WHEN IS HYPONORMALITY FOR 2-VARIABLE WEIGHTED SHIFTS INVARIANT UNDER POWERS?

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ABSTRACT. For 2-variable weighted shifts $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ we study the invariance of (joint) k-hyponormality under the action $(h,\ell) \mapsto W_{(\alpha,\beta)}^{(h,\ell)} := (T_1^h, T_2^\ell)$ $(h,\ell \geq 1)$. We show that for every $k \geq 1$ there exists $W_{(\alpha,\beta)}$ such that $W_{(\alpha,\beta)}^{(h,\ell)}$ is k-hyponormal (all $h \geq 2$, $\ell \geq 1$) but $W_{(\alpha,\beta)}$ is not k-hyponormal. On the positive side, for a class of 2-variable weighted shifts with tensor core we find a computable necessary condition for invariance. Next, we exhibit a large nontrivial class for which hyponormality is indeed invariant under *all* powers; moreover, for this class 2-hyponormality automatically implies subnormality. Finally, we show that there exists a 2-hyponormal $W_{(\alpha,\beta)}$ such that $W_{(\alpha,\beta)}^{(2,1)}$ is not 2-hyponormal. Our results partially depend on new formulas for the determinant of generalized Hilbert matrices and on criteria for their positive semi-definiteness.

1. INTRODUCTION

Given a pair $\mathbf{T} \equiv (T_1, T_2)$ of commuting subnormal Hilbert space operators, the Lifting Problem for Commuting Subnormals (LPCS) calls for necessary and sufficient conditions for the existence of a commuting pair $\mathbf{N} \equiv (N_1, N_2)$ of normal extensions of T_1 and T_2. In previous work ([CLY1], [CLY2], [CLY3], [CLY4], [CuYo1], [CuYo2], [CuYo3]) we have studied the relevance of (joint) k-hyponormality to LPCS. In particular, one asks to what extent the existence of liftings for the powers $\mathbf{T}^{(h,\ell)} \equiv (T_1^h, T_2^\ell)$ $(h,\ell \geq 1)$ can guarantee a lifting for $\mathbf{T}$. For the class of 2-variable weighted shifts $W_{(\alpha,\beta)}$, it is often the case that the powers are less complex than the initial pair; thus it becomes especially significant to unravel the invariance of k-hyponormality under the action $(h,\ell) \mapsto W_{(\alpha,\beta)}^{(h,\ell)}$ $(h,\ell \geq 1)$.

Our aim in this paper is to shed new light on some of the intricacies associated with LPCS and k-hyponormality for powers of commuting subnormals. To describe our results we need some notation; we further expand on our terminology and basic results in Section 2. We use $\mathfrak{H}_0$ (resp. $\mathfrak{H}_\infty$) to denote the set of commuting pairs of subnormal operators (resp. subnormal pairs) on Hilbert space. For $k \geq 1$, we let $\mathfrak{H}_k$ denote the class of k-hyponormal pairs in $\mathfrak{H}_0$. Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$. The main results in [CuYo1] and [CLY1] show that these inclusions are all proper. In our previous research we have shown that detecting these proper inclusions can be done within classes of 2-variable weighted shifts with relatively simple weight structure, as we now describe.

For a sequence $\alpha \equiv \{\alpha_k\}_{k=0}^\infty \in \ell^\infty(\mathbb{Z}_+)$ of positive numbers, we let $W_\alpha \equiv \text{shift}\ (\alpha_0, \alpha_1, \cdots)$ denote the weighted shift on $\ell^2(\mathbb{Z}_+)$ given by $W_\alpha e_k := \alpha_k e_{k+1}$ $(k \geq 0)$. We also let $U_+ := \text{shift}\ (1, 1, \cdots)$

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(the (unweighted) unilateral shift), and for $0 < a < 1$ we let $S_a := \text{shift}\,(a, 1, 1, \cdots)$. Multivariable weighted shifts are defined in an analogous manner. For instance, on $\ell^2(\mathbb{Z}_+^2)$ we let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ denote the 2-variable weighted shift associated with weight sequences α and β, defined by $T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$ and $T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}$ ($\mathbf{k} \in \mathbb{Z}_+^2$).

For an arbitrary 2-variable weighted shift $W_{(\alpha,\beta)}$, we let $\mathcal{M}_i$ (resp. $\mathcal{N}_j$) be the subspace of $\ell^2(\mathbb{Z}_+^2)$ which is spanned by the canonical orthonormal basis associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geq 0$ and $k_2 \geq i$ (resp. $k_1 \geq j$ and $k_2 \geq 0$). We will often write $\mathcal{M}_1$ simply as $\mathcal{M}$ and $\mathcal{N}_1$ as $\mathcal{N}$. The core $c(W_{(\alpha,\beta)})$ of $W_{(\alpha,\beta)}$ is the restriction of $W_{(\alpha,\beta)}$ to the invariant subspace $\mathcal{M} \bigcap \mathcal{N}$. A 2-variable weighted shift $W_{(\alpha,\beta)}$ is said to be *of tensor form* if it is of the form $(I \otimes W_\alpha, W_\beta \otimes I)$. The class of all 2-variable weighted shifts $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ whose core is of tensor form will be denoted by $\mathcal{TC}$; in symbols, $\mathcal{TC} := \{W_{(\alpha,\beta)} \in \mathfrak{H}_0 : c(W_{(\alpha,\beta)}) \text{ is of tensor form}\}$.

We now consider the class $\mathcal{S} := \{W_{(\alpha,\beta)} \in \mathfrak{H}_0 : \alpha_{(k_1,0)} = \alpha_{(k_1+1,0)}$ and $\beta_{(0,k_2)} = \beta_{(0,k_2+1)}$ for some $k_1 \geq 1$and $k_2 \geq 1\}$ and we let $\mathcal{S}_1 := \mathcal{S} \cap \mathfrak{H}_1$. From propagation phenomena for 1- and 2-variable weighted shifts (see [CuYo2] and [CLY4]), we observe that, without loss of generality, we can always assume that the restrictions of each $W_{(\alpha,\beta)} \in \mathcal{S}_1$ to the invariant subspace $\mathcal{M}$ (resp. $\mathcal{N}$) is of the form $(I \otimes S_a, U_+ \otimes I)$ (resp. $(I \otimes U_+, S_b \otimes I)$); cf. Figure 3(i)). In particular, the core $c(W_{(\alpha,\beta)})$ of a 2-variable weighted shift in $\mathcal{S}_1$ is always the doubly commuting pair $(I \otimes U_+, U_+ \otimes I)$ and, a fortiori, $W_{(\alpha,\beta)} \in \mathcal{TC}$. Observe also that if $W_{(\alpha,\beta)} \in \mathcal{S}_1$, then $W_{(\alpha,\beta)}$ is completely determined by the three parameters $x := \alpha_{(0,0)}$, $y := \beta_{(0,0)}$ and $a := \alpha_{(0,1)}$. Thus we shall often denote a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{S}_1$ by $\langle x, y, a \rangle$.

Between $\mathcal{S}_1$ and $\mathcal{TC}$ there is a class that provides significant information about LPCS, and we now define it. Let $\mathcal{A} := \{W_{(\alpha,\beta)} \in \mathcal{TC} : c(W_{(\alpha,\beta)}) \text{ is 1-atomic}\}$. Clearly $\mathcal{S}_1 \subsetneq \mathcal{A} \subsetneq \mathcal{TC}$. In [CLY3] we solved LPCS within the class $\mathcal{TC}$, and in particular we gave a simple test for subnormality within $\mathcal{A}$.

To prove that the k-hyponormality of all powers may not guarantee the k-hyponormality of the initial pair, we build an example that uses weights related to those of the Bergman shift. The reader will recall that the moment matrix associated with the Bergman shift is the classical Hilbert matrix. Thus to deal with our situation we need to describe positivity and the calculation of determinants for *generalized* Hilbert matrices; we do this in Theorem 3.1. Although Section 3 has intrinsic and independent value since it deals with matrices that arise in various contexts, the main reason for including it here is as a tool for producing some of the examples in subsequent sections.

It is well known that for a general operator T on Hilbert space, the hyponormality of T does not imply the hyponormality of T^2 [Hal]. However, for a unilateral weighted shift W_α, the hyponormality of W_α (detected by the condition $\alpha_j \leq \alpha_{j+1}$ for all $j \geq 0$) does imply the hyponormality of every power W_α^n ($n \geq 2$). It is also well known that the subnormality of T implies the subnormality of T^n (all $n \geq 2$), but the converse implication is not true, even if T is a unilateral weighted shift [Sta]. Since k-hyponormality lies between hyponormality and subnormality, it is then natural to consider

Problem 1.1. *Let T be an operator and let $k \geq 2$.*
(i) Does the k-hyponormality of T imply the k-hyponormality of T^2?
(ii) Does the k-hyponormality of T^2 imply the k-hyponormality of T?

At the beginning of Section 4 we consider this problem, and we then study its multivariable analogue. It is worth noting that, in the multivariable case, the standard assumption on a pair $\mathbf{T} \equiv (T_1, T_2)$ is that each component T_i is subnormal ($i = 1, 2$). With this in mind, comparing the k-hyponormality of a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ to the k-hyponormality of its powers $W_{(\alpha,\beta)}^{(h,\ell)}$ is highly nontrivial. We now formulate the relevant problems in the multivariable case.

Problem 1.2. *Given $k \geq 1$ and $W_{(\alpha,\beta)} \in \mathfrak{H}_k$, does it follow that $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_k$ for all $h, \ell \geq 1$?*

In Section 5 we establish that the class $\mathfrak{H}_k$ ($k \geq 1$) is *not* invariant under powers. Concretely, we prove that there exists $W_{(\alpha,\beta)} \in \mathfrak{H}_2$ such that $W_{(\alpha,\beta)}^{(2,1)} \notin \mathfrak{H}_2$ (Theorem 5.4). Conversely, we can ask

Problem 1.3. *Given $k \geq 1$, assume that for all $h \geq 2$ and $\ell \geq 1$, $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_k$. Does it follow that $W_{(\alpha,\beta)} \in \mathfrak{H}_k$?*

In Theorem 4.8 we answer Problem 1.3 in the negative; that is, for each $k \geq 1$ we build a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathfrak{H}_0 \setminus \mathfrak{H}_k$ such that $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_k$ (all $h \geq 2$ and $\ell \geq 1$).

Next, for $k = 1, 2$, we find a computable necessary condition for the k-hyponormality of $W_{(\alpha,\beta)}$ to remain invariant under all powers (Theorem 5.1). We then show that this necessary conditions is not sufficient (Remark 5.3(ii)).

Section 6 is devoted to the study of the class $\mathcal{S}_1$. We show that for $\langle x, y, a \rangle \in \mathcal{S}_1$, all powers $\langle x, y, a \rangle^{(h,\ell)}$ are hyponormal (Theorem 6.6). Moreover, a shift $\langle x, y, a \rangle \in \mathcal{S}_1$ is 2-hyponormal if and only if it is subnormal.

As we mentioned before, for single operators it is an open problem whether the 2-hyponormality of T implies the 2-hyponormality of T^2. Although this problem is intimately related to Theorem 5.4, we observe that the latter does not provide an answer to Problem 1.1 when $k = 2$, since our pairs consist of commuting subnormal operators.

Problem 1.2 is a special case of a much more general problem, that of determining necessary and sufficient conditions for the weak k-hyponormality of a commuting pair. We say that a pair $\mathbf{T} \in \mathfrak{H}_0$ is *weakly k-hyponormal* if

$$\mathbf{p}(\mathbf{T}) := (p_1(T_1, T_2), p_2(T_1 T_2)),$$

is hyponormal for all polynomials p_1, $p_2 \in \mathbb{C}[z, w]$ with deg p_1, deg $p_2 \leq k$, where $\mathbf{p} \equiv (p_1, p_2)$. To verify that $\mathbf{T}$ is weakly k-hyponormality is highly nontrivial. Thus Problems 1.2 and 1.3 can be regarded as suitably multivariable analogues of [Shi, Question 33]: If T is a hyponormal unilateral shift and if p is a polynomial, must $p(T)$ be hyponormal? If T is subnormal, the answer is clearly yes, but we note that polynomial hyponormality is strictly weaker than subnormality, as proved in [CuPu].

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2. Notation and Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S, T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [Ath], [CMX]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal

n-tuple to a common invariant subspace. For $k \geq 1$, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is said to be *k-hyponormal* ([CLY1]) if

$$\mathbf{T}(k) := (T_1, T_2, T_1^2, T_2T_1, T_2^2, \cdots, T_1^k, T_2T_1^{k-1}, \cdots, T_2^k)$$

is hyponormal, or equivalently

$$[\mathbf{T}(k)^*, \mathbf{T}(k)] = ([(T_2^qT_1^p)^*, T_2^mT_1^n])_{\substack{1\leq n+m\leq k \\ 1\leq p+q\leq k}} \geq 0.$$

Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ k-hyponormal. The Bram-Halmos criterion states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if the k-tuple $(T, T^2, \cdots, T^k)$ is hyponormal for all $k \geq 1$.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1, & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2, & \text{if } k > 0 \end{array} \right\}.$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \text{ (all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{2.1}$$

In an entirely similar way one can define multivariable weighted shifts. Trivially, a pair of unilateral weighted shifts W_a and W_β gives rise to a 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+$). In this case, $W_{(\alpha,\beta)}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ with $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, we have $T_1 \cong I \bigotimes W_a$ and $T_2 \cong W_\beta \bigotimes I$, and $W_{(\alpha,\beta)}$ is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed.

Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha,\beta) := \begin{cases} 1, & \text{if } \mathbf{k} = \mathbf{0} \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

We remark that, due to the commutativity condition (2.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).

We now recall a well known characterization of subnormality for multivariable weighted shifts [JeLu], due to C. Berger (cf. [Con, III.8.16]) and independently established by R. Gellar and L.J. Wallen [GeWa]) in the single variable case: $W_{(\alpha,\beta)}$ admits a commuting normal extension if and only if there is a probability measure μ (which we call the *Berger measure* of $W_{(\alpha,\beta)}$) defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ (where $a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \int_R \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}) :=$

$\int_R t_1^{k_1} t_2^{k_2} d\mu(\mathbf{t})$, for all $\mathbf{k} \in \mathbb{Z}_+^2$. Observe that U_+ and S_a are subnormal, with Berger measures δ_1 and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support the singleton set $\{p\}$. Also, a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{S}_1$ has a core with Berger measure $\delta_1 \times \delta_1$.

