J. Yoon (cf. [16]), and of J. Gleason ([23]) show that the reverse implications in (2.4) do not necessarily hold.

In [19, Theorem 2.2], R.E. Curto and J. Yoon showed that the spherically quasinormal commuting pairs are precisely the fixed points of the spherical Aluthge transform; moreover, it follows from the results in [18, Section 2] that if $\mathbf{T}$ is spherically quasinormal then (V_1, V_2) is a commuting pair. In [18] it was also shown that every spherically quasinormal 2-variable weighted shift is a positive multiple of a spherical isometry (see Theorem 3.4). In order to state this result, we need a brief discussion of unilateral and 2-variable weighted shifts, which follows.

2.5. **Unilateral weighted shifts.** For $\omega \equiv \{\omega_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\omega \equiv \text{shift}(\omega_0, \omega_1, \cdots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\omega e_n := \omega_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of $\omega \equiv \{\omega_n\}_{n=0}^{\infty}$ are given as

$$\gamma_k \equiv \gamma_k(W_\omega) := \begin{cases} 1, & \text{if } k = 0 \\ \omega_0^2 \cdots \omega_{k-1}^2, & \text{if } k > 0. \end{cases} \tag{2.5}$$

The (unweighted) unilateral shift is $U_+ := \text{shift}(1,1,1,\cdots)$, and for $0 < a < 1$ we let $S_a := \text{shift}(a,1,1,\cdots)$.

We now recall a well known characterization of subnormality for unilateral weighted shifts, due to C. Berger (cf. [10, III.8.16]) and independently established by Gellar and Wallen ([22]): W_ω is subnormal if and only if there exists a probability measure σ supported in $[0, \|W_\omega\|^2]$ (called the *Berger measure* of W_ω) such that

$$\gamma_k(W_\omega) = \omega_0^2 \cdot \ldots \cdot \omega_{k-1}^2 = \int t^k d\sigma(t) \ (k \geq 1).$$

Observe that U_+ and S_a are subnormal, with Berger measures δ_1 and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support the singleton set $\{p\}$. On the other hand, the Berger measure of the Bergman shift B_+ (acting on $A^2(\mathbb{D})$, and with weights $\omega_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$) is the Lebesgue measure on the interval $[0,1]$.

2.6. **2-variable weighted shifts.** Consider now two double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \text{ and } T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}, \tag{2.6}$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

$$T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \ \left(\text{all } \mathbf{k} \in \mathbb{Z}_+^2\right). \tag{2.7}$$

Moreover, for $\mathbf{k} \in \mathbb{Z}_+^2$ we have

$$T_1^* e_{0,k_2} = 0 \quad \text{and} \quad T_1^* e_{\mathbf{k}} = \alpha_{\mathbf{k}-\varepsilon_1} e_{\mathbf{k}-\varepsilon_1} \ (k_1 \geq 1); \tag{2.8}$$

$$T_2^* e_{k_1,0} = 0 \quad \text{and} \quad T_2^* e_{\mathbf{k}} := \beta_{\mathbf{k}-\boldsymbol{\varepsilon_2}} e_{\mathbf{k}-\boldsymbol{\varepsilon_2}} \ (k_2 \geq 1). \tag{2.9}$$

In an entirely similar way one can define multivariable weighted shifts. The weight diagram of a generic 2-variable weighted shift is shown in Figure 1.

When all weights are equal to 1 we obtain the so-called Helton-Howe shift; that is, the shift that corresponds to the pair of multiplications by the coordinate functions in the Hardy space $H^2(\mathbb{T} \times \mathbb{T})$ of the 2-torus, with respect to normalized arclength measure on each unit circle $\mathbb{T}$ (cf. [23]). This shift can also be represented as $(U_+ \otimes I, I \otimes U_+)$, acting on $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$.

We now recall the definition of *moments* for a commuting 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$. Given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the moment of $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(W_{(\alpha,\beta)}) := \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \tag{2.10}$$

We remark that, due to the commutativity condition (2.7), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1, k_2).

To detect hyponormality, there is a simple criterion:

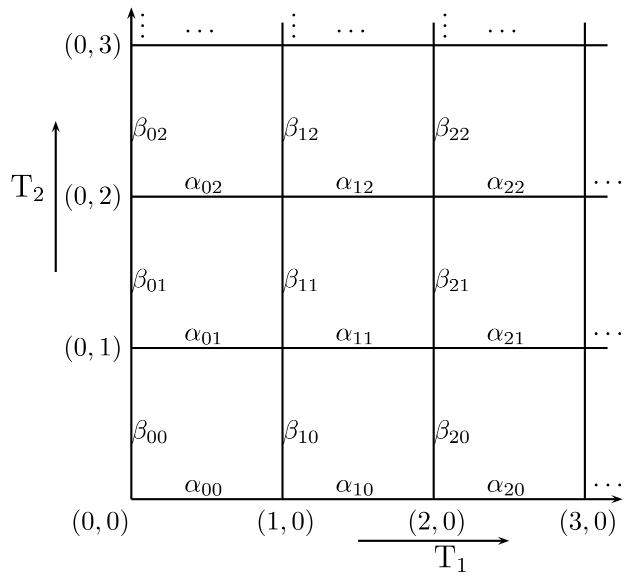

FIGURE 1. Weight diagram of a generic 2-variable weighted shift.

