EXTENSIONS AND EXTREMALITY
OF RECURSIVELY GENERATED WEIGHTED SHIFTS

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ABSTRACT. Given an \( n \)-step extension \( \alpha : x_0, \cdots, x_1, (a_0, \cdots, a_k)^\wedge \) of a recursively generated weight sequence \( 0 < a_0 < \cdots < a_k \), and if \( W_\alpha \) denotes the associated unilateral weighted shift, we prove that

\[
W_\alpha \text{ is subnormal } \iff \begin{cases} 
W_\alpha \text{ is } ([k+1]/2) + 1\text{-hyponormal} & (n = 1) \\
W_\alpha \text{ is } ([k+1]/2) + 2\text{-hyponormal} & (n > 1). 
\end{cases}
\]

In particular, the subnormality of an extension of a recursively generated weighted shift is independent of its length if the length is bigger than 1. As a consequence we see that if \( \alpha(x) \) is a canonical rank-one perturbation of the recursive weight sequence \( \alpha \), then subnormality and \( k \)-hyponormality for \( W_{\alpha(x)} \) eventually coincide. We then show that \( \alpha(x) \) is recursively generated, i.e., \( W_{\alpha(x)} \) is recursive subnormal.

INTRODUCTION

Let \( \mathcal{H} \) be a separable infinite dimensional complex Hilbert space and let \( \mathcal{L}(\mathcal{H}) \) be the algebra of bounded linear operators on \( \mathcal{H} \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be \emph{normal} if \( T^*T = TT^* \) and \emph{hyponormal} if \( T^*T \geq TT^* \). Given a bounded sequence of positive numbers \( \alpha : \alpha_0, \alpha_1, \cdots \) (called \emph{weights}), the (unilateral) \emph{weighted shift} \( W_{\alpha} \) associated with \( \alpha \) is the operator on \( \ell^2(\mathbb{Z}_+) \) defined by \( W_{\alpha}e_n := \alpha_ne_{n+1} \) for all \( n \geq 0 \), where \( \{e_n\}_{n=0}^\infty \) is the canonical orthonormal basis for \( \ell^2(\mathbb{Z}_+) \). It is straightforward to check that \( W_{\alpha} \) can never be \emph{normal}, and that \( W_{\alpha} \) is \emph{hyponormal} if and only if \( \alpha_n \leq \alpha_{n+1} \) for all \( n \geq 0 \). The Bram-Halmos criterion for subnormality states that an operator \( T \) is \emph{subnormal} if and only if

\[
\sum_{i,j} (T^*x_j, T^jx_i) \geq 0
\]

for all finite collections \( x_0, x_1, \cdots, x_k \in \mathcal{H} \) ([Br],[Con, III.1.9]). Using Choleski’s algorithm for operator matrices, it is easy to see that this is equivalent to the following positivity test:

\[
(0.1) \quad \begin{pmatrix}
I & T^* & \cdots & T^k
T & T^*T & \cdots & T^{*k}
\vdots & \vdots & \ddots & \vdots 
T^k & T^{*k} & \cdots & T^{*k}T^k
\end{pmatrix} \geq 0 \quad \text{(all } k \geq 1).\]

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Condition (0.1) provides a measure of the gap between hyponormality and subnormality, and $k$-hyponormality has been introduced and studied in an attempt to bridge the gap between subnormality and hyponormality ([At], [Cu1], [Cu2], [CF1], [CF2], [CF3], [CL1], [CMX], [McCP]). In fact, the positivity condition (0.1) for $k = 1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (0.1) for all $k$. If we denote by $[A,B] := AB - BA$ the commutator of two operators $A$ and $B$, and if we define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix

$$(0.2) \quad M_k(T) := ([T^j, T^i])_{i,j=1}^k$$

is positive, or equivalently, the $(k + 1) \times (k + 1)$ operator matrix in (0.1) is positive, then the Bram-Halmos criterion can be rephrased as saying that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([CMX]).

If $W_\alpha$ is the weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the moments of $W_\alpha$ are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ $(n \geq 0)$ [Shi]; however, we reserve this term for the sequence $\gamma_n := \beta_n^2$ $(n \geq 0)$. A criterion for $k$-hyponormality can be given in terms of moments ([Cu1, Theorem 4]): if we build a $(k + 1) \times (k + 1)$ Hankel matrix $A(n;k)$ by

$$(0.3) \quad A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0),$$

then

$$(0.4) \quad W_\alpha \text{ is } k\text{-hyponormal} \iff A(n;k) \geq 0 \quad (n \geq 0).$$

In [Sta], J. Stampfli showed that given $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}$ with $0 < a < b < c$, there always exists a subnormal completion of $\alpha$, but that for $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$ $(a < b < c < d)$ such a subnormal completion may not exist. There are instances where $k$-hyponormality implies subnormality for weighted shifts. For example, in [CF3], it was shown that if $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ $(a < b < c)$ then $W_{\alpha(x)}$ is 2-hyponormal if and only if it is subnormal: more concretely, $W_{\alpha(x)}$ is 2-hyponormal if and only if

$$\sqrt{x} \leq H_2(\sqrt{a}, \sqrt{b}, \sqrt{c}) := \sqrt{\frac{ab(c-b)}{(b-a)^2 + b(c-b)}},$$

in which case $W_{\alpha(x)}$ is subnormal. In this paper we extend the above result to weight sequences of the form $\alpha : x_1, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ with $0 < \alpha_0 < \cdots < \alpha_k$. Our main results are as follows.