3. The determinant of a generalized Hilbert matrix

Given positive real numbers x and h, and an integer $k \geq 1$, we define the generalized Hilbert matrix $A_k(x,h)$ as follows:

$$(A_k(x,h))_{i,j} := \begin{cases} x, & \text{if } i=j=1 \\ \frac{1}{(i+j-2)h+1}, & \text{otherwise} \end{cases} \qquad (1 \leq i,j \leq k+1).$$

(Observe that $A_k(1,1)$ is the classical Hilbert matrix.) In this section we calculate the determinant of, and establish positivity properties for, the generalized Hilbert matrix $A_k(x,h)$.

To describe our results, we need some notation. We let $0! := 1$, $k! := k(k-1)!$, and $k^! := \Pi_{i=1}^k i!$. We also let

$$\begin{aligned} f_0 &:= x, \\ f_{\ell+1} &:= f_\ell \left(\frac{(k-\ell)h+1}{(k-\ell)\,h} \right)^2 - \frac{2(k-\ell)h+1}{((k-\ell)\,h)^2} \; (0 \leq \ell \leq k-1), \end{aligned}$$

$$g(h,k) := \left(\tfrac{1}{(k+1)h+1}\right)^k \prod_{j=0}^{k-1} \left(\tfrac{1}{(jh+1)(2kh-jh+1)}\right)^{j+1}, \tag{3.1}$$

and

$$\begin{aligned} f(x,h,k) := {} & x \left(\tfrac{kh+1}{kh}\right)^2 \left(\tfrac{(k-1)h+1}{(k-1)h}\right)^2 \cdots \left(\tfrac{3h+1}{3h}\right)^2 \left(\tfrac{2h+1}{2h}\right)^2 \left(\tfrac{(h+1)^2}{h}\right) \\ & - \left(\tfrac{2(kh)+1}{(kh)^2}\right) \left(\tfrac{(k-1)h+1}{(k-1)h}\right)^2 \cdots \left(\tfrac{3h+1}{3h}\right)^2 \left(\tfrac{2h+1}{2h}\right)^2 \tfrac{(h+1)^2}{h} \\ & - \left(\tfrac{2((k-1)h)+1}{((k-1)h)^2}\right) \left(\tfrac{(k-2)h+1}{(k-2)h}\right)^2 \cdots \left(\tfrac{3h+1}{3h}\right)^2 \left(\tfrac{2h+1}{2h}\right)^2 \tfrac{(h+1)^2}{h} \\ & - \cdots - \left(\tfrac{2(2h)+1}{(2h)^2}\right) \tfrac{(h+1)^2}{h} - \left(\tfrac{2h+1}{h}\right). \end{aligned} \tag{3.2}$$

Theorem 3.1. *For $x,h>0$ and $k \geq 1$, we have*

$$\det\ A_k(x,h) = h^{k(k+1)} \left(k^!\right)^2 g(h,k) f_k, \tag{3.3}$$

where $f_k = f_{k-1} \left(\frac{h+1}{h}\right)^2 - \frac{2h+1}{h^2}$. Moreover,

$$f_k = f(x,h,k).$$

Proof. Consider the $(k+1)\times(k+1)$ matrix

$$A_k(x,h) = \begin{pmatrix} x & \frac{1}{h+1} & \frac{1}{2h+1} & \cdots & \frac{1}{(k-1)h+1} & \frac{1}{kh+1} \\ \frac{1}{h+1} & \frac{1}{2h+1} & \frac{1}{3h+1} & \cdots & \frac{1}{kh+1} & \frac{1}{(k+1)h+1} \\ \frac{1}{2h+1} & \frac{1}{3h+1} & \frac{1}{4h+1} & \cdots & \frac{1}{(k+1)h+1} & \frac{1}{(k+2)h+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(k-1)h+1} & \frac{1}{kh+1} & \frac{1}{(k+1)h+1} & \cdots & \frac{1}{(2k-2)h+1} & \frac{1}{(2k-1)h+1} \\ \frac{1}{kh+1} & \frac{1}{(k+1)h+1} & \frac{1}{(k+2)h+1} & \cdots & \frac{1}{(2k-1)h+1} & \frac{1}{2kh+1} \end{pmatrix}.$$

Let us first subtract the $(k+1)$-st row from each row above it. The entry in the j-th column of the i-th row becomes

$$\begin{cases} x - \frac{1}{kh+1}, & \text{if } (i,j) = (1,1) \\ \frac{1}{(i+j-2)h+1} - \frac{1}{(k+j-1)h+1} = \frac{(k-i+1)h}{[(i+j-2)h+1][(k+j-1)h+1]}, & \text{if } (i,j) \neq (1,1). \end{cases}$$

The new $(k+1) \times (k+1)$ matrix is

$$B_k(x,h) := \begin{pmatrix} x - \frac{1}{kh+1} & \frac{kh}{[h+1][(k+1)h+1]} & \cdots & \frac{kh}{[(k-1)h+1][(2k-1)h+1]} & \frac{kh}{[kh+1][2kh+1]} \\ \frac{(k-1)h}{[h+1][kh+1]} & \frac{(k-1)h}{[2h+1][(k+1)h+1]} & \cdots & \frac{(k-1)h}{[kh+1][(2k-1)h+1]} & \frac{(k-1)h}{[(k+1)h+1][2kh+1]} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{h}{[(k-1)h+1][kh+1]} & \frac{h}{[kh+1][(k+1)h+1]} & \cdots & \frac{h}{[(2k-2)h+1][(2k-1)h+1]} & \frac{h}{[(2k-1)h+1][2kh+1]} \\ \frac{1}{kh+1} & \frac{1}{(k+1)h+1} & \cdots & \frac{1}{(2k-1)h+1} & \frac{1}{2kh+1} \end{pmatrix}.$$

Note that det $A_k(x,h) =$ det $B_k(x,h)$. To compute det $B_k(x,h)$, we observe that one can factor out $(k-(i-1))h$ from the i-th row $(1 \leq i < k)$ and $\frac{1}{(k+j-1)h+1}$ from the j-th column $(1 \leq j \leq k+1)$ in the matrix $B_k(x,h)$. Hence we obtain

$$\det\ A_k(x,h) = k!h^k \cdot \frac{1}{kh+1} \cdot \frac{1}{(k+1)h+1} \cdots \frac{1}{2kh+1} \cdot \det\ C_k(x,h),$$

where

$$C_k(x,h) := \begin{pmatrix} \left(x - \frac{1}{kh+1}\right) \frac{kh+1}{kh} & \frac{1}{h+1} & \cdots & \frac{1}{kh+1} \\ \frac{1}{h+1} & \frac{1}{2h+1} & \cdots & \frac{1}{(k+1)h+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(k-1)h+1} & \frac{1}{kh+1} & \cdots & \frac{1}{(2k-1)h+1} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Next, let us subtract the last column from each of the preceding columns in the $(k+1) \times (k+1)$ matrix $C_k(x,h)$. We obtain

$$D_k(x,h) := \begin{pmatrix} \left(x - \frac{1}{kh+1}\right) \frac{kh+1}{kh} - \frac{1}{kh+1} & \frac{(k-1)h}{(h+1)[kh+1]} & \cdots & \frac{h}{[(k-1)h+1][kh+1]} & \frac{1}{kh+1} \\ \frac{kh}{(h+1)[(k+1)h+1]} & \frac{(k-1)h}{(2h+1)[(k+1)h+1]} & \cdots & \frac{h}{[kh+1][(k+1)h+1]} & \frac{1}{(k+1)h+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{kh}{[(k-1)h+1][(2k-1)h+1]} & \frac{(k-1)h}{[kh+1][(2k-1)h+1]} & \cdots & \frac{h}{[(2k-2)h+1][(2k-1)h+1]} & \frac{1}{(2k-1)h+1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Note that det $C_k(x,h) =$ det $D_k(x,h)$. As we have done before, let us factor out $(k-(j-1))h$ from the j-th column $(1 \leq j \leq k)$ and $\frac{1}{(k+i-1)h+1}$ from the i-th row $(1 \leq i \leq k)$ in the matrix $D_k(x,h)$. Let

$$f_1 \equiv f_1(x,h,k) := x\left(\frac{kh+1}{kh}\right)^2 - \frac{2kh+1}{(kh)^2}.$$

Then we have

$$\det\ A_k(x,h) = (k!)^2\, h^{2k} \left(\tfrac{1}{kh+1}\right)^2 \left(\tfrac{1}{(k+1)h+1}\right)^2 \cdots \left(\tfrac{1}{(2k-1)h+1}\right)^2 \left(\tfrac{1}{2kh+1}\right) \det\ A_{k-1}(f_1,h),$$

where

$$A_{k-1}(f_1,h) := \begin{pmatrix} f_1 & \frac{1}{h+1} & \cdots & \frac{1}{(k-1)h+1} \\ \frac{1}{h+1} & \frac{1}{2h+1} & \cdots & \frac{1}{[kh+1]} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{[(k-1)h+1]} & \frac{1}{[kh+1]} & \cdots & \frac{1}{(2k-2)h+1} \end{pmatrix}$$

is now a $k \times k$ matrix. Continuing in this way we have

$$\det\ A_k(x,h) = (k!(k-1)!)^2\, h^{2(k+(k-1))} \left(\tfrac{1}{(k-1)h+1}\right)^2 \left(\tfrac{1}{kh+1}\right)^2 \left(\tfrac{1}{(k+1)h+1}\right)^4 \cdots$$

$$\left(\tfrac{1}{(2k-3)h+1}\right)^4 \left(\tfrac{1}{(2k-2)h+1}\right)^3 \left(\tfrac{1}{(2k-1)h+1}\right)^2 \left(\tfrac{1}{2kh+1}\right) \det\ A_{k-2}(f_2,h),$$

where

$$f_2 \equiv f_2(x,h,k) := f_1 \left(\frac{(k-1)h+1}{(k-1)h}\right)^2 - \left(\frac{2(k-1)h+1}{(k-1)^2h^2}\right).$$

In general, we see that $\det\ A_k(x,h)$ can be expressed in terms of $\det\ A_{k-\ell-1}(f_{\ell+1},h)$, where

$$f_{\ell+1} = f_\ell \left(\frac{(k-\ell)h+1}{(k-\ell)\,h}\right)^2 - \frac{2(k-\ell)h+1}{((k-\ell)\,h)^2} \quad (0 \le \ell \le k-1). \tag{3.4}$$

Thus, by direct calculation we have

$$\det\ A_k(x,h) = \left(k!\right)^2 h^{k(k+1)} g(h,k) f_k,$$

where $g(h,k)$ is given by (3.1) and

$$f_k = \det\ A_0(f_k,h) = f_{k-1} \left(\frac{h+1}{h}\right)^2 - \frac{2h+1}{h^2}. \tag{3.5}$$

On the other hand, careful inspection of the recursive definition of f_k (cf. (3.5) and (3.4)) and of the formula for $f(x,h,k)$ (see (3.2)) shows that $f_k = f(x,h,k)$ (all $x, h > 0$ and $k \ge 1$). The proof is now complete. □

Corollary 3.2. *For $k \ge 1$ and $h \ge 1$,*

$$\det A_k(x,h) < \det A_{k-1}(x,h).$$

Proof. We consider two cases.

Case 1: $k = 1$. Note that $\det A_0(x,h) = x$ and $\det A_1(x,h) = \frac{x}{2h+1} - \left(\frac{1}{h+1}\right)^2$. Thus we have

$$\frac{\det\ A_1(x,h)}{\det\ A_0(x,h)} < \frac{1}{2h+1} < 1.$$

Case 2: $k \ge 2$. Consider the quotient

$$\frac{\det\ A_k(x,h)}{\det\ A_{k-1}(x,h)} = \frac{h^{k(k+1)}\left(k!\right)^2 g(h,k) f_k}{h^{k(k-1)}\,(k-1!)^2\, g(h,k-1) f_{k-1}} = \frac{h^{2k} k!^2 g(h,k) f_k}{g(h,k-1) f_{k-1}}$$

and observe, using (3.1), that

$$\frac{g(h,k)}{g(h,k-1)} = \frac{\left(\frac{1}{kh+1}\right)^k \prod_{j=0}^{k-1}\left(\frac{1}{(jh+1)(2kh-jh+1)}\right)^{j+1}}{\left(\frac{1}{(k-1)h+1}\right)^{k-1} \prod_{j=0}^{k-2}\left(\frac{1}{(jh+1)(2(k-1)h-jh+1)}\right)^{j+1}}$$

$$= \left(\frac{1}{2kh+1}\right)\prod_{j=0}^{k-1}\left(\frac{1}{(k+j)\,h+1}\right)^2.$$

Hence

$$\begin{aligned}\frac{\det\ A_k(x,h)}{\det\ A_{k-1}(x,h)} &= h^{2k}k!^2\left(\frac{1}{2kh+1}\right)\prod_{j=0}^{k-1}\left(\frac{1}{(k+j)+1}\right)^2\frac{f_k}{f_{k-1}}\\ &< k!^2\left(\frac{1}{2kh}\right)\prod_{j=0}^{k-1}\left(\frac{1}{(k+j)}\right)^2\left(\frac{f_{k-1}\left(\frac{h+1}{h}\right)^2-\frac{2h+1}{h^2}}{f_{k-1}}\right)\\ &< k!^2\left(\frac{1}{2kh}\right)\prod_{j=0}^{k-1}\left(\frac{1}{(k+j)}\right)^2\left(\frac{h+1}{h}\right)^2\\ &\le \frac{1}{4}\left(\frac{1}{2h}\right)\left(\frac{h+1}{h}\right)^2 < 1\end{aligned}$$

whenever $h \geq 1$. Therefore, we have $\det\ A_k(x,h) < \det\ A_{k-1}(x,h)$ (all $h, k \geq 1$), as desired. □

Remark 3.3. As we have mentioned before, the matrix $A_k(1,1)$ is the classical Hilbert matrix. Specializing the above results to the case $x = h = 1$ in Theorem 3.1, we obtain

$$\begin{aligned}f(1,1,k) &= (k+1)^2 - \{(2k-1)+(2k-3)+\cdots+(2\cdot 2+1)+(2\cdot 1+1)\}\\ &= (k+1)^2 - k\frac{6+(k-1)2}{2} = 1\end{aligned}$$

and

$$\begin{aligned}g(1,k) &= \frac{0!}{(2k+1)!}\cdot\frac{1!}{(2k)!}\cdot\frac{2!}{(2k-1)!}\cdot\frac{3!}{(2k-2)!}\cdots\frac{(k-1)!}{(k+2)!}\cdot\frac{k!}{(k+1)!}\\ &= \frac{0!}{(2k+1)!}\cdot\frac{1!}{(2k)!}\cdot\frac{2!}{(2k-1)!}\cdot\frac{3!}{(2k-2)!}\cdots\frac{(k-1)!}{(k+2)!}\cdot\frac{k!}{(k+1)!}\cdot\frac{k!}{k!}\\ &= \frac{(k!)^2}{(2k+1)!}.\end{aligned}$$

We now use (3.3) and we recover the classical identity

$$\det\ A_k(1,1) = \frac{(k!)^4}{(2k+1)!} \quad \text{(cf. [PoSz, Part VII, Problem 4], [Choi, Solution to Problem 1]).}$$

Theorem 3.4. *Assume $x > 0$ and $h, k \geq 1$. The following statements are equivalent.*
(i) $A_k(x,h) \geq 0$;
(ii) $\det\ A_k(x,h) \geq 0$;
(iii) $f_k \equiv f(x,k,h) \geq 0$;

(iv) $x \geq b(k,h)$*, where*

$$\begin{cases} b(1,h) & := & \frac{2h+1}{(h+1)^2} \quad \text{and} \\ b(j,h) & := & \left[b(j-1,h) + \frac{2jh+1}{(jh)^2} \right] \cdot \left(\frac{jh}{jh+1} \right)^2 \quad (1 \leq j \leq k). \end{cases} \tag{3.6}$$

Proof. (i) $\Rightarrow$ (ii) This is trivial.