Theorem 2.4. *([13]) (Six-point Test) Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then*

$$\mathbf{T} \textit{ is hyponormal } \Leftrightarrow \begin{pmatrix} \alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geq 0 \; (\textit{all } \mathbf{k} \in \mathbb{Z}^2_+).$$

A straightforward generalization of the above mentioned Berger-Gellar-Wallen result was proved in [25]. That is, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ admits a commuting normal extension if and only if there is a probability measure μ (which we call the Berger measure of $\mathbf{T}$) defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ (where $a_i := \|T_i\|^2$) such that

$$W_\alpha \text{ is subnormal } \Leftrightarrow \gamma_{\mathbf{k}} := \begin{cases} 1, & \text{if } \mathbf{k} = 0 \\ \alpha^2_{(0,0)} \cdot \ldots \cdot \alpha^2_{(k_1-1,0)}, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta^2_{(0,0)} \cdot \ldots \cdot \beta^2_{(0,k_2-1)}, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha^2_{(0,0)} \cdot \ldots \cdot \alpha^2_{(k_1-1,0)} \cdot \beta^2_{(k_1,0)} \cdot \ldots \cdot \beta^2_{(k_1,k_2-1)}, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{cases}$$

$$= \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \;\; (\text{all } \mathbf{k} \in \mathbb{Z}^2_+).$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

3. Spherically Quasinormal 2-variable Weighted Shifts

In this section we present a characterization of spherical quasinormality for 2-variable weighted shifts, which was announced in [17] and proved in [18] and [19]. Before we state it, we list some simple facts about quasinormality for 2-variable weighted shifts.

Remark 3.1. (cf. [16]) We first observe that no 2-variable weighted shift can be matricially hyponormal, as a simple calculation shows. Also, a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ is (jointly) quasinormal if and only if $\alpha_{(k_1,k_2)} = \alpha_{(0,0)}$ and $\beta_{(k_1,k_2)} = \beta_{(0,0)}$ for all $k_1, k_2 \geq 0$. This can be seen via a simple application of (2.7) and (2.8). As a result, up to a scalar multiple in each component, a quasinormal 2-variable weighted shift is identical to the so-called Helton-Howe shift. This fact is entirely consistent with the one-variable result: a unilateral weighted shift W_ω is quasinormal if and only if $W_\omega = cU_+$ for some $c > 0$.

The following result describes the weight diagram of $\Delta_{\mathrm{sph}}(\mathbf{T}) \equiv (\widehat{T}_1, \widehat{T}_2)$.

Proposition 3.2. *([18]) Let $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ be a 2-variable weighted shift. Then*

$$\widehat{T}_1 e_{\mathbf{k}} = \alpha_{\mathbf{k}} \frac{(\alpha^2_{\mathbf{k}+\epsilon_1} + \beta^2_{\mathbf{k}+\epsilon_1})^{1/4}}{(\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}})^{1/4}} e_{\mathbf{k}+\epsilon_1} \tag{3.1}$$

$$\widehat{T}_2 e_{\mathbf{k}} = \beta_{\mathbf{k}} \frac{(\alpha^2_{\mathbf{k}+\epsilon_2} + \beta^2_{\mathbf{k}+\epsilon_2})^{1/4}}{(\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}})^{1/4}} e_{\mathbf{k}+\epsilon_2} \tag{3.2}$$

for all $\mathbf{k} \in \mathbb{Z}^2_+$.

We now recall the class of spherically isometric commuting pairs of operators (cf. [6], [7], [23]). A commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is a spherical isometry if $T_1^* T_1 + T_2^* T_2 = I$.

Lemma 3.3. *[6, Proposition 2] Any spherical isometry is subnormal.*

Theorem 3.4. *([16, Theorem 3.1]; cf. [18, Lemma 10.3]) For a commuting 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$, the following statements are equivalent:*
(i) $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ is a spherically quasinormal 2-variable weighted shift;
(ii) (algebraic condition) $T_1^* T_1 + T_2^* T_2 = C \cdot I$, for some positive constant C;
(iii) (weight condition) for all $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}^2_+$, $\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)} = C$, for some positive constant $C > 0$;
(iv) (moment condition) for all $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}^2_+$, $\gamma_{\mathbf{k}+\varepsilon_1} + \gamma_{\mathbf{k}+\varepsilon_2} = C\gamma_{\mathbf{k}}$, for some positive constant $C > 0$.