(ii) $\Rightarrow$ (i) Since

$$0 \leq \det\ A_k(x,h) < \det\ A_{k-1}(x,h) < \cdots < \det\ A_0(x,h)$$

(by Corollary 3.2), it follows from Choleski's Algorithm [Atk] that $A_k(x,h) \geq 0$.

(ii) $\Leftrightarrow$ (iii) This follows easily from the identity $\det\ A_k(x,h) = h^{k(k+1)} \left(k!\right)^2 g(k,h) f_k$ in Theorem 3.1, since $g(h,k)$ is clearly positive.

(iii) $\Leftrightarrow$ (iv) For $k = 1$, observe that $f_1 \geq 0 \iff \det\ A_1(x,h) \geq 0 \iff x \equiv f_0 \geq b(1,h)$. For $k \geq 2$, recall that $f_{\ell+1} = f_\ell \left(\frac{(k-\ell)h+1}{(k-\ell)h} \right)^2 - \frac{2(k-\ell)h+1}{((k-\ell)h)^2}$ $(0 \leq \ell \leq k-1)$. Thus

$$\begin{aligned} & f(x,k,h) \equiv f_k \geq 0 \\ & \iff f_{k-1} \geq \frac{2h+1}{(h+1)^2} \equiv b(1,h) \\ & \iff f_{k-2} \geq \left[b(1,h) + \frac{2(2h)+1}{(2h)^2} \right] \left(\frac{2h}{2h+1} \right)^2 \equiv b(2,h) \\ & \iff \cdots \iff f_0 \geq \left[b(k-1,h) + \frac{2kh+1}{(kh)^2} \right] \cdot \frac{kh}{(kh+1)^2} \equiv b(k,h) \\ & \iff x \equiv f_0 \geq b(k,h). \end{aligned}$$

(Observe in passing that $b(k,h) > 0$ (all $h, k \geq 1$) and that $\lim_{h\to\infty} b(k,h) = 0$ (all $k \geq 1$).) $\square$

4. The class $\mathfrak{H}_k$ $(k \geq 1)$ is not invariant under powers

For a general operator T on Hilbert space, it is well known that the subnormality of T implies the subnormality of T^m $(m \geq 2)$. The converse implication, however, is false; in fact, the subnormality of all powers T^m $(m \geq 2)$ does not necessarily imply the subnormality of T, even if T is a unilateral weighted shift [Sta, p. 378]. Consider for instance $W_\alpha \equiv \text{shift}\ (a, b, 1, 1, \cdots)$ where $0 < a < b < 1$. Clearly W_α is not 2-hyponormal (and therefore not subnormal), but W_α^m is subnormal for all $m \geq 2$. Thus it is indeed possible for a weighted shift W_α to have all powers W_α^m $(m \geq 2)$ k-hyponormal without W_α being k-hyponormal. The example above illustrates the case $k \geq 2$. When $k = 1$, it suffices to consider $W_\alpha \equiv \text{shift}\ (1, 1-x, y, y, \cdots)$ where $0 < x < 1 < y$. Then W_α^m $(m \geq 2)$ is hyponormal, but W_α is not hyponormal.

In the multivariable case, the standard assumption on a pair $\mathbf{T} \equiv (T_1, T_2)$ is that each component T_i is subnormal $(i = 1, 2)$. With this in mind, comparing the k-hyponormality of a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ to the k-hyponormality of its powers $W_{(\alpha,\beta)}^{(h,\ell)}$ is highly nontrivial. In [CLY2] we first considered this problem, for the case of 1-hyponormality. Specifically, if we let $W_{(\alpha,\beta)}$ denote the 2-variable weighted shift whose weight diagram is given in Figure 2(i), we proved in [CLY2, Lemma 2.4 and Theorem 2.6] that (see Figure 1): (i) $W_{(\alpha,\beta)} \in \mathfrak{H}_1$ and $W_{(\alpha,\beta)}^{(2,1)} \notin \mathfrak{H}_1 \iff a_{int} < a \leq \sqrt[4]{\frac{3}{5}}$ and $h_{21}(a) < y \leq h_1(a)$; and (ii) $W_{(\alpha,\beta)} \notin \mathfrak{H}_1$ and $W_{(\alpha,\beta)}^{(2,1)} \in \mathfrak{H}_1 \iff 0 < a < a_{int}$ and $h_1(a) < y \leq h_{21}(a)$.

In this section we extend the above mentioned result to arbitrary $\mathfrak{H}_k$ $(k \geq 2)$ and we also give a negative answer to Problem 1.3. Our main result, Theorem 4.8, gives necessary and sufficient conditions for $W_{(\alpha,\beta)}$ as above to have the property $W_{(\alpha,\beta)} \notin \mathfrak{H}_k$ and $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_k$, for each $k \geq 2$.

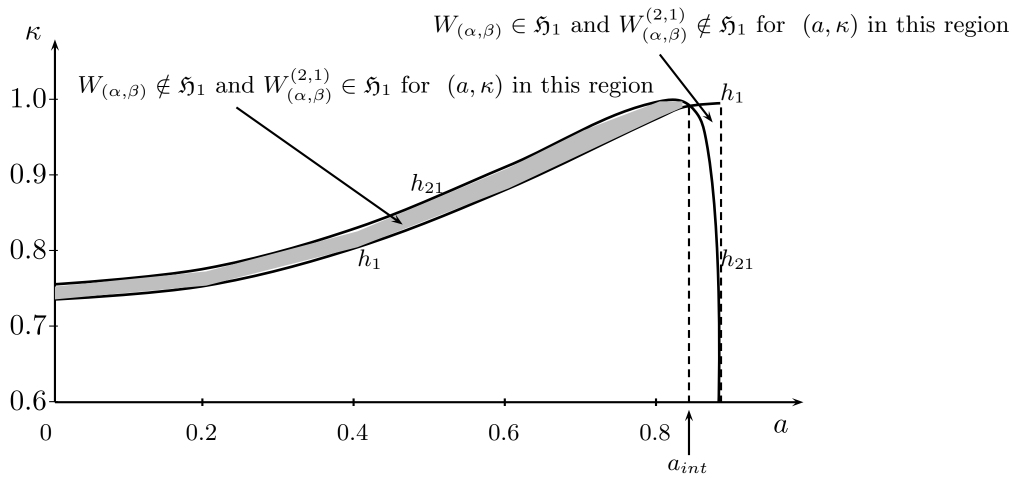

FIGURE 1. Graphs of h_1 and h_{21} on the interval $[0, \sqrt[4]{\frac{3}{5}}]$

To study k-hyponormality of multivariable weighted shifts, we first recall that, in one variable, the n-th power of a weighted shift is unitarily equivalent to the direct sum of n weighted shifts. Something similar happens in two variables, as we will see in the proof of Theorem 4.8 below. First, we need some terminology.

Let $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+) = \bigvee_{j=0}^{\infty}\{e_j\}$. Given integers i and m ($m \geq 1$, $0 \leq i \leq m-1$), define $\mathcal{H}_i := \bigvee_{j=0}^{\infty}\{e_{mj+i}\}$; clearly, $\mathcal{H} = \bigoplus_{i=0}^{m-1}\mathcal{H}_i$. Following the notation in [CuPa], for a weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ we let

$$W_{\alpha(m:i)} := \text{shift } (\Pi_{n=0}^{m-1}\alpha_{mj+i+n})_{j=0}^{\infty}; \tag{4.1}$$

that is, $W_{\alpha(m:i)}$ denotes the sequence of products of weights in adjacent packets of size m, beginning with $\alpha_i \cdots \alpha_{i+m-1}$. For example, given a weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$, we have $W_{\alpha(2:0)} =$ shift $(\alpha_0\alpha_1, \alpha_2\alpha_3, \cdots)$, $W_{\alpha(2:1)} =$ shift $(\alpha_1\alpha_2, \alpha_3\alpha_4, \cdots)$ and $W_{\alpha(3:2)} =$ shift $(\alpha_2\alpha_3\alpha_4, \alpha_5\alpha_6\alpha_7, \cdots)$.

Lemma 4.1. *(*[CuPa, Corollary 2.8]*) (i) Let $k \geq 1$. Then W_α^m is k-hyponormal $\Leftrightarrow$ $W_{\alpha(m:i)}$ is k-hyponormal for $0 \leq i \leq m-1$.*
(ii) W_α^m is subnormal $\Leftrightarrow$ $W_{\alpha(m:i)}$ is subnormal for $0 \leq i \leq m-1$.

We now introduce a key family of examples. Given $0 < \kappa < 1$, we let $x \equiv \{x_n\}_{n=0}^{\infty}$ be given by

$$x_n := \begin{cases} \kappa\sqrt{\frac{3}{4}}, & \text{if } n = 0 \\ \frac{\sqrt{(n+1)(n+3)}}{(n+2)}, & \text{if } n \geq 1. \end{cases} \tag{4.2}$$

It is easy to see that $W_x \equiv$ shift $(x_0, x_1, x_2, \cdots)$ is subnormal, with Berger measure

$$d\xi_x(s) := (1-\kappa^2)d\delta_0(s) + \frac{\kappa^2}{2}ds + \frac{\kappa^2}{2}d\delta_1(s) \text{ ([CLY1, Proposition 4.2]).}$$

Consider now the 2-variable weighted shift given in Figure 2(i), where $W_x \equiv$ shift $(x_0, x_1, x_2, \cdots)$ and $y := \kappa$.

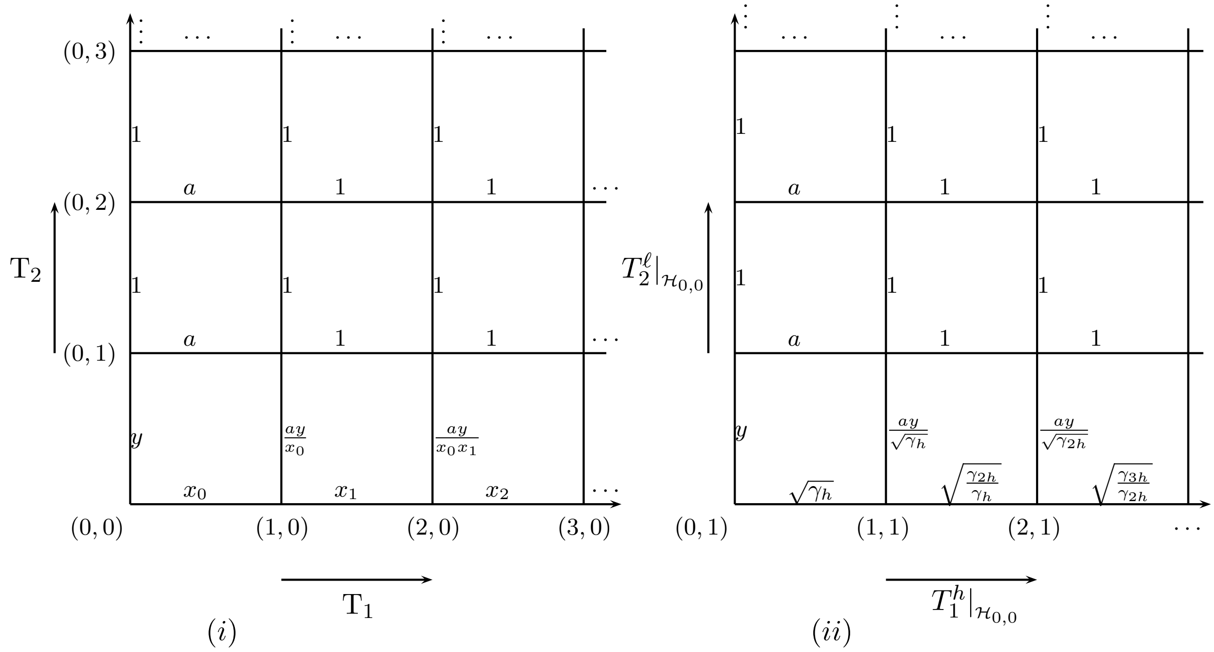

Figure 2. Weight diagram of the 2-variable weighted shifts in Theorem 4.2, 5.1, 5.4 and weight diagram of the 2-variable weighted shift $W^{(h,\ell)}_{(\alpha,\beta)}$, respectively.

Theorem 4.2. *([CLY1]) For $0 < a \le \frac{1}{\sqrt{2}}$, $0 < \kappa < 1$, x_n as in (4.2) and $y := \kappa$, let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be the 2-variable weighted shift given by Figure 2(i). For $k \ge 2$, let*

$$F(a,k) := \sqrt{\frac{\frac{(k+1)^2}{2k(k+2)} - a^2}{a^4 - \frac{5}{2}a^2 + \frac{(k+1)^2}{2k(k+2)} + \frac{2k^2+4k+3}{4(k+1)^2}}}. \tag{4.3}$$

Then
(i) T_1 and T_2 are subnormal;
(ii) $W_{(\alpha,\beta)} \in \mathfrak{H}_1 \Leftrightarrow 0 < \kappa \le \sqrt{\frac{32-48a^4}{59-72a^2}}$;
(iii) $W_{(\alpha,\beta)} \in \mathfrak{H}_k \Leftrightarrow 0 < \kappa \le F(a,k)$ $(k \ge 2)$;
(iv) $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty \Leftrightarrow 0 < \kappa \le \sqrt{\frac{1}{2-a^2}}$.
In particular, $W_{(\alpha,\beta)}$ is hyponormal and not subnormal if and only if $\sqrt{\frac{1}{2-a^2}} < \kappa \le \sqrt{\frac{32-48a^4}{59-72a^2}}$.

We now recall that, by (3.6),

$$b(k,h) = \left[b(k-1,h) + \frac{2kh+1}{(kh)^2}\right] \cdot \left(\frac{kh}{kh+1}\right)^2 \text{ and } b(1,h) = \frac{2h+1}{(h+1)^2}.$$

Using mathematical induction we can see that

$$b(k,1) = \frac{k(k+2)}{(k+1)^2}. \tag{4.4}$$

Remark 4.3. If $x = \frac{2(1-\kappa^2)}{\kappa^2}$ in Theorem 3.4, then for $h \geq 1$ and $k \geq 1$,

$$\det\ A_k(x,h) \geq 0 \Leftrightarrow x \geq b(k,h) \Leftrightarrow \kappa \leq \sqrt{\frac{2}{2+b(k,h)}}.$$

Lemma 4.4. *For $h \geq 1$ and $k \geq 1$, we have $b(k,h) \leq b(k,1)$.*

Proof. We fix $h \geq 1$ and use induction on k. First, observe that, on the interval $[1,+\infty)$, $b(1,h) \equiv \frac{2h+1}{(h+1)^2}$ is a decreasing function of h, so we clearly have $b(1,h) \leq b(1,1)$. For the induction step, assume now that $k \geq 2$ and that $b(k-1,h) \leq b(k-1,1)$. Then

$$\begin{aligned} b(k,h) &= \left[b(k-1,h) + \frac{2kh+1}{(kh)^2}\right] \cdot \left(\frac{kh}{kh+1}\right)^2 \\ &\leq \left[b(k-1,1) + \frac{2kh+1}{(kh)^2}\right] \cdot \left(\frac{kh}{kh+1}\right)^2 \\ &= \left[\frac{(k-1)(k+1)}{k^2} + \frac{2kh+1}{(kh)^2}\right] \cdot \left(\frac{kh}{kh+1}\right)^2 \quad \text{(by (4.4))} \\ &= \frac{(kh+1)^2 - h^2}{(kh+1)^2} \leq \frac{k(k+2)}{(k+1)^2}, \end{aligned}$$

since the next-to-the-last expression is a decreasing function of h on the interval $[1,+\infty)$. We therefore have $b(k,h) \leq \frac{k(k+2)}{(k+1)^2} \equiv b(k,1)$, as desired. □

Corollary 4.5. *For $h \geq 1$ and $k \geq 1$,*

$$F\left(\sqrt{\tfrac{1}{2}}, k\right) = \sqrt{\frac{2}{2+b(k,1)}} \leq \sqrt{\frac{2}{2+b(k,h)}}. \tag{4.5}$$

Proof. From Lemma 4.4 we know that $b(k,h) \leq b(k,1)$. Thus it suffices to establish in (4.5). A direct calculation using (4.3) shows that $F\left(\sqrt{\frac{1}{2}}, k\right)^2 = \frac{2(k+1)^2}{3k^2+6k+2}$, and from (4.4) we know that $b(k,1) = \frac{k(k+2)}{(k+1)^2}$. It follows that

$$\begin{aligned} \frac{2}{2+b(k,1)} - F\left(\sqrt{\tfrac{1}{2}}, k\right)^2 &= \frac{2}{2+\frac{k(k+2)}{(k+1)^2}} - \frac{2(k+1)^2}{3k^2+6k+2} \\ &= \frac{2(k+1)^2}{2(k+1)^2 + k(k+2)} - \frac{2(k+1)^2}{3k^2+6k+2} = 0, \end{aligned}$$

as desired. □

Remark 4.6. From Lemma 4.4 and Corollary 4.5 we see at once that for $h \geq 1$ and $k \geq 1$, the following statements are equivalent:
(i) $b(k,h) < b(k,1)$;
(ii) $F\left(\sqrt{\frac{1}{2}}, k\right) < \sqrt{\frac{2}{2+b(k,h)}}$.

Lemma 4.7. *Let $G(h) := \frac{2h^3+7h^2+8h+3}{2h^3+7h^2+10h+4}$. Then G is an increasing function of h on $[1,\infty)$, $G(1) = \frac{20}{23}$ and $\lim_{h\to\infty} G(h) = 1$.*

Proof. $\lim_{h\to\infty} G(h) = 1$ is clear. To establish that G is increasing, observe that

$$G'(h) = \frac{2\left(4h^3 + 10h^2 + 7h + 1\right)}{\left(2h^3 + 7h^2 + 10h + 4\right)^2} > 0$$

on $[1, \infty)$. $\square$

We are now ready to prove our main result of this section.

Theorem 4.8. *Let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be the 2-variable weighted shift whose weight diagram is given in Figure 2(i) (where $a = \sqrt{\frac{1}{2}}$ and W_x is as in (4.2)). Then given $k, \ell \geq 1$ and $h \geq 2$,*

$$W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_k \text{ but } W_{(\alpha,\beta)} \notin \mathfrak{H}_k \Leftrightarrow \begin{cases} \sqrt{\frac{20}{23}} < \kappa \leq \sqrt{\frac{63}{68}}, & \text{if } k = 1 \\ \sqrt{\frac{2(k+1)^2}{3k^2+6k+2}} < \kappa \leq \sqrt{\frac{2}{2+b(k,h)}}, & \text{if } k \geq 2. \end{cases}$$

Proof. From Lemma 7.2, we recall that a 2-variable weighted shift $W_{(\alpha,\beta)}$ is k-hyponormal if and only if

$$M_{\mathbf{k}}(k) = (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{0\leq n+m\leq k \\ 0\leq p+q\leq k}} \geq 0, \tag{4.6}$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$.

We first let $\mathcal{H}_{(m,n)} := \bigvee_{i,j=0}^{\infty}\{e_{(hi+m,\ell j+n)} : h, \ell \geq 1\}$, for $0 \leq m \leq h-1$ and $0 \leq n \leq \ell - 1$. Then we have $\ell^2(\mathbb{Z}_+^2) \equiv \bigoplus_{m=0}^{h-1}\bigoplus_{n=0}^{\ell-1}\mathcal{H}_{(m,n)}$. Observe that $\mathcal{H}_{(m,n)}$ reduces T_1^h and T_2^ℓ. Thus if a 2-variable weighted shift $W_{(\alpha,\beta)}$ is given as in Figure 2(i), then for $h, \ell \geq 1$, we can write

$$W_{(\alpha,\beta)}^{(h,\ell)} \equiv (T_1^h, T_2^\ell) \cong (W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_0}) \bigoplus \bigoplus_{i=1}^{h-1}(W_{\alpha(h:i)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_i}),$$

where

$$W_{\alpha(h:i)} = \text{shift}\ (\sqrt{\frac{\gamma_{(i+1)h}}{\gamma_{ih}}}, \sqrt{\frac{\gamma_{(i+2)h}}{\gamma_{(i+1)h}}}, \cdots) \text{ and } \mathcal{H}_i := \bigoplus_{n=0}^{\ell-1}\mathcal{H}_{(i,n)} \quad (0 \leq i \leq k-1).$$

Clearly, $\left\|W_{\alpha(h:i)}\right\| = 1$, and the Berger measure of $W_{\alpha(h:i)}$ has an atom at 1, so by Lemma 7.4 we see that $(W_{\alpha(h:i)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_i})$ $(1 \leq i \leq h-1)$ is subnormal. Thus, for $k \geq 1$, the k-hyponormality of $W_{(\alpha,\beta)}^{(h,\ell)}$ is equivalent to the k-hyponormality of $(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_0})$. Observe that

$$(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_0}) \cong \bigoplus_{n=0}^{\ell-1}(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2^\ell|_{\mathcal{H}_{(0,n)}})$$

and

$$\bigoplus_{n=0}^{\ell-1}(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2^\ell|_{\mathcal{H}_{(0,n)}}) \cong (W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2^\ell|_{\mathcal{H}_{(0,0)}}) \bigoplus \bigoplus_{n=0}^{\ell-1}(I \otimes S_{\sqrt{\frac{1}{2}}}, U_+ \otimes I).$$

Observe that the second summand is clearly subnormal; thus, for $h, \ell \geq 1$, the k-hyponormality of (T_1^h, T_2^ℓ) is equivalent to the k-hyponormality of the first summand, $(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2^\ell|_{\mathcal{H}_{(0,0)}})$. Observe also that

$$(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2^\ell|_{\mathcal{H}_{(0,0)}}) \cong (W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_{(0,0)}}).$$

Thus

$$W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_k \Leftrightarrow (W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_{(0,0)}}) \in \mathfrak{H}_k.$$

We consider two cases.

Case 1: $k = 1$. To check hyponormality, by Lemma 7.1 and Lemma 7.4 it suffices to apply the Six-point Test at $\mathbf{k} = (0,0)$. A direct calculation shows that

$$H_{(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_{(0,0)}})}(0,0) \geq 0 \Leftrightarrow \kappa \leq G(h) \equiv \sqrt{\tfrac{2h^3+7h^2+8h+3}{2h^3+7h^2+10h+4}}$$

(cf. Lemma 4.7). Therefore, for all $h, \ell \geq 1$, we have

$$W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_1 \Leftrightarrow \kappa \leq G(h).$$

Since $G(h)$ is an increasing function, we see that if $\sqrt{\frac{20}{23}} = G(1) < \kappa \leq G(2) = \sqrt{\frac{63}{68}}$, we simultaneously get $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_1$ and $W_{(\alpha,\beta)} \notin \mathfrak{H}_1$ (all $h \geq 2, \ell \geq 1$).

Case 2: $k \geq 2$. Note that

$$W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_k \Leftrightarrow (W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_{(0,0)}}) \in \mathfrak{H}_k.$$

To check the k-hyponormality of $(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_{(0,0)}})$, we observe that it suffices to apply Lemma 7.2(ii) at $\mathbf{k} = (0,0)$. Now, the moments associated with $(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_{(0,0)}})$ are

$$\gamma_{\mathbf{k}}(W_{(\alpha,\beta)}^{(h,\ell)}) = \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \gamma_{k_1 h}(W_{(\alpha,\beta)}), & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \kappa^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \frac{\kappa^2}{2}, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \tag{4.7}$$

By direct computation (i.e., interchanging rows and columns, discarding some redundant rows and columns, and multiplying by $\frac{2}{\kappa^2}$ in the moment matrix of $(W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_{(0,0)}})$), we see that for $0 < \kappa < 1$ and $h, \ell \geq 1$,

$$\begin{aligned} & W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_k \\ & \Leftrightarrow W_{(\alpha,\beta)}^{(h,\ell)}|_{\mathcal{H}_{(0,0)}} \in \mathfrak{H}_k \\ & \Leftrightarrow (W_{\alpha(h:0)} \oplus (I \otimes S_{\sqrt{\frac{1}{2}}}), T_2|_{\mathcal{H}_{(0,0)}}) \in \mathfrak{H}_k \\ & \Leftrightarrow J_k(\kappa, h) \geq 0 \\ & \Leftrightarrow L_k(\kappa, h) \geq 0, \end{aligned}$$

where

$J_k(\kappa, h) :=$

$$\begin{pmatrix} 1 & \frac{\kappa^2(h+2)}{2(h+1)} & \kappa^2 & \frac{\kappa^2(2h+2)}{2(2h+1)} & \frac{\kappa^2}{2} & \frac{\kappa^2(3h+2)}{2(3h+1)} & \cdots & \frac{\kappa^2(kh+2)}{2(kh+1)} \\ \frac{\kappa^2(h+2)}{2(h+1)} & \frac{\kappa^2(2h+2)}{2(2h+1)} & \frac{\kappa^2}{2} & \frac{\kappa^2(3h+2)}{2(3h+1)} & \frac{\kappa^2}{2} & \frac{\kappa^2(4h+2)}{2(4h+1)} & \cdots & \frac{\kappa^2((k+1)h+2)}{2((k+1)h+1)} \\ \kappa^2 & \frac{\kappa^2}{2} & \kappa^2 & \frac{\kappa^2}{2} & \frac{\kappa^2}{2} & \frac{\kappa^2}{2} & \cdots & \frac{\kappa^2}{2} \\ \frac{\kappa^2(2h+2)}{2(2h+1)} & \frac{\kappa^2(3h+2)}{2(3h+1)} & \frac{\kappa^2}{2} & \frac{\kappa^2(4h+2)}{2(4h+1)} & \frac{\kappa^2}{2} & \frac{\kappa^2(5h+2)}{2(5h+1)} & \cdots & \frac{\kappa^2((k+2)h+2)}{2((k+2)h+1)} \\ \frac{\kappa^2}{2} & \frac{\kappa^2}{2} & \frac{\kappa^2}{2} & \frac{\kappa^2}{2} & \frac{\kappa^2}{2} & \frac{\kappa^2}{2} & \cdots & \frac{\kappa^2}{2} \\ \frac{\kappa^2(3h+2)}{2(3h+1)} & \frac{\kappa^2(4h+2)}{2(4h+1)} & \frac{\kappa^2}{2} & \frac{\kappa^2(5h+2)}{2(5h+1)} & \frac{\kappa^2}{2} & \frac{\kappa^2(6h+2)}{2(6h+1)} & \cdots & \frac{\kappa^2((k+3)h+2)}{2((k+3)h+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{y^2(kh+2)}{2(kh+1)} & \frac{y^2((k+1)h+2)}{2((k+1)h+1)} & \frac{\kappa^2}{2} & \frac{y^2((k+2)h+2)}{2((k+2)h+1)} & \frac{\kappa^2}{2} & \frac{\kappa^2((k+3)h+2)}{2((k+3)h+1)} & \cdots & \frac{\kappa^2(2kh+2)}{2(2kh+1)} \end{pmatrix}_{(k+3)\times(k+3)}$$

and

$L_k(\kappa,h) :=$

$$\left(\begin{array}{cc} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 1 & \cdots & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} \frac{2}{\kappa^2} & \frac{1}{h+1}+1 & \cdots & \frac{1}{(k-1)h+1}+1 & \frac{1}{kh+1}+1 \\ \frac{1}{h+1}+1 & \frac{1}{2h+1}+1 & \cdots & \frac{1}{kh+1}+1 & \frac{1}{(k+1)h+1}+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(k-1)h+1}+1 & \frac{1}{kh+1}+1 & \cdots & \frac{1}{(2k-2)h+1}+1 & \frac{1}{(2k-1)h+1}+1 \\ \frac{1}{kh+1}+1 & \frac{1}{(k+1)h+1}+1 & \cdots & \frac{1}{(2k-1)h+1}+1 & \frac{1}{2kh+1}+1 \end{pmatrix} \end{array}\right)_{(k+3)\times(k+3)}.$$