3.1. **Construction of Spherically Quasinormal 2-variable Weighted Shifts.**

As observed in [18], within the class of 2-variable weighted shifts there is a simple description of spherical isometries, in terms of the weight sequences $\alpha \equiv \{\alpha_{(k_1,k_2)}\}$ and $\beta \equiv \{\beta_{(k_1,k_2)}\}$. Indeed, since spherical isometries are (jointly) subnormal, we know that the unilateral weighted shift associated with the 0-th row in the weight diagram must be subnormal. Thus, without loss of generality, we can always assume that the 0-th row corresponds to a subnormal unilateral weighted shift, and denote its weights by $\{\alpha_{(k,0)}\}_{k=0,1,2,\cdots}$. Also, in view of Theorem 3.4 we can assume that $c = 1$. Using the identity

$$\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}} = 1 \quad (\mathbf{k} \in \mathbb{Z}^2_+) \tag{3.3}$$

and the above mentioned 0-th row, we can compute $\beta_{(k,0)} := \sqrt{1-\alpha_{k,0}^2}$ for $k = 0, 1, 2, \cdots$. With these new values at our disposal, we can use the commutativity property (2.7) to generate the values of α in the first row; that is,

$$\alpha_{(k,1)} := \alpha_{(k,0)}\beta_{(k+1,0)}/\beta_{(k,0)}.$$

We can now repeat the algorithm, and calculate the weights $\beta_{(k,1)}$ for $k = 0, 1, 2, \cdots$, again using the identity (3.3). This in turn leads to the α weights for the second row, and so on. For more on this construction, the reader is referred to [19]. In particular, it is worth noting that the construction may stall if the sequence $\{\alpha_{(k,0)}\}_{k\geq 0}$ is not strictly increasing.

Proposition 3.5. *([14, Proposition 12.14]) Let*

$$\alpha_{(0,0)} := \sqrt{p},\ \ \alpha_{(1,0)} := \sqrt{q},\ \ \alpha_{(2,0)} := \sqrt{r}\ \ \ \textit{and}\ \ \alpha_{(3,0)} := \sqrt{r},$$

and assume that $0 < p < q < r < 1$. Then the algorithm described in this section fails at some stage. As a consequence, there does not exist a spherical isometry interpolating these initial data.

Remark 3.6. In Proposition 3.5 the reader may have noticed that the 0-th row is not subnormal; for, it is well known that, up to a constant, the only subnormal unilateral weighted shifts with two equal weights are U_+ and S_a ([33, Theorem 6]). Thus, save for these two special (trivial) cases, assuming subnormality of the 0-th row will automatically guarantee that $\alpha_{(k,0)}$ is strictly increasing; therefore, in the sequel we will always assume that the 0-th row is subnormal.

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4. Iterates of the Spherical Aluthge Transform

For notational convenience, in this section we will switch from pairs (T_1, T_2) to pairs (S, T). Given a 2-variable weighted shift $(S,T) \equiv W_{\alpha,\beta}$, recall that the spherical Aluthge transform is given by

$$(\Delta_{\rm sph}(S,T))_1 e_{\mathbf{k}} = \alpha_{\mathbf{k}} \frac{(\alpha^2_{\mathbf{k}+\epsilon_1} + \beta^2_{\mathbf{k}+\epsilon_1})^{1/4}}{(\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}})^{1/4}} e_{\mathbf{k}+\epsilon_1}$$

and

$$(\Delta_{\rm sph}(S,T))_2 e_{\mathbf{k}} = \beta_{\mathbf{k}} \frac{(\alpha^2_{\mathbf{k}+\epsilon_2} + \beta^2_{\mathbf{k}+\epsilon_2})^{1/4}}{(\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}})^{1/4}} e_{\mathbf{k}+\epsilon_2}$$

for all $\mathbf{k} \in \mathbb{Z}^2_+$.