Note that $\det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} > 0$, and let

$$M_k(\kappa,h) := \begin{pmatrix} \frac{2}{\kappa^2} & \frac{1}{h+1}+1 & \cdots & \frac{1}{(k-1)h+1}+1 & \frac{1}{kh+1}+1 \\ \frac{1}{h+1}+1 & \frac{1}{2h+1}+1 & \cdots & \frac{1}{kh+1}+1 & \frac{1}{(k+1)h+1}+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(k-1)h+1}+1 & \frac{1}{kh+1}+1 & \cdots & \frac{1}{(2k-2)h+1}+1 & \frac{1}{(2k-1)h+1}+1 \\ \frac{1}{kh+1}+1 & \frac{1}{(k+1)h+1}+1 & \cdots & \frac{1}{(2k-1)h+1}+1 & \frac{1}{2kh+1}+1 \end{pmatrix}_{(k+1)\times(k+1)}.$$

Then we have

$$M_k(\kappa,h) - \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 1 & \cdots & 1 & 1 \end{pmatrix}^*_{(k+1)\times(k+1)} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 1 & \cdots & 1 & 1 \end{pmatrix}_{(k+1)\times(k+1)} = A_k(x,h).$$

where $x := \frac{2(1-\kappa^2)}{\kappa^2}$ and $A_k(x,h)$ is as in Theorem 3.1. Thus, after we apply Smul'jan Lemma (Lemma 7.6) to $L_k(\kappa,h)$, we show that for $0<\kappa<1$ and $h,\ell \geq 1$, $L_k(\kappa,h) \geq 0 \Leftrightarrow A_k(x,h) \geq 0$. Therefore,

$$W^{(h,\ell)}_{(\alpha,\beta)} \in \mathfrak{H}_k \Leftrightarrow W^{(h,\ell)}_{(\alpha,\beta)}|_{\mathcal{H}_{(0,0)}} \in \mathfrak{H}_k \Leftrightarrow L_k(\kappa,h) \geq 0 \Leftrightarrow A_k(x,h) \geq 0. \tag{4.8}$$

From Remark 4.6(ii), for $k,h \geq 2$, we recall that

$$b(k,h) < b(k,1) \Leftrightarrow F\left(\sqrt{\tfrac{1}{2}},k\right) = \sqrt{\tfrac{2(k+1)^2}{3k^2+6k+2}} < \sqrt{\tfrac{2}{2+b(k,2)}}. \tag{4.9}$$

By Theorem 4.2(iii),

$$W_{(\alpha,\beta)} \in \mathfrak{H}_k \Leftrightarrow 0<\kappa \leq F\left(\sqrt{\tfrac{1}{2}},k\right) \ (k \geq 2). \tag{4.10}$$

Now, Remark 4.3(i) and Theorem 3.4 imply that for $h \geq 1$ and $k \geq 2$,

$$\det\ A_k(x,h) \geq 0 \Leftrightarrow A_k(x,h) \geq 0 \Leftrightarrow x \geq b(k,h) \Leftrightarrow \kappa \leq \sqrt{\frac{2}{2+b(k,h)}}. \tag{4.11}$$

Thus, by (4.8) and (4.11), we have that for $h,\ell \geq 1$,

$$W^{(h,\ell)}_{(\alpha,\beta)} \in \mathfrak{H}_k \Leftrightarrow 0<\kappa \leq \sqrt{\frac{2}{2+b(k,h)}} \ (k \geq 2). \tag{4.12}$$

Therefore, by (4.8), (4.9), (4.10), (4.11) and (4.12), for $\ell \geq 1$ and $h, k \geq 2$, we have

$$W^{(h,\ell)}_{(\alpha,\beta)} \in \mathfrak{H}_k \text{ but } W_{(\alpha,\beta)} \notin \mathfrak{H}_k \text{ if and only if } F\left(\sqrt{\tfrac{1}{2}}, k\right) < \kappa \leq \sqrt{\tfrac{2}{2+b(k,h)}}, \tag{4.13}$$

as desired. □

Remark 4.9. (i) We know that for $k, h \geq 1$, $b(k,h) \leq b(k,1) = \frac{k(k+2)}{(k+1)^2}$, so that $\limsup_k b(k,h) \leq 1$. As an application of (4.12) we can establish that $\lim_k\ b(k,h)$ exists. Recall that $W_{(\alpha,\beta)} \in \mathfrak{H}_{k+1} \Rightarrow W_{(\alpha,\beta)} \in \mathfrak{H}_k$, so that from (4.12) we see that for each fixed $h \geq 1$, $b(k,h)$ must be a nondecreasing function of k, and therefore $b(h) := \lim_k\ b(k,h) = \limsup_k b(k,h) \leq 1$.
(ii) We believe it is nontrivial to show that for $h \geq 1$, $\lim_{k\to\infty}\ b(k,h) = 1$. We now provide an operator-theoretic proof of this fact. By (4.10) and (4.12), for $k \geq 2$, we have $W^{(h,\ell)}_{(\alpha,\beta)} \in \mathfrak{H}_k \Leftrightarrow 0 < \kappa \leq \sqrt{\frac{2}{2+b(k,h)}}$. Since $b(k,h)$ is a nondecreasing function of k, and $\lim_{k\to\infty} b(k,h) = b(h) \leq 1$, we easily see that

$$W^{(h,\ell)}_{(\alpha,\beta)} \in \mathfrak{H}_\infty \iff 0 < \kappa \leq \sqrt{\frac{2}{2+b(h)}}. \tag{4.14}$$

We now let $\mathcal{M}_1(0,0)$ denote the subspace of $\mathcal{H}_{(0,0)}$ spanned by canonical orthonormal basis vectors with indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geq 0$ and $k_2 \geq 1$. We have $W^{(h,\ell)}_{(\alpha,\beta)}|_{\mathcal{M}_1(0,0)} \cong (I \otimes S_a, U_+ \otimes I) \in \mathfrak{H}_\infty$ with Berger measure $\mu_{\mathcal{M}_1(0,0)} := [(1-a^2)\delta_0 + a^2\delta_1] \times \delta_1$. Thus, by Lemma 7.3 and a direct calculation, we see that

$$W^{(h,\ell)}_{(\alpha,\beta)} \in \mathfrak{H}_\infty \Leftrightarrow W^{(h,\ell)}_{(\alpha,\beta)}|_{\mathcal{H}(0,0)} \in \mathfrak{H}_\infty \Leftrightarrow \kappa^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}_1(0,0)})} (\mu_{\mathcal{M}_1(0,0)})^X_{ext} \leq \xi_x \Leftrightarrow 0 < \kappa \leq \sqrt{\tfrac{2}{3}},$$

that is, for $h, \ell \geq 1$,

$$W^{(h,\ell)}_{(\alpha,\beta)} \in \mathfrak{H}_\infty \Leftrightarrow 0 < \kappa \leq \sqrt{\tfrac{2}{3}}. \tag{4.15}$$

From (4.14) and (4.15) we see at once that $b(h) = 1$, as desired.

Example 4.10. As specific instances of Theorem 4.8, we have
(i)

$$\begin{aligned}(W^{(9,1)}_{(\alpha,\beta)} &\in \mathfrak{H}_2 \text{ and } W_{(\alpha,\beta)} \notin \mathfrak{H}_1) \\ &\Leftrightarrow 0.932505 \simeq \sqrt{\tfrac{20}{23}} < \kappa \leq \sqrt{\frac{2}{2+b(2,9)}} = \sqrt{\tfrac{9025}{10257}} \simeq 0.938023;\end{aligned}$$

(ii) for $h, \ell \geq 1$,

$$\begin{aligned}W^{(h,\ell)}_{(\alpha,\beta)} &\in \mathfrak{H}_1 \text{ and } W_{(\alpha,\beta)} \notin \mathfrak{H}_\infty \\ &\Leftrightarrow \sqrt{\tfrac{2}{3}} < \kappa \leq \sqrt{G(h)} = \sqrt{\frac{2h^3+7h^2+8h+3}{2h^3+7h^2+10h+4}};\end{aligned}$$

(iii) for $h, \ell \geq 1$,

$$\begin{aligned}W^{(h,\ell)}_{(\alpha,\beta)} &\in \mathfrak{H}_2 \text{ and } W_{(\alpha,\beta)} \notin \mathfrak{H}_\infty \\ &\Leftrightarrow \sqrt{\tfrac{2}{3}} < \kappa \leq \sqrt{\frac{2}{2+b(2,h)}} = \sqrt{\frac{8h^4+24h^3+26h^2+12h+2}{8h^4+36h^3+39h^2+18h+3}}.\end{aligned}$$

5. Hyponormal Invariance Under Powers in the Class $\mathcal{A}$

In this section we study a large class $\mathcal{C}$ of nontrivial pairs of commuting subnormals such that $W_{(\alpha,\beta)} \in \mathfrak{H}_1 \bigcap \mathcal{C} \Rightarrow W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_1$ (all $h, \ell \geq 1$). The class $\mathcal{C}$ is a subclass of the class $\mathcal{A}$, and it consists of 2-variable weighted shifts whose weight diagrams are given in Figure 2(i). Motivated by the necessary condition for LPCS found in [CuYo2] (see Lemma 7.5), we observe that the Berger measure ξ_x of the unilateral weighted shift $W_x \equiv \text{shift}(\alpha_{00}, \alpha_{10}, \cdots)$ admits a unique decomposition as

$$\xi_x \equiv p\delta_0 + q\delta_1 + (1-p-q)\rho,$$

where $0 < p, q < 1$, $p+q \leq 1$, and ρ a probability measure with $\rho(\{0,1\}) = 0$. As a result, a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{C}$ can be parameterized as $W_{(\alpha,\beta)} \equiv \langle p, q, \rho, y, a \rangle$, with $0 < a, y \leq 1$. In Theorem 5.1 below we characterize the shifts $W_{(\alpha,\beta)}$ which remain hyponormal, 2-hyponormal or subnormal under the action $(h,\ell) \mapsto W_{(\alpha,\beta)}^{(h,\ell)}$ $(h, \ell \geq 1)$.

Theorem 5.1. *Let $W_{(\alpha,\beta)} \equiv \langle p, q, \rho, y, a \rangle \in \mathcal{C}$ be the 2-variable weighted shift whose weight diagram is given in Figure 2(i). The following assertions hold.*

(i) Assume that $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_1$ (all $h, \ell \geq 1$). Then $0 < y \leq m_1(a,q) := \sqrt{\frac{q(1-q)}{(a^2-q)^2+q(1-q)}}$.

(ii) Assume that $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_2$ (all $h, \ell \geq 1$). Then $y \leq m_2(a,q) := \min\left\{\sqrt{\frac{1-q}{1-a^2}}, \sqrt{\frac{q}{a^2}}\right\}$.

(iii) $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty \Longleftrightarrow y \leq m_\infty(a,p,q) := \min\left\{\sqrt{\frac{p}{1-a^2}}, \sqrt{\frac{q}{a^2}}\right\}$.

We need an auxiliary lemma, of independent interest.

Lemma 5.2. *Let W_x be a subnormal unilateral weighted shift, with Berger measure $\xi_x \equiv p\delta_0 + q\delta_1 + [1-(p+q)]\rho$, and recall that γ_n is the n-th moment of ξ_x, that is, $\gamma_n = \int s^n d\xi_x(s)$ $(n \geq 0)$. Then $\lim_{n\to\infty} \gamma_n = q$.*

Proof. For $n \geq 0$, let $f_n(s) := s^n$ $(0 \leq s \leq 1)$. Consider the sequence of nonnegative functions $\{f_n\}_{n\geq 0}$. Clearly $f_n \searrow \chi_{\{1\}}$ pointwise, and $|f_n| \leq 1$ (all $n \geq 0$). By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int f_n(s) d\rho(s) = \int \chi_{\{1\}} d\rho(s) = \rho(\{1\}) = 0$$

(recall that $\rho(\{0\} \cup \{1\}) = 0$). Thus

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \int s^n d\xi_x(s) = \lim_{n\to\infty} \int f_n(s) d\xi_x(s) = q + [1-(p+q)] \cdot \lim_{n\to\infty} \int f_n(s) d\rho(s) = q,$$

as desired. $\square$

Proof of Theorem 5.1. For fixed $h, \ell \geq 1$, $0 \leq m \leq h-1$ and $0 \leq n \leq \ell - 1$, we recall that $\mathcal{H}_{(m,n)} = \bigvee_{i,j=0}^{\infty} \{e_{(hi+m,\ell j+n)} : h, \ell \geq 1\}$ and $\ell^2(\mathbb{Z}_+^2) \equiv \bigoplus_{m=0}^{h-1} \bigoplus_{n=0}^{\ell-1} \mathcal{H}_{(m,n)}$. For $W_{(\alpha,\beta)}^{(h,\ell)}|_{\mathcal{H}_{(0,0)}}$, we refer to the weight diagram in Figure 2(ii). In the decomposition $\xi_x \equiv p\delta_0 + q\delta_1 + [1-(p+q)]\rho$, we may assume, without loss of generality, that $q < 1$; for, the condition $q = 1$ and hyponormality immediately imply the subnormality of $W_{(\alpha,\beta)}$.

Given $h, \ell \geq 1$, we consider the moments associated with $W_{(\alpha,\beta)}^{(h,\ell)}$ of order $\mathbf{k}$, that is,

$$\gamma_{\mathbf{k}}(W_{(\alpha,\beta)}^{(h,\ell)}) = \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \gamma_{k_1 h}(W_{(\alpha,\beta)}), & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ y^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ a^2 y^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \tag{5.1}$$

(i) From Lemma 7.4, we observe that for $h, \ell \geq 1$, $W_{(\alpha,\beta)}^{(h,\ell)}|_{\mathcal{M}_1} \cong (I \otimes S_a, U_+ \otimes I) \in \mathfrak{H}_\infty$ and $W_{(\alpha,\beta)}^{(h,\ell)}|_{\mathcal{N}_1} \in \mathfrak{H}_\infty$. Thus, by Lemma 7.2, to verify the hyponormality of $W_{(\alpha,\beta)}^{(h,\ell)}$, it suffices to apply the Six-point Test (Lemma 7.1) to $W_{(\alpha,\beta)}^{(h,\ell)}$ at $\mathbf{k} = (0,0)$. We then have

$$\begin{aligned} M_{(0,0)}(1)(W_{(\alpha,\beta)}^{(h,\ell)}) &= \begin{pmatrix} 1 & \gamma_h(W_{(\alpha,\beta)}) & y^2 \\ \gamma_h(W_{(\alpha,\beta)}) & \gamma_{2h}(W_{(\alpha,\beta)}) & a^2y^2 \\ y^2 & a^2y^2 & y^2 \end{pmatrix} \geq 0 \ \text{(all } h \geq 1) \\ &\Rightarrow \quad H := \begin{pmatrix} 1 & q & y^2 \\ q & q & a^2y^2 \\ y^2 & a^2y^2 & y^2 \end{pmatrix} \geq 0 \ \text{(by Lemma 5.2)} \\ &\iff \quad 0 < y \leq m_1(a,q) := \sqrt{\frac{q(1-q)}{(a^2-q)^2 + q(1-q)}}, \end{aligned}$$

as desired. Observe that the function m_1 satisfies the following properties:
(i$_1$) $0 < m_1(a,q) \leq 1$ on the square $(0,1] \times (0,1)$;
(i$_2$) $\lim_{q \to 0^+} m_1(a,q) = 0 = \lim_{q \to 1^+} m_1(a,q) = 0$ (for all a);
(i$_3$) $\lim_{a \to 0^+} m_1(a,q) = \sqrt{1-q}$ (for all q);
(i$_4$) $m_1(1,q) = \sqrt{q}$ (for all q); and
(i$_5$) $m_1(a,q) = 1 \iff q = a^2$.
Thus near the edges of the square the hyponormality of $W_{(\alpha,\beta)}^{(h,\ell)}$ for all h and ℓ forces y to be small, while along the parabola $q = a^2$ the values of y can reach 1.

(ii) From Lemmas 7.4 and 7.2, and the fact that $W_{(\alpha,\beta)}^{(h,\ell)}|_{\mathcal{M}_1}$, $W_{(\alpha,\beta)}^{(h,\ell)}|_{\mathcal{N}_1} \in \mathfrak{H}_\infty$, to verify the 2-hyponormality of $W_{(\alpha,\beta)}^{(h,\ell)}$ $(h, \ell \geq 1)$ it suffices to apply the 15-point Test to $W_{(\alpha,\beta)}^{(h,\ell)}$ at $\mathbf{k} = (0,0)$. By direct computation (i.e., interchanging rows and columns, and discarding some redundant rows and columns), it is straightforward to observe that the positivity of the 10×10 matrix $M_{(0,0)}(2)(W_{(\alpha,\beta)}^{(h,\ell)})$ is determined by that of the following 5×5 matrix:

$$P(h) := \begin{pmatrix} \begin{pmatrix} 1 & \gamma_h(W_{(\alpha,\beta)}) \\ \gamma_h(W_{(\alpha,\beta)}) & \gamma_{2h}(W_{(\alpha,\beta)}) \end{pmatrix} & \begin{pmatrix} y^2 & \gamma_{2h}(W_{(\alpha,\beta)}) & a^2y^2 \\ a^2y^2 & \gamma_{3h}(W_{(\alpha,\beta)}) & a^2y^2 \end{pmatrix} \\ \begin{pmatrix} y^2 & a^2y^2 \\ \gamma_{2h}(W_{(\alpha,\beta)}) & \gamma_{3h}(W_{(\alpha,\beta)}) \\ a^2y^2 & a^2y^2 \end{pmatrix} & \begin{pmatrix} y^2 & a^2y^2 & a^2y^2 \\ a^2y^2 & \gamma_{4h}(W_{(\alpha,\beta)}) & a^2y^2 \\ a^2y^2 & a^2y^2 & a^2y^2 \end{pmatrix} \end{pmatrix}.$$

Thus the assumption $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_2$ (all $h, \ell \geq 1$) readily implies that

$$P \equiv P(\infty) := \begin{pmatrix} \begin{pmatrix} 1 & q \\ q & q \end{pmatrix} & \begin{pmatrix} y^2 & q & a^2y^2 \\ a^2y^2 & q & a^2y^2 \end{pmatrix} \\ \begin{pmatrix} y^2 & a^2y^2 \\ q & q \\ a^2y^2 & a^2y^2 \end{pmatrix} & \begin{pmatrix} y^2 & a^2y^2 & a^2y^2 \\ a^2y^2 & q & a^2y^2 \\ a^2y^2 & a^2y^2 & a^2y^2 \end{pmatrix} \end{pmatrix} \geq 0 \text{ (using Lemma 5.2).}$$

Since $\begin{pmatrix} 1 & q \\ q & q \end{pmatrix}$ is positive and invertible, we can apply Smul'jan Lemma (Lemma 7.6) to P:

$$\begin{aligned} P &\geq 0 \Longleftrightarrow \begin{pmatrix} \frac{y^2\left(q\left(1-(1-2a^2)y^2\right)-q^2-a^4y^2\right)}{(1-q)q} & 0 & a^2y^2 - \frac{a^4y^4}{q} \\ 0 & 0 & 0 \\ a^2y^2 - \frac{a^4y^4}{q} & 0 & a^2y^2 - \frac{a^4y^4}{q} \end{pmatrix} \geq 0 \\ &\Longleftrightarrow \left\{ \begin{array}{lcl} (1-a^2)y^2 & \leq & 1-q \\ a^2y^2 & \leq & q \end{array} \right. \Leftrightarrow y \leq \min\left\{ \sqrt{\frac{1-q}{1-a^2}}, \sqrt{\frac{q}{a^2}} \right\}. \end{aligned}$$

Therefore, $W_{(\alpha,\beta)}^{(h,\ell)} \in \mathfrak{H}_2$ (all h, ℓ) $\Rightarrow y \leq \min\left\{\sqrt{\frac{1-q}{1-a^2}}, \sqrt{\frac{q}{a^2}}\right\}$, as desired. (The reader will notice that $\min\left\{\sqrt{\frac{1-q}{1-a^2}}, \sqrt{\frac{q}{a^2}}\right\} \leq 1$; for, if $q > a^2$ then $1-q < 1-a^2$.)
(iii) From Figure 2(i), we observe that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with Berger measure

$$\mu_{\mathcal{M}} = \left((1-a^2)\delta_0 + a^2\delta_1\right) \times \delta_1.$$

Note that $\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$ and $(\mu_{\mathcal{M}})_{ext}^X = (1-a^2)\delta_0 + a^2\delta_1$. We now apply Lemma 7.3 to the 2-variable weighted shift $W_{(\alpha,\beta)}$ and to the subspace $\mathcal{M}$. It follows that the necessary and sufficient condition for $W_{(\alpha,\beta)}$ to be subnormal is

$$y^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \xi_x = p\delta_0 + q\delta_1 + [1-(p+q)]\rho,$$

or equivalently,

$$\left\{ \begin{array}{lcl} (1-a^2)y^2 & \leq & p \\ a^2y^2 & \leq & q. \end{array} \right.$$

Thus we have the desired result. The proof of the theorem is now complete. □

Remark 5.3. (i) By a direct calculation, we note that

$$a^2 \leq q \Leftrightarrow \tfrac{1-q}{1-a^2} \leq \tfrac{q(1-q)}{(a^2-q)^2+q(1-q)} \text{ and } a^2 > q \Leftrightarrow \tfrac{q}{a^2} < \tfrac{q(1-q)}{(a^2-q)^2+q(1-q)}.$$

Thus it is always true that $\min\left\{\sqrt{\frac{q}{a^2}}, \sqrt{\frac{1-q}{1-a^2}}\right\} \leq \sqrt{\frac{q(1-q)}{(a^2-q)^2+q(1-q)}}$.
(ii) For $h \geq 2$ and $k, \ell \geq 1$, the necessary conditions in (i) and (ii) in Theorem 5.1 are not sufficient for power invariance. To show this, we let $W_{(\alpha,\beta)}$ denote the 2-variable weighted shift whose weight diagram is given in Figure 2(i), with $a = \sqrt{\frac{1}{2}}$ and W_x is as in (4.2). Furthermore, for given small $\varepsilon > 0$ and $h \geq 1$, we let

$$\kappa := \left\{ \begin{array}{ll} \sqrt{\frac{20}{23}}, & \text{if } k = 1 \\ \sqrt{\frac{2}{2+b(k,h)}} + \varepsilon(h), & \text{if } k \geq 2 \end{array} \right\} \tag{5.2}$$

provided that $\sqrt{\frac{2}{2+b(k,h)}} + \varepsilon(h) < 1$. (Since, for $k, h \geq 1$, $b(k,h) > 0$, $\lim_{k\to\infty} b(k,h) = 1$ and $\lim_{h\to\infty} b(k,h) = 0$, it is possible to choose κ given in (5.2)). By Theorem 4.8, we note that for $h \geq 2$, $k = 1, 2$ and $\ell \geq 1$, $W_{(\alpha,\beta)}^{(h,\ell)} \notin \mathfrak{H}_k$ $(k = 1, 2)$. Observe that

$$\left\{ \begin{array}{ll} 0 < \kappa \leq \sqrt{\frac{q(1-q)}{a^4+q-2a^2q}} \Leftrightarrow \kappa \leq 1, & \text{if } k = 1 \\ \left\{a^2\kappa^2 \leq q \leq \left(1-\kappa^2\right) + a^2\kappa^2\right\} \Leftrightarrow \kappa \leq 1, & \text{if } k \geq 2. \end{array} \right. \tag{5.3}$$

If we choose κ given in (5.2), then (5.3) is always true. Thus the 2-variable weighted shift $W_{(\alpha,\beta)}$ given in Figure 2(i) satisfies the necessary conditions in Theorem 5.1, but for $h \geq 2$ and $\ell \geq 1$, $W_{(\alpha,\beta)}^{(h,\ell)} \notin \mathfrak{H}_1$.

Throughout this section, we have focused on the question of hyponormality for shifts in the class $\mathcal{A}$. We now turn our attention to 2-hyponormality, in the hope of detecting to what extent one can expect invariance under powers in this class. Along the way we will discover that there is a large subclass, $\mathcal{S}_1$, for which things work extremely well.

As we saw in Section 4, for a general operator T on Hilbert space and for all $m \geq 2$, we know that $T^m \in \mathfrak{H}_k$ $(k \geq 2)$ need not imply $T \in \mathfrak{H}_k$. But it is still unknown whether the k-hyponormality of T $(k \geq 2)$ implies the k-hyponormality of T^m $(m \geq 2)$, even when T is a weighted shift (see Problem 1.1). We now show that there exists a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{A}$ for which the 2-hyponormality of $W_{(\alpha,\beta)}$ does not imply the 2-hyponormality of $W_{(\alpha,\beta)}^{(2,1)}$.

The motivation behind the construction in Theorem 5.4 comes from Figure 1. Indeed, inspection of the values of a that illustrate the gap between the hyponormality of $W_{(\alpha,\beta)}$ and that of its powers suggests that something similar may work for k-hyponormality. We saw in Theorem 4.8 that a value smaller than a_{int} (namely, $a = \sqrt{\frac{1}{2}}$) did the job in separating the k-yponormality of the powers from the k-hyponormality of the pair. For the converse, it is then natural to consider a value of a bigger than a_{int}. This is the approach in Theorem 4.8), with $a = \frac{17}{20}$; for, recall that $a_{int} \approx 0.8386$ [CLY2, paragraph preceding Theorem 2.7], so our choice of a satisfies $a > a_{int}$.

Theorem 5.4. *Let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be the 2-variable weighted shift whose weight diagram is given in Figure 2(i) (where $a = \frac{17}{20}$ and W_x is as in (4.2)). Then*
(i) $W_{(\alpha,\beta)} \in \mathfrak{H}_0$*;*
(ii) $W_{(\alpha,\beta)} \in \mathfrak{H}_2 \Leftrightarrow 0 < \kappa \leq F(\frac{17}{20}, 2) = \sqrt{\frac{230400}{279311}} \simeq 0.908$*;*
(iii) $W_{(\alpha,\beta)}^{(2,1)} \notin \mathfrak{H}_2$.