Thus, it is clear that each of the iterates of $\Delta_{\rm sph}(S,T)$ is a 2-variable weighted shift. We now define, recursively, two weight sequences, $S_n(i,j)$ and $T_n(i,j)$ using the horizontal and vertical components of the iterates of the spherical Aluthge transform. For $n = 0$ we let $S_0(i,j) := \alpha_{(i,j)}$ and $T_0(i,j) := \beta_{(i,j)}$. For $n > 0$, $S_n(i,j)$ and $T_n(i,j)$ are the weights of the horizontal and vertical actions of $\Delta^{(n)}_{\rm sph}(S,T)$. This easily leads to the following expressions:

$$\begin{array}{rcl} (\Delta_{\rm sph}(S_n,T_n))_1 e_{(i,j)} & = & S_{n+1}(i,j) e_{(i,j)} \\ (\Delta_{\rm sph}(S_n,T_n))_2 e_{(i,j)} & = & T_{n+1}(i,j) e_{(i,j)}. \end{array}$$

It follows that

$$S_{n+1}e_{(0,0)} = S_n(0,0)\frac{(S_n(1,0)^2+T_n(1,0)^2)^{1/4}}{(S_n(0,0)^2+T_n(0,0)^2)^{1/4}}e_{(1,0)}$$

and

$$T_{n+1}e_{(0,0)} = T_n(0,0)\frac{(S_n(0,1)^2+T_n(0,1)^2)^{1/4}}{(S_n(0,0)^2+T_n(0,0)^2)^{1/4}}e_{(0,1)}.$$

As in the 1 variable case [32], one observes that the asymptotic behavior anywhere impacts the asymptotic behavior at the origin $(0,0)$; as a result, and without loss of generality, we can focus attention on the recursively defined sequences

$$S_{n+1}(0,0) = S_n(0,0)\frac{(S_n(1,0)^2+T_n(1,0)^2)^{1/4}}{(S_n(0,0)^2+T_n(0,0)^2)^{1/4}} \tag{4.1}$$

and

$$T_{n+1}(0,0) = T_n(0,0)\frac{(S_n(0,1)^2+T_n(0,1)^2)^{1/4}}{(S_n(0,0)^2+T_n(0,0)^2)^{1/4}}. \tag{4.2}$$

We will now restrict attention to finite rank perturbations of spherical isometries. We will prove that the iterates of Δ_{sph} converge in the WOT to a spherical isometry. The proof entails consideration of cases of increasing complexity. First, we need some notation.

Let $\mathbf{k} \in \mathbb{Z}_+^2$ and let $\mathcal{L}_\mathbf{k} := \bigvee\{e_{\mathbf{k}+\mathbf{p}} : \mathbf{p} \in \mathbb{Z}_+^2\}$; that is, $\mathcal{L}_\mathbf{k}$ is the closed subspace generated by the orthonormal canonical basis vectors in the quadrant determined by the lattice point $\mathbf{k}$. Alternatively, $\mathcal{L}_\mathbf{k} = \mathbf{k} + \mathbb{Z}_+^2$.

Remark 4.1. (i) Observe that if $\mathcal{M}_{k_2}$ represents the range of T^{k_2} and if $\mathcal{N}_{k_1}$ represents the range of S^{k_1}, then $\mathcal{L}_\mathbf{k} = \mathcal{M}_{k_2} \bigcap \mathcal{N}_{k_1}$. Also, for $k_1 = k_2 = 1$, the space $\mathcal{L}_{(1,1)}$ is the *core* of the 2-variable weighted shift (cf. [18, paragraph immediately following Lemma 3.4]).

(ii) It is easy to show that all iterates of the spherical Aluthge transform leave the subspaces $\mathcal{L}_\mathbf{k}$ invariant.

Theorem 4.2. *(Case 1: 1-cell perturbation.) Consider the 2-variable weighted shift given by the weight diagram in Figure 2. Then the iterates of the spherical Aluthge transform of (S,T) converge in the WOT to a spherical isometry.*

Proof. Since the spherical Aluthge transform leaves invariant the subspace where (S,T) is a spherical isometry (that is, the subspace $\mathcal{M}_1 \wedge \mathcal{N}_1$), it is enough to focus attention on the asymptotic behavior of the iterates at the origin. It is not hard to see that $\Delta_{\text{sph}}(S,T)$ has the same structure, and the same is true of $\Delta_{\text{sph}}^2(S,T)$, $\Delta_{\text{sph}}^3(S,T)$, ...

Thus, for this special case, the asymptotic behavior of the spherical Aluthge iterates is controlled by the pair

$$\begin{cases} p_n & := & S_n(0,0) \\ q_n & := & T_n(0,0). \end{cases}$$

Observe that

$$\begin{cases} p_1 & = & p(p^2+q^2)^{-1/4} \\ q_1 & = & q(p^2+q^2)^{-1/4}, \end{cases}$$