Proof. (i) This follows easily once we know that W_x is subnormal.
(ii) It is clear from Theorem 4.2.
(iii) Recall that for $n = 0, 1$, $\mathcal{H}_n \equiv \bigvee_{i=0}^{\infty}\{e_{(2i+n,j)} : j = 0, 1, 2, \cdots\}$ and $\ell^2(\mathbb{Z}_+^2) \equiv \mathcal{H}_0 \bigoplus \mathcal{H}_1$. Note that

$$W_{(\alpha,\beta)}^{(2,1)} \equiv (T_1^2, T_2) \cong W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_0} \bigoplus W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_1}.$$

By Lemma 7.4, we observe that $W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_1} \cong (W_{\alpha(2:1)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_1})$ is subnormal, because $W_{\alpha(2:1)} = \text{shift }(x_1x_2, x_3x_4, \cdots)$ has an atom at $\{1\}$. Hence $W_{(\alpha,\beta)}^{(2,1)} \in \mathfrak{H}_2$ if and only if $W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_0} := (T_1^2, T_2)|_{\mathcal{H}_0} \in \mathfrak{H}_2$. Let $\mathcal{M}_1(0)$ (resp. $\mathcal{N}_1(0)$) be the subspace of $\mathcal{H}_0$ spanned by canonical orthonormal basis vectors associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geq 0$ and $k_2 \geq 1$ (resp. $k_1 \geq 1$ and $k_2 \geq 0$). By Lemma 7.4, we note that $W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_0}$ on $\mathcal{M}_1(0)$ (resp. $\mathcal{N}_1(0)$) is subnormal, because shift $(x_2x_3, x_4x_5, \cdots)$ (resp. shift $(y, 1, 1 \cdots)$) has an atom at $\{1\}$. Thus, by Lemma 7.2, to verify the 2-hyponormality of $W_{(\alpha,\beta)}^{(2,1)}$ it suffices to apply the 15-point Test to $W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_0}$ at $\mathbf{k} = (0, 0)$. Note that the moments associated with $(T_1^2, T_2)|_{\mathcal{H}_0}$ of order $\mathbf{k}$ are

$$\gamma_{\mathbf{k}}(W^{(2,1)}_{(\alpha,\beta)}|_{\mathcal{H}_0}) = \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \prod_{i=1}^{k_1} x^2_{2(i-1)} x^2_{2i-1}, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \kappa^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \frac{289}{400}\kappa^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{cases}.$$

Since the third and sixth rows of $M_{(0,0)}(2)|_{\mathcal{H}_0}$ are identical, if we multiply $\frac{1}{y^2}$ and then apply row and column operations to $M_{(0,0)}(2)|_{\mathcal{H}_0}$, then we have $M_{(0,0)}(2)|_{\mathcal{H}_0} \geq 0 \Leftrightarrow \tilde{M}_{(0,0)} \geq 0$, where

$$\tilde{M}_{(0,0)} := \left(\begin{array}{cc} \begin{pmatrix} \frac{3}{5} & \frac{4}{7} & a^2 \\ \frac{4}{7} & \frac{5}{9} & a^2 \\ a^2 & a^2 & 1 \end{pmatrix} & \begin{pmatrix} \frac{2}{3} & a^2 \\ \frac{3}{5} & a^2 \\ 1 & a^2 \end{pmatrix} \\ \begin{pmatrix} \frac{2}{3} & \frac{3}{5} & 1 \\ a^2 & a^2 & a^2 \end{pmatrix} & \begin{pmatrix} \frac{1}{y^2} & a^2 \\ a^2 & a^2 \end{pmatrix} \end{array} \right) \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

We now apply Lemma 7.6 to $\tilde{M}_{(0,0)}$. Since A is invertible and $\det A = \frac{15353}{88200000} > 0$ (where $a = \frac{17}{20}$), we have

$$\tilde{M}_{(0,0)} \geq 0 \Longleftrightarrow Q := C - W^* A W \geq 0,$$

where

$$W := \begin{pmatrix} \frac{340403}{14172} & -\frac{1122765}{61412} \\ -\frac{184401}{4724} & \frac{2020977}{61412} \\ \frac{41980}{3543} & -\frac{151147}{15353} \end{pmatrix} \text{ and } B = AW.$$

A direct calculation shows that

$$Q = \begin{pmatrix} \frac{212580-944003y^2}{212580y^2} & \frac{1422169}{472400} \\ \frac{1422169}{472400} & -\frac{16777317}{6141200} \end{pmatrix} =: \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}.$$

From the argument above, we note that

$$W^{(2,1)}_{(\alpha,\beta)} \in \mathfrak{H}_2 \Longleftrightarrow W^{(2,1)}_{(\alpha,\beta)}|_{\mathcal{H}_0} \in \mathfrak{H}_2 \Longleftrightarrow \tilde{M}_{(0,0)} \geq 0 \Longleftrightarrow Q \geq 0.$$

Since $q_{22} < 0$, Q is not positive semidefinite, so for $0 < y < 1$, we have $W^{(2,1)}_{(\alpha,\beta)} \notin \mathfrak{H}_2$, as desired. □

Corollary 5.5. *(i) By Theorem 5.4, we note that* $W_{(\alpha,\beta)} \in \mathfrak{H}_2$ *but* $W^{(2,1)}_{(\alpha,\beta)} \notin \mathfrak{H}_2$ *if*

$$0 < \kappa \leq \sqrt{\frac{230400}{279311}} \simeq 0.908.$$

(ii) By Theorem 4.2 we see that

$$W_{(\alpha,\beta)} \in \mathfrak{H}_1 \Leftrightarrow 0 < \kappa \leq \sqrt{\frac{69437}{69800}} \simeq 0.997.$$

and

$$W_{(\alpha,\beta)} \in \mathfrak{H}_\infty \Leftrightarrow 0 < \kappa \leq \frac{20}{\sqrt{511}} \simeq 0.885.$$

Remark 5.6. Looking at Theorem 5.4, if seems natural to conjecture that a similar result should work for k-hyponormality ($k > 2$). That is, perhaps we have $W_{(\alpha,\beta)} \in \mathfrak{H}_k \Leftrightarrow 0 < \kappa \le F(\frac{17}{20}, k)$, but $W^{(2,1)}_{(\alpha,\beta)} \notin \mathfrak{H}_k$ for $k \ge 1$, especially based on the matrix $J_k(\kappa, h)$ in Theorem 4.8. However, it is highly nontrivial to establish that $W^{(2,1)}_{(\alpha,\beta)} \notin \mathfrak{H}_k$ for $k \ge 1$. For, in Theorem 4.8 the choice of $a = \sqrt{\frac{1}{2}}$ leads to the generalized Hilbert matrix $J_k(\kappa, h)$. However, if we choose $a = \frac{17}{20}$ then the moment matrix is not a generalized Hilbert matrix, and it becomes unwieldy to check its positivity.

6. The Class $\mathcal{S}_1$ Is Invariant Under All Powers

In Section 5 we dealt with 2-variable weighted shifts of the form $W_{(\alpha,\beta)} \equiv \langle p, q, \rho, y, a \rangle$ and established some results about hyponormality, 2-hyponormality and subnormality. We now restrict attention to the case $\rho = 0$, and assume that $W_{(\alpha,\beta)} \in \mathfrak{H}_1$; that is, $W_{(\alpha,\beta)} \in \mathcal{S}_1$. Under this assumption, we will now sharpen the hyponormality results. Recall that, without loss of generality, every 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{S}_1$ is completely determined by the three parameters $x := \alpha_{(0,0)}$, $y := \beta_{(0,0)}$ and $a := \alpha_{(0,1)}$; cf. Figure 3(i). As before, we shall denote such a shift by $\langle x, y, a \rangle$; of course, we always assume $0 < x, y, a \le 1$, and moreover $ay \le x$ (since we need to ensure that shift $(\beta_{10}, \beta_{11}, \beta_{12}, \cdots) \equiv$ shift $(\frac{ay}{x}, 1, 1, \cdots)$ is subnormal).

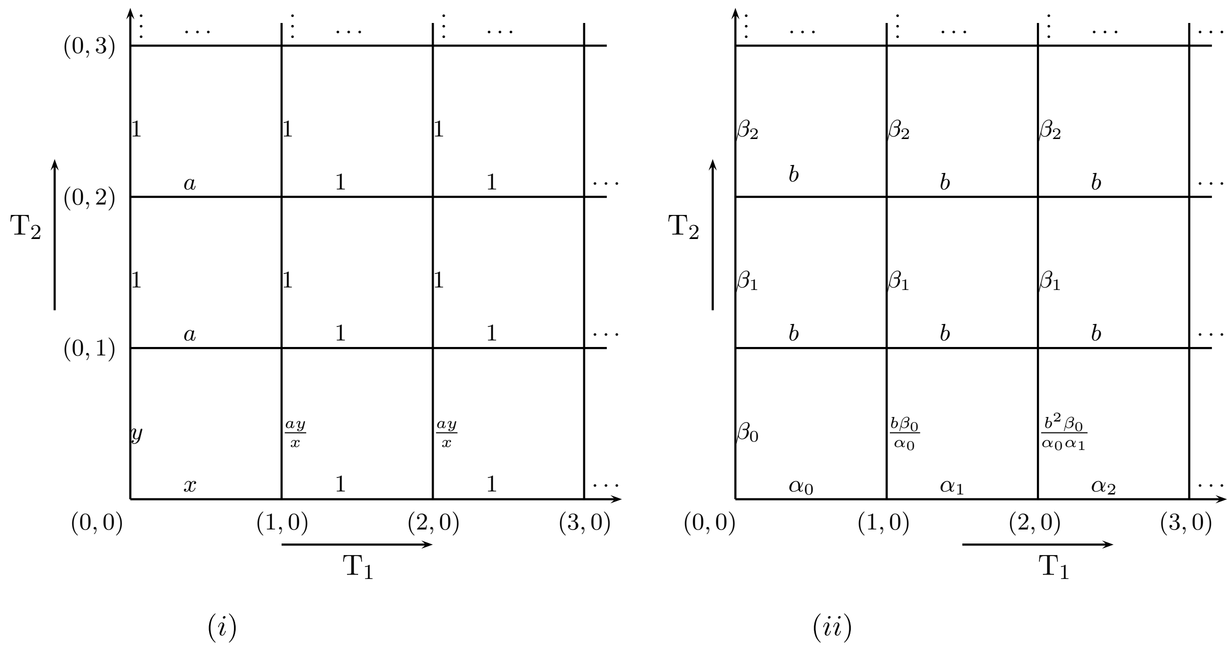

Figure 3. Weight diagram of a generic 2-variable weighted shift in $\mathcal{S}_1$ and the 2-variable weighted shift in Lemma 7.4, respectively.

First, we wish to obtain a canonical representation for the powers $\langle x, y, a \rangle^{(h,\ell)}$ as an orthogonal direct sum of 2-variable weighted shifts in $\mathcal{S}_1$. In what follows, we abbreviate the orthogonal direct sums of m copies of a shift $\langle x, y, a \rangle$ by $m \cdot \langle x, y, a \rangle$.

Proposition 6.1. *Let $\langle x, y, a \rangle \in \mathcal{S}_1$ and let $h, \ell \ge 1$. Then*

$$\langle x, y, a \rangle^{(h,\ell)} \cong \langle x, y, a \rangle \bigoplus (h-1) \cdot \left\langle 1, \frac{ay}{x}, 1 \right\rangle \bigoplus (\ell - 1) \cdot \langle a, 1, a \rangle \bigoplus (h-1)(\ell-1) \cdot \langle 1, 1, 1 \rangle .$$

Proof. We decompose the space $\ell^2(\mathbb{Z}_+^2)$ as the orthogonal direct sum of $h\ell$ subspaces $\mathcal{H}_{(m,n)}$, each isometrically isomorphic to $\ell^2(\mathbb{Z}_+^2)$, namely $\mathcal{H}_{(m,n)} := \bigvee_{i,j=0}^{\infty}\{e_{(hi+m,\ell j+n)}\}$ $(0 \le m \le h-1, 0 \le n \le \ell-1)$. This particular decomposition allows us to write the power $\langle x,y,a\rangle^{(h,\ell)}$ as the orthogonal direct sum $\bigoplus_{0\le m\le h-1, 0\le n\le \ell-1} \langle x,y,a\rangle^{(h,\ell)} |_{\mathcal{H}_{(m,n)}}$. We will now identify each of the summands $\langle x,y,a\rangle^{(h,\ell)} |_{\mathcal{H}_{(m,n)}}$ $(0 \le m \le h-1, 0 \le n \le \ell-1)$.

Case 1: $(m = 0,\ n = 0)$ Direct inspection of the weight families α and β shows that

$$\langle x,y,a\rangle^{(h,\ell)} e_{(hi,\ell j)} = \langle x,y,a\rangle e_{(hi,\ell j)},$$

and therefore

$$\langle x,y,a\rangle^{(h,\ell)} |_{\mathcal{H}_{(0,0)}} \cong \langle x,y,a\rangle .$$

Case 2: $(m > 0,\ n = 0)$ In this case the generic basis vector of $\mathcal{H}_{(m,0)}$is $e_{(hi+m,\ell j)}$, so that $\langle x,y,a\rangle^{(h,\ell)} e_{(hi+m,\ell j)} = \left\langle 1, \frac{ay}{x}, 1\right\rangle e_{(hi+m,\ell j)}$. It follows that $\langle x,y,a\rangle^{(h,\ell)} |_{\mathcal{H}_{(m,0)}} \cong \left\langle 1, \frac{ay}{x}, 1\right\rangle$.

Case 3: $(m = 0,\ n > 0)$ In this case the generic basis vector of $\mathcal{H}_{(0,n)}$is $e_{(hi,\ell j+n)}$, and therefore $\langle x,y,a\rangle^{(h,\ell)} e_{(hi,\ell j+n)} = \langle a,1,a\rangle e_{(hi,\ell j+n)}$. It follows that $\langle x,y,a\rangle^{(h,\ell)} |_{\mathcal{H}_{(0,n)}} \cong \langle a,1,a\rangle$.

Case 4: $(m > 0,\ n > 0)$ Since $\mathcal{H}_{(m,n)} \subseteq \mathcal{M} \bigcap \mathcal{N}$, and the core of $W_{(\alpha,\beta)}$ is trivial, it is clear that all relevant weights are equal to 1, so $\langle x,y,a\rangle^{(h,\ell)} e_{(hi+m,\ell j+n)} = \langle 1,1,1\rangle e_{(hi+m,\ell j+n)}$, and therefore $\langle x,y,a\rangle^{(h,\ell)} |_{\mathcal{H}_{(m,n)}} \cong \langle 1,1,1\rangle$.

The proof is complete. ☐

We now recall the characterization of hyponormality, 2-hyponormality and subnormality for 2-variable weighted shifts in $\mathcal{S}_1$ found in [CLY4, Proofs of Theorems 3.1 and 3.3]. Recall that $\mathcal{S}_1 = \mathcal{S} \bigcap \mathfrak{H}_1$, $\mathcal{S}_2 = \mathcal{S} \bigcap \mathfrak{H}_2$ and $\mathcal{S}_\infty = \mathcal{S} \bigcap \mathfrak{H}_\infty$.

Theorem 6.2. *(cf.* [CLY4]*) Let* $\langle x,y,a\rangle \in \mathcal{S}_1$. *Then*
(i) $\langle x,y,a\rangle \in \mathfrak{H}_2 \iff f_2(x,y,a) := (1-x^2) - y^2(1-a^2) \ge 0$.
(ii) $\langle x,y,a\rangle \in \mathfrak{H}_\infty \iff f_2(x,y,a) \ge 0$.

Corollary 6.3. *Let* $\langle x,y,a\rangle \in \mathcal{S}_1$. *The following statements are equivalent.*
(i) $\langle x,y,a\rangle \in \mathfrak{H}_2$*;*
(ii) $\langle x,y,a\rangle \in \mathfrak{H}_\infty$*;*
(iii) $y \le \sqrt{\frac{1-x^2}{1-a^2}}$.

Corollary 6.4. *Let* $0 < a < 1$. *Then* $\langle a,1,a\rangle \in \mathfrak{H}_\infty$.