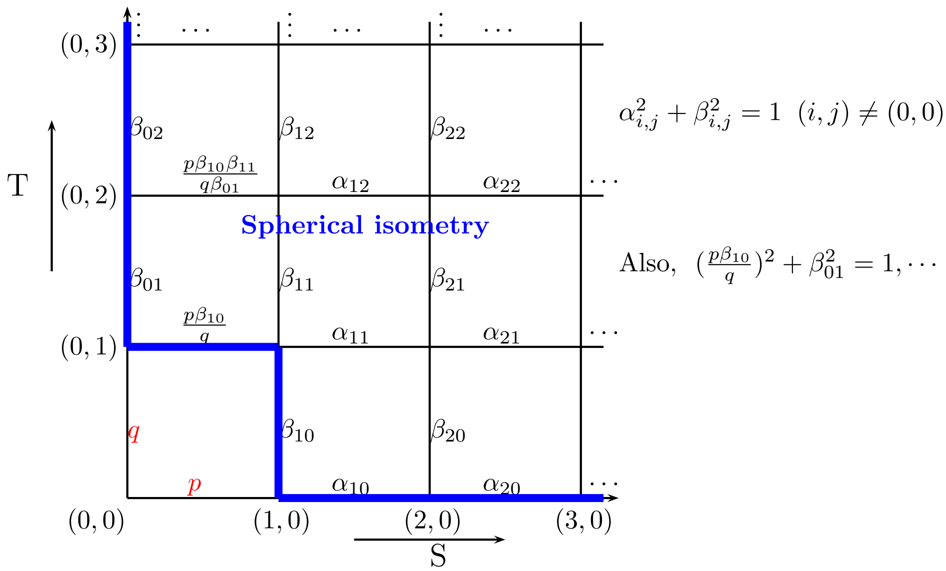

FIGURE 2. Weight diagram for the 2-variable weighted shift in Theorem 4.2

$$\begin{cases} p_2 &= p(p^2+q^2)^{-3/8} \\ q_2 &= q(p^2+q^2)^{-3/8} \end{cases}$$

and, in general, for $n > 2$,

$$\begin{cases} p_n &= p(p^2+q^2)^{-\sum_{k=2}^{n+1}(\frac{1}{2})^k} \\ q_n &= q(p^2+q^2)^{-\sum_{k=2}^{n+1}(\frac{1}{2})^k}. \end{cases}$$

From this it readily follows that, in the limit, we obtain

$$\begin{cases} p_\infty &= p(p^2+q^2)^{-\frac{1}{2}} \\ q_\infty &= q(p^2+q^2)^{-\frac{1}{2}}. \end{cases}$$

Since

$$p_\infty^2 + q_\infty^2 = 1,$$

we see that the sequence of iterates does converge to a spherical isometry. $\square$

Remark 4.3. For future use, we record the following identity involving p_n and q_n in the Proof of Theorem 4.2:

$$p_n^2 + q_n^2 = (p^2+q^2)^{2^{-n}}. \tag{4.3}$$

Theorem 4.4. *(Case 2: 2-cell perturbation.) Consider the 2-variable weighted shift given by the weight diagram in Figure 3. Then the iterates of the spherical Aluthge transform of (S,T) converge in the WOT to a spherical isometry.*

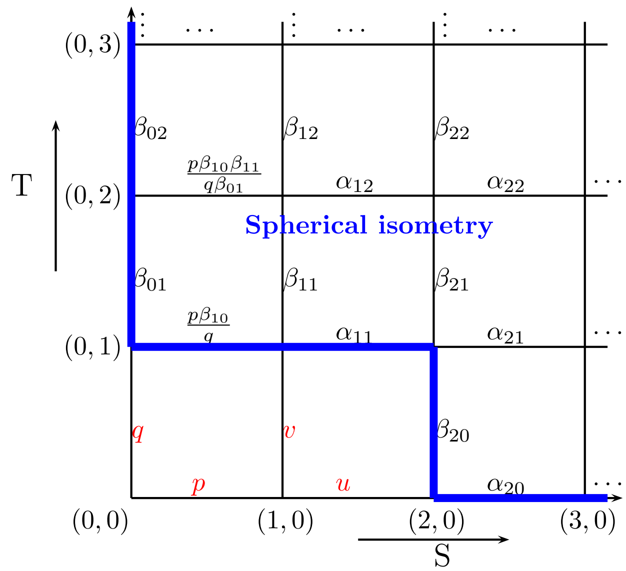

FIGURE 3. Weight diagram for the 2-variable weighted shift in Theorem 4.4

Proof. Observe first that the restriction of (S, T) to the invariant subspace $\mathcal{L}_{(1,0)}$ is a 2-variable weighted shift satisfying the conditions in Theorem 4.2, with the parameters u and v taking the place of p and q. In particular, we know from (4.3) that

$$u_n^2 + v_n^2 = (u^2 + v^2)^{2^{-n}}. \tag{4.4}$$

Moreover, by (4.1) the values of $S_{n+2}(0,0)$ are determined by the values of $S_{n+1}(0,0)$, $T_{n+1}(0,0)$, $S_{n+1}(1,0)$ and $T_{n+1}(1,0)$, and the last two values follow the pattern for the weights in Theorem 4.2, since the lattice point $(1,0)$ is in the subspace $\mathcal{L}_{(1,0)}$. We now observe that

$$S_{n+1}(0,0)^2 + T_{n+1}(0,0)^2 = \frac{S_n(0,0)^2\sqrt{u_n^2 + v_n^2} + T_n(0,0)^2}{\sqrt{S_n(0,0)^2 + T_n(0,0)^2}}.$$

(Notice that $T_n(0,0)^2$ appears without another factor in the numerator because $T_{n+1}(0,0)$ uses information about the lattice points $(0,0)$ and $(0,1)$, and of course the restriction of (S,T) to $\mathcal{L}_{(0,1)}$ is a spherical isometry.)

It follows that both the expressions for $S_{n+1}(0,0)^2 + T_{n+1}(0,0)^2$ and $u_{n+1}^2 + v_{n+1}^2$, which are needed for $S_{n+2}(0,0)$ and $T_{n+2}(0,0)$, depend directly on the quantity $u_n^2 + v_n^2$, whose asymptotic behavior is given by (4.4). It is now not hard to check that $S_{n+2}(0,0)$ and $T_{n+2}(0,0)$ converges to 1 as $n \to \infty$. At the same time, the reader will notice that convergence does not easily follow from the convergence of the sequence $\{u_n^2 + v_n^2\}$, but the concrete asymptotic pattern in (4.4) is important; that is, one has a sequence of the form $c^{2^{-n}}$, where c is a positive constant. $\square$

Theorem 4.5. *(Case 3: 3-cell perturbation.) Consider the 2-variable weighted shift given by the weight diagram in Figure 4. Then the iterates of the spherical Aluthge transform of (S,T) converge in the WOT to a spherical isometry.*

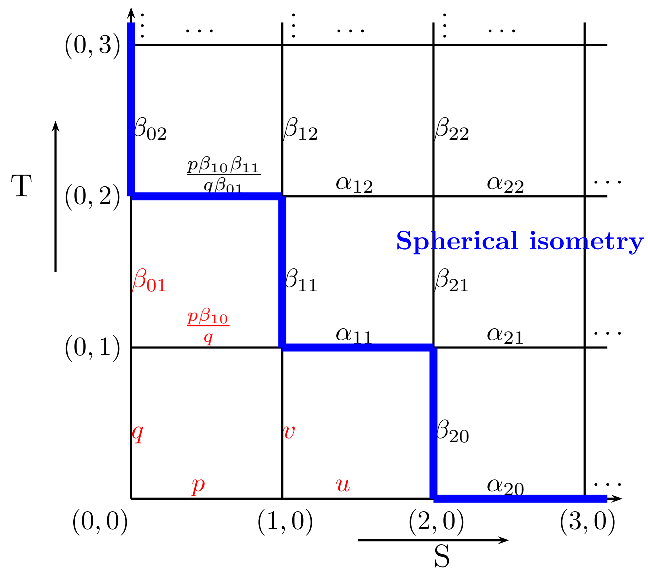

FIGURE 4. Weight diagram for the 2-variable weighted shift in Theorem 4.5

Proof. Observe that the restriction of (S,T) to the invariant subspace $\mathcal{L}_{(0,1)}$ satisfies the hypotheses in Theorem 4.2. Using this information, one now needs to imitate the Proof of Theorem 4.4 to reach the desired conclusion. $\square$

Theorem 4.6. *(Case 4: multi-cell perturbation.) Consider the 2-variable weighted shift given by the weight diagram in Figure 5. Then the iterates of the spherical Aluthge transform of (S,T) converge in the WOT to a spherical isometry.*

Proof. As the reader will surely anticipate, this case reduces to the previous cases, through a series of steps. For instance, the restriction of (S,T) to $\mathcal{L}_{(1,0)}$ fits Case 3, and once this information is incorporated, Case 4 becomes similar to Case 2. $\square$

We conclude this section with two open questions, which we plan to discuss in a separate paper.