Proof. We apply Corollary 6.3 with $x := a$ and $y := 1$. Since condition (iii) is satisfied, it follows that $\langle a,1,a\rangle \in \mathfrak{H}_\infty$. ☐

Lemma 6.5. *Let* $0 < y < 1$. *Then* $\langle 1,y,1\rangle \in \mathfrak{H}_\infty$.

Proof. Here $T_1 \cong I \bigotimes U_+$ and $T_2 \cong S_y \bigotimes I$, so $\langle 1,y,1\rangle \equiv W_{(\alpha,\beta)}$ is clearly subnormal. ☐

Theorem 6.6. *Let* $\langle x,y,a\rangle \in \mathcal{S}_1$. *The following statements are equivalent.*
(i) $\langle x,y,a\rangle^{(h,\ell)} \in \mathfrak{H}_1$ *for all* $h,\ell \ge 1$.
(ii) $\langle x,y,a\rangle^{(h_0,\ell_0)} \in \mathfrak{H}_1$ *for some* $h_0,\ell_0 \ge 1$.

Proof. It is clearly sufficient to establish (ii) ⇒ (i). Assume therefore that $\langle x,y,a\rangle^{(h_0,\ell_0)} \in \mathfrak{H}_1$ for some $h_0,\ell_0 \ge 1$. By Proposition 6.1, we know that

$$\langle x,y,a\rangle^{(h_0,\ell_0)} \cong \langle x,y,a\rangle \bigoplus (h-1)\cdot\left\langle 1, \frac{ay}{x}, 1\right\rangle \bigoplus (\ell-1)\cdot\langle a,1,a\rangle \bigoplus (h-1)(\ell-1)\cdot\langle 1,1,1\rangle .$$

An application of Corollary 6.4 and Lemma 6.5 shows that $\langle x, y, a\rangle \in \mathfrak{H}_1$, Now, let $h, \ell \geq 1$ be arbitrary. A new application of Proposition 6.1 (this time using h and ℓ) shows that $\langle x, y, a\rangle^{(h,\ell)} \in \mathfrak{H}_1$. The proof is complete. □

Corollary 6.7. *Let $\langle x, y, a\rangle \in \mathcal{S}_1$, and let $k \geq 2$ be given. The following statements are equivalent.*
(i) For some $h_0, \ell_0 \geq 1$, $\langle x, y, a\rangle^{(h_0,\ell_0)} \in \mathfrak{H}_k$.
(ii) For all $h, \ell \geq 1$, $\langle x, y, a\rangle^{(h,\ell)} \in \mathfrak{H}_k$.
(iii) For some $h_0, \ell_0 \geq 1$ $\langle x, y, a\rangle^{(h_0,\ell_0)} \in \mathfrak{H}_\infty$.
(iv) For all $h, \ell \geq 1$ $\langle x, y, a\rangle^{(h,\ell)} \in \mathfrak{H}_\infty$.

Proof. Straightforward from Proposition 6.1 and the Proof of Theorem 6.6. □

We conclude this section with a problem of independent interest. Recall that $\mathcal{A} = \{W_{(\alpha,\beta)} \in \mathcal{TC}$: the Berger measure of $c(W_{(\alpha,\beta)})$ is one-atomic$\}$.

Problem 6.8. *Is $\mathcal{S}_1$ the largest class in $\mathcal{A}$ for which the implication*

$$W_{(\alpha,\beta)}^{(h_0,\ell_0)} \in \mathfrak{H}_2 \text{ for some } h_0, \ell_0 \geq 1 \Rightarrow W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$$

holds?

7. Appendix

For the reader's convenience, in this section we gather several well known auxiliary results which are needed for the proofs of the main results in this article. First, to detect hyponormality for 2-variable weighted shifts we use a simple criterion involving a base point $\mathbf{k}$ in $\mathbb{Z}_+^2$ and its five neighboring points in $\mathbf{k} + \mathbb{Z}_+^2$ at path distance at most 2.

Lemma 7.1. *([Cur, Theorem 6.1]) (Six-point Test) Let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then*

$$\begin{aligned} &[W_{(\alpha,\beta)}^*, W_{(\alpha,\beta)}] \geq 0 \\ &\iff H(k_1,k_2)(1) := \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0 \ (\textit{for all } \mathbf{k} \in \mathbb{Z}_+^2). \end{aligned}$$

Next, we present an analogous criterion to detect the k-hyponormality of 2-variable weighted shifts.

Lemma 7.2. *([CLY1, Theorem 2.4]) Let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be a 2-variable weighted shift with weight sequence α and β. The following statements are equivalent:*
(i) $W_{(\alpha,\beta)}$ is k-hyponormal;
(ii) $M_{\mathbf{k}}(k) := (\gamma_{\mathbf{k}+(n,m)+(p,q)})_{\substack{0\leq n+m\leq k \\ 0\leq p+q\leq k}} \geq 0$ for all $\mathbf{k} \in \mathbb{Z}_+^2$.

In particular, a commuting pair (T_1, T_2) is 2-hyponormal if and only if the 5-tuple $(T_1, T_2, T_1^2, T_1T_2, T_2^2)$ is hyponormal. For 2-variable weighted shifts, this is equivalent to the condition (Fifteen-point Test)

$$M_{\mathbf{k}}(2) := (\gamma_{\mathbf{k}+(n,m)+(p,q)})_{\substack{0\leq n+m\leq 2 \\ 0\leq p+q\leq 2}} \geq 0 \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2);$$

that is,

$$M_{\mathbf{k}}(2) \equiv \begin{pmatrix} \gamma_{k_1,k_2} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2} \\ \gamma_{k_1+1,k_2} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1+3,k_2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} \\ \gamma_{k_1,k_2+1} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1,k_2+3} \\ \gamma_{k_1+2,k_2} & \gamma_{k_1+3,k_2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+4,k_2} & \gamma_{k_1+3,k_2+1} & \gamma_{k_1+2,k_2+2} \\ \gamma_{k_1+1,k_2+1} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1+3,k_2+1} & \gamma_{k_1+2,k_2+2} & \gamma_{k_1+1,k_2+3} \\ \gamma_{k_1,k_2+2} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1,k_2+3} & \gamma_{k_1+2,k_2+2} & \gamma_{k_1+1,k_2+3} & \gamma_{k_1,k_2+4} \end{pmatrix} \geq 0.$$

This takes into account a base point k and its 14 neighbors at path distance at most 4.

To check subnormality of 2-variable weighted shifts, we introduce some definitions.

(i) Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \leq \nu$ on $X := \mathbb{R}_+$, if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on $\mathbb{R}_+$.

(ii) Let μ be a probability measure on $X \times Y$, and assume that $\frac{1}{t} \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := (1-\delta_0(t))\frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t)$.

(iii) Given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. Then we have:

Lemma 7.3. [CuYo1, Proposition 3.10] *(Subnormal backward extension) Let $W_{(\alpha,\beta)}$ be a 2-variable weighted shift, and assume that $W_{(\alpha,\beta)}|_{\mathcal{M}_1}$ is subnormal with associated measure $\mu_{\mathcal{M}_1}$ and that $W_0 :=$* shift $(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal with associated measure ξ_0. Then $W_{(\alpha,\beta)}$ is subnormal if and only if*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}_1})$;

(ii) $\beta_{00}^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}_1})})^{-1}$;

(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}_1})} (\mu_{\mathcal{M}_1})_{ext}^X \leq \xi_0$.

Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}_1})} = 1$, then $(\mu_{\mathcal{M}_1})_{ext}^X = \xi_0$. In the case when $W_{(\alpha,\beta)}$ is subnormal, the Berger measure μ of $W_{(\alpha,\beta)}$ is given by

$$d\mu(s,t) = \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}_1})} d(\mu_{\mathcal{M}_1})_{ext}(s,t) + (d\xi_0(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}_1})} d(\mu_{\mathcal{M}_1})_{ext}^X(s)) d\delta_0(t).$$

Lemma 7.4. *(*[Yoon, Theorem 2.8]*) Let $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ be a 2-variable weighted shift whose weight diagram is given in Figure 3(ii), so that $W_{(\alpha,\beta)}|_{\mathcal{M}_1} \cong (I \otimes$* shift $(\beta_1, \beta_2, \cdots), U_+ \otimes bI)$. *Assume that $\|W_\alpha\| = b > 0$, where $W_\alpha \equiv$* shift $(\alpha_0, \alpha_1, \alpha_2, \cdots)$. *Then $W_{(\alpha,\beta)} \in \mathfrak{H}_1 \iff W_{(\alpha,\beta)} \in \mathfrak{H}_\infty \iff$ the Berger measure μ_α of W_α has an atom at b^2.*

Given a subnormal 2-variable weighted shift $W_{(\alpha,\beta)}$ with Berger measure μ, we let $W_{\alpha^{(j)}} (j \geq 0)$ (resp. $W_{\beta^{(i)}}$ $(i \geq 0)$) denote the associated j-th horizontal (resp. i-th vertical) slice of $W_{(\alpha,\beta)}$. Clearly, $W_{\alpha^{(j)}}$ (resp. $W_{\beta^{(i)}}$) is subnormal, and we let ξ_j (resp. η_i) denote its Berger measure. We proved in [CuYo2] that $d\xi_j(s) := \{\frac{1}{\gamma_{0j}} \int t^j \, d\Phi_s(t)\} \, d\xi(s)$, where $d\mu(s,t) \equiv d\Phi_s(t) d\xi(s)$ is the canonical disintegration of μ by vertical slices (resp. $d\eta_i(t) = \{\frac{1}{\gamma_{i0}} \int s^i \, d\Psi_t(s)\} \, d\eta(t)$, where $d\mu(s,t) \equiv d\Psi_t(s) d\eta(t)$ is the canonical disintegration of μ by horizontal slices).

Lemma 7.5. *(*[CuYo2, Theorem 3.3]*) Let μ, ξ_j and η_i be as above. If $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$, then for every $i, j \geq 0$ we have*

$$\xi_{j+1} \ll \xi_j \text{ and } \eta_{i+1} \ll \eta_i. \tag{7.1}$$

Lemma 7.6. *(cf.* [Smu], [CuFi, Proposition 2.2]*) Let* $M \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be a* 2×2 *operator matrix, where* A *and* C *are square matrices and* B *is a rectangular matrix. Then*

$$M \geq 0 \iff \text{there exists } W \text{ such that } \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW. \end{cases}$$

References

[Ath] A. Athavale, On joint hyponormality of operators, *Proc. Amer. Math. Soc.* 103(1988), 417-423.

[Atk] K. Atkinson, *Introduction to Numerical Analysis*, Wiley and Sons, 2nd. Ed. 1989.

[Choi] M.D. Choi, Tricks or treats with the Hilbert matrix, *Amer. Math. Monthly* 90 (1983), 301–312.

[Con] J. Conway, *The Theory of Subnormal Operators,* Mathematical Surveys and Monographs, vol. 36, Amer. Math. Soc. Providence, 1991.

[Cur] R. Curto, Joint hyponormality: A bridge between hyponormality and subnormality, *Proc. Symposia Pure Math.* 51(1990), 69-91.

[CuFi] R. Curto and L. Fialkow, Solution of the truncated complex moment problem with flat data, *Memoirs Amer. Math. Soc.* no. 568, Amer. Math. Soc., Providence, 1996.

[CLY1] R. Curto, S.H. Lee and J. Yoon, k-hyponormality of multivariable weighted shifts, *J. Funct. Anal.* 229(2005), 462-480.

[CLY2] R. Curto, S.H. Lee and J. Yoon, Hyponormality and subnormality for powers of commuting pairs of subnormal operators, *J. Funct. Anal.* 245(2007), 390-412.

[CLY3] R. Curto, S.H. Lee and J. Yoon, Reconstruction of the Berger measure when the core is of tensor form, *Actas del XVI Coloquio Latinoamericano de Álgebra, Bibl. Rev. Mat. Iberoamericana*, (2007), 317-331.

[CLY4] R. Curto, S.H. Lee and J. Yoon, Which 2-hyponormal 2-variable weighted shifts are subnormal?, *Linear Algebra Appl., 429(2008) 2227-2238.*

[CMX] R. Curto, P. Muhly and J. Xia, Hyponormal pairs of commuting operators, *Operator Theory: Adv. Appl.* 35(1988), 1-22.

[CuPa] R. Curto, S. Park, k-hyponormality of powers of weighted shifts, *Proc. Amer. Math. Soc.* 131(2002), 2762-2769.

[CuPu] R. Curto and M. Putinar, Nearly subnormal operators and moments problems, *J. Funct. Anal.* 115(1993), 480-497.

[CuYo1] R. Curto, J. Yoon, Jointly hyponormal pairs of subnormal operators need not be jointly subnormal, Trans. *Amer. Math. Soc.* 358(2006), 5139-5159.

[CuYo2] R. Curto and J. Yoon, Disintegration-of-measure techniques for multivariable weighted shifts, *Proc. London Math. Soc.*, 93(2006), 381-402.

[CuYo3] R. Curto and J. Yoon, Propagation phenomena for hyponormal 2-variable weighted shifts, *J. Operator Theory,* 58:1(2007), 101-130.

[GeWa] Gellar and Wallen, Subnormal weighted shifts and the Halmos-Bram criterion, *Proc. Japan Acad.*, 46(1970), 375-378.

[Hal] P.R. Halmos, *A Hilbert Space Problem Book*, Second edition, Graduate Texts in Mathematics, 19, Springer-Verlag, New York-Berlin, 1982.

[JeLu] N.P. Jewell and A.R. Lubin, Commuting weighted shifts and analytic function theory in several variables, *J. Operator Theory* 1(1979), 207-223.

[PoSz] G. Pólya and G. Szegö, *Problems and Theorems in Analysis,* vol. II: *Theory of functions, zeros, polynomials, determinants, number theory, geometry,* Springer-Verlag, New York-Heidelberg, 1976. xi+391 pp.

[Shi] A.L. Shields, Weighted shift operators and analytic function theory, *Math. Surveys* 13 (1974), 49-128.

[Smu] Ju. L. Smul'jan, An operator Hellinger integral, *Mat. Sb. (N.S.)* 49 (1959), 381-430 (in Russian).

[Sta] J. Stampfli, Which weighted shifts are subnormal?, *Pacific J. Math.* 17(1966), 367-379.

[Wol] Wolfram Research, Inc. *Mathematica*, Version 4.2, *Wolfram Research Inc.*, Champaign, IL, 2002.

[Yoon] J. Yoon, Schur product techniques for commuting multivariable weighted shifts, *J. Math. Anal. Appl.*, 333(2007), 626-641.

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