Question 4.7. *Let (T_1,T_2) be a commuting pair of operators on a finite dimensional Hilbert space. Does the sequence of iterates $\Delta^n_{sph}(T_1,T_2)$ converge in the norm?*

Remark 4.8. One very special case of Theorem 4.2 has to do with taking the Helton-Howe shift and altering only the weights $\alpha_{(0,0)}$ and $\beta_{(0,0)}$. By commutativity, we must

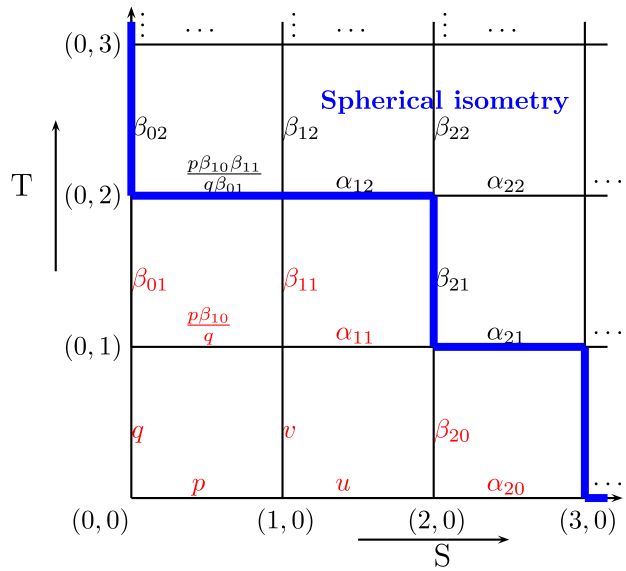

FIGURE 5. Weight diagram for the 2-variable weighted shift in Theorem 4.6

have $x := \alpha^2_{(0,0)} = \beta^2_{(0,0)}$. Call this new shift (S_x, T_x). (Strictly speaking, (S_x, T_x) does not satisfy the hypotheses of Theorem 4.2, since the Helton-Howe shift (S_1, T_1) is not a spherical isometry, but $(\frac{1}{\sqrt{2}}S_x, \frac{1}{\sqrt{2}}T_x)$ is.) One can then prove that the Berger measure of (S_x, T_x) is $(1-x)\delta_{(0,0)} + x\delta_{(1,1)}$. When we take the spherical Aluthge transform, the atoms remain unchanged, but the densities become $1 - \sqrt{x}$ and $\sqrt{x}$, respectively. As we keep iterating, the square root becomes fourth root, eighth root, etc., so the Berger measure of the n-th iterate is given by $(1 - \sqrt[2^n]{x})\delta_{(0,0)} + \sqrt[2^n]{x}\delta_{(1,1)}$. As the number of iterates grow, this expression converges to 1, so in the limit we get only $\delta_{(1,1)}$, that is, the Berger measure of the Helton-Howe shift.

Remark 4.9. The reader must have surely noticed that in Theorem 4.2 the parameters p and q determine the asymptotic behavior of the iterates. On the other hand, due to the commutativity of (S, T) those parameters are directly related, that is, $q\alpha_{01} = p\beta_{10}$; in other words, q depends on p and the data encapsulated by the spherical isometry $(S, T) \restriction_{\mathcal{L}_{(1,0)} \bigvee \mathcal{L}_{(0,1)}}$. That is, the asymptotic behavior in that case depends on one degree of freedom, given by, for instance, p. In Theorem 4.4, the number of degrees of freedom is two (think about the parameters p and u as being free), while in Theorem 4.5 the number of degrees of freedom is three. We leave it to the reader to determine the number of degrees of freedom in Theorem 4.6 and in more general cases.

Question 4.10. *What is the asymptotic behavior of the iterates of the spherical Aluthge transform of 2-variable weighted shifts with finitely atomic Berger measures?*

Acknowledgments. The authors are deeply grateful to the referee for a careful reading of the paper and for several suggestions and edits that helped improve the presentation. Some of the calculations in this paper were made with the software tool *Mathematica* [35].

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, *Integral Equations Operator Theory* **13** (1990), 307–315.
[2] T. Ando, Aluthge transforms and the convex hull of the spectrum of a Hilbert space operator, in *Recent advances in operator theory and its applications*, *Oper. Theory Adv. Appl.* **160**(2005), 21–39.
[3] T. Ando and T. Yamazaki, The iterated Aluthge transforms of a 2×2 matrix converge, *Linear Algebra Appl.* **375**(2003), 299–309.
[4] J. Antezana, E. Pujals and D. Stojanoff, Convergence of iterated Aluthge transform sequence for diagonalizable matrices, *Adv. Math.* **216** (2007), 255–278.
[5] J. Antezana, E. Pujals and D. Stojanoff, Iterated Aluthge transform: A brief survey, *Rev. Un. Mat. Argentina* **49**(2008), 29–41.
[6] A. Athavale, On the intertwining of joint isometries, *J. Operator Theory,* **23**(1990), 339–350.
[7] A. Athavale and S. Podder, On the reflexivity of certain operator tuples, *Acta Math. Sci. (Szeged)* **81**(2015), 285–291.
[8] C. Benhida, R.E. Curto, S.H. Lee and J. Yoon, Joint spectra of spherical Aluthge transforms of commuting n-tuples of Hilbert space operators, C. R. Math. Acad. Sci. Paris, in press.
[9] M. Chō, I. B. Jung and W. Y. Lee, On Aluthge transforms of p-hyponormal operators, *Integral Equations Operator Theory* **53**(2005), 321–329.
[10] J. Conway, *The Theory of Subnormal Operators,* Mathematical Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.
[11] R. Curto, On the connectedness of invertible n-tuple, *Indiana Univ. Math. J.* **29**(1980), 393–406.
[12] R. Curto, Applications of several complex variables to multi-parameter spectral theory, in *Surveys of Recent Results in Operator Theory*, Vol. II, J.B. Conway and B.B. Morrel, editors, Longman Publishing Co., London, 1988, 25–90.
[13] R.E. Curto, Joint hyponormality: A bridge between hyponormality and subnormality, *Proc. Symposia Pure Math.* **51**(1990), Part II, 69–91.
[14] R.E. Curto, Two-variable weighted shifts in multivariable operator theory, in *Handbook of Analytic Operator Theory*, K. Zhu (ed.), CRC Press, 2019; 17–63.
[15] R.E. Curto and G. Exner, Berger measure for some transformations of subnormal weighted shifts, *Integral Equations Operator Theory* **84**(2016), 429–450.
[16] R.E. Curto, S.H. Lee and J. Yoon, Quasinormality of powers of commuting pairs of bounded operators, preprint 2019.
[17] R.E. Curto and J. Yoon, Toral and spherical Aluthge transforms for 2-variable weighted shifts, *C. R. Acad. Sci. Paris* **354**(2016), 1200–1204.
[18] R.E. Curto and J. Yoon, Aluthge transforms of 2-variable weighted shifts, *Integral Equations Operator Theory*(2018) **90**:52.
[19] R.E. Curto and J. Yoon, Spherically quasinormal pairs of commuting operators, *Analysis of Operators on Function Spaces (The Serguei Shimorin Memorial Volume)*, A. Aleman, H. Hedenmalm, D. Khavinson and M. Putinar (eds.), *Trends in Mathematics*, Birkhäuser Verlag, 2019; 213–237.
[20] K. Dykema and H. Schultz, Brown measure and iterates of the Aluthge transform for some operators arising from measurable actions, *Trans. Amer. Math. Soc.* **361**(2009), 6583–6593.
[21] G.R. Exner, Aluthge transforms and n-contractivity of weighted shifts, *J. Operator Theory* **61**(2009), 419–438.
[22] R. Gellar and L.J. Wallen, Subnormal weighted shifts and the Halmos-Bram criterion, *Proc. Japan Acad.* **46**(1970), 375–378.

[23] J. Gleason, Quasinormality of Toeplitz tuples with analytic symbols, *Houston J. Math.* **32**(2006), 293–298.
[24] P.R. Halmos, *A Hilbert Space Problem Book*, Second Edition, Springer-Verlag, New York, 1992.
[25] N.P. Jewell and A.R. Lubin, Commuting weighted shifts and analytic function theory in several variables, *J. Operator Theory* **1**(1979), 207–223.
[26] I.B. Jung, E. Ko, and C. Pearcy, Aluthge transform of operators, *Integral Equations Operator Theory* **37**(2000), 437–448.
[27] I.B. Jung, E. Ko and C. Pearcy, Spectral pictures of Aluthge transforms of operators, *Integral Equations Operator Theory* **40**(2001), 52–60.
[28] I.B. Jung, E. Ko and C. Pearcy, The iterated Aluthge transform of an operator, *Integral Equations Operator Theory* **45**(2003), 375–387.
[29] M.K. Kim and E. Ko, Some connections between an operator and its Aluthge transform, *Glasg. Math. J.* **47**(2005), 167–175.
[30] F. Kimura, Analysis of non-normal operators via Aluthge transformation, *Integral Equations Operator Theory* **50**(2004), 375–384.
[31] S.H. Lee, W.Y. Lee and J. Yoon, Subnormality of Aluthge transform of weighted shifts, *Integral Equations Operator Theory* **72**(2012), 241–251.
[32] K. Rion, Convergence properties of the Aluthge sequence of weighted shifts, *Houston J. Math.* **42**(2016), 1217–1226.
[33] J. Stampfli, Which weighted shifts are subnormal, Pacific J. Math. **17**(1966), 367–379.
[34] J. L. Taylor, A joint spectrum for several commuting operators, *J. Funct. Anal.* **6**(1970), 172–191.
[35] Wolfram Research, Inc., *Mathematica*, Version 11.3, Wolfram Research Inc., Champaign, IL, 2018.

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