**Extensions of Recursively Generated Weighted Shifts.** If $\alpha : x_1, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ then

$W_\alpha$ is subnormal $\iff \begin{cases} W_\alpha \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 1)\text{-hyponormal} & (n = 1) \\
W_\alpha \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 2)\text{-hyponormal} & (n > 1). \end{cases}$

In particular, the above theorem shows that the subnormality of an extension of the recursive shift is independent of its length if the length is bigger than 1.

**Canonical Rank–One Perturbations.** Let $\alpha = \{\alpha_n\}_{n=0}^\infty = (\alpha_0, \cdots, \alpha_k)^\wedge$. If $W_{\alpha'}$ is a perturbation of $W_\alpha$ at the $j$-th weight then

$W_{\alpha'}$ is subnormal $\iff \begin{cases} W_{\alpha'} \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 1)\text{-hyponormal} & (j = 0) \\
W_{\alpha'} \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 2)\text{-hyponormal} & (j \geq 1). \end{cases}$

**Extremality Criterion.** Let $\alpha(x)$ be a canonical rank-one perturbation of a weight sequence $\alpha$. If $(k + 1)$-hyponormality and $k$-hyponormality for $W_{\alpha(x)}$ coincide, then $\alpha(x)$ is recursively generated, i.e., $W_{\alpha(x)}$ is recursive subnormal.
1. Extensions of Recursively Generated Shifts

C. Berger’s characterization of subnormality for unilateral weighted shifts (cf. [Hal], [Con, III.8.16]) states that \(W_{\alpha}\) is subnormal if and only if there exists a Borel probability measure \(\mu\) such that

\[
\gamma_n = \int t^n d\mu(t) \quad \text{for all } n \geq 0.
\]

Given an initial segment of weights \(\alpha: \alpha_0, \ldots, \alpha_r\), the sequence \(\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)\) such that \(\hat{\alpha}_i = \alpha_i (i = 0, \ldots, r)\) is said to be recursively generated by \(\alpha\) if there exist \(r \geq 1\) and \(\varphi_0, \ldots, \varphi_{r-1} \in \mathbb{R}\) such that

\[
\gamma_{n+r} = \varphi_0 \gamma_n + \cdots + \varphi_{r-1} \gamma_{n+r-1} \quad \text{for all } n \geq 0,
\]

where \(\gamma_0 := 1, \gamma_n := \alpha_0^2 \cdots \alpha_n^2 (n \geq 1)\). In this case the weighted shift \(\hat{W}_{\alpha}\) with a weight sequence \(\hat{\alpha}\) is said to be recursively generated (or simply recursive). If

\[
g(t) := t^r - (\varphi_{r-1} t^{r-1} + \cdots + \varphi_0),
\]

then \(g\) has \(r\) distinct real roots \(0 < s_0 < \cdots < s_{r-1}\) (cf. [CF2, Theorem 3.9]). Let

\[
V := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
s_0 & s_1 & \cdots & s_{r-1} \\
\vdots & \vdots & \ddots & \vdots \\
s_0^{r-1} & s_1^{r-1} & \cdots & s_{r-1}^{r-1}
\end{pmatrix}
\]

and let

\[
\begin{pmatrix}
\rho_0 \\
\vdots \\
\rho_{r-1}
\end{pmatrix} := V^{-1} \begin{pmatrix}
\gamma_0 \\
\vdots \\
\gamma_{r-1}
\end{pmatrix}.
\]

If \(\hat{W}_{\alpha}\) is a recursively generated subnormal shift then the Berger measure of \(\hat{W}_{\alpha}\) is of the form

\[
\mu := \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{s_{r-1}}.
\]

Given an initial segment of weights

\[
\alpha : \alpha_0, \ldots, \alpha_{2k} \quad (k \geq 0),
\]

suppose \(\hat{\alpha} \equiv (\alpha_0, \ldots, \alpha_{2k})^\wedge\), i.e., \(\hat{\alpha}\) is recursively generated by \(\alpha\). Write

\[
v_n := \begin{pmatrix}
\gamma_n \\
\vdots \\
\gamma_{n+k}
\end{pmatrix} \quad (0 \leq n \leq k + 1).
\]

Then \(\{v_0, \ldots, v_{k+1}\}\) is linearly dependent in \(\mathbb{R}^{k+1}\). Now the rank of \(\alpha\) is defined by the smallest integer \(i (1 \leq i \leq k + 1)\) such that \(v_i\) is a linear combination of \(v_0, \ldots, v_{i-1}\). Since \(\{v_0, \ldots, v_{i-1}\}\) is linearly independent, there exists a unique \(i\)-tuple \(\varphi \equiv (\varphi_0, \ldots, \varphi_{i-1}) \in \mathbb{R}^i\) such that \(v_i = \varphi_0 v_0 + \cdots + \varphi_{i-1} v_{i-1}\), or equivalently,

\[
\gamma_j = \varphi_{i-1} \gamma_{j-1} + \cdots + \varphi_0 \gamma_{j-i} \quad (i \leq j \leq k + i),
\]

which says that \((\alpha_0, \ldots, \alpha_{k+i})\) is recursively generated by \((\alpha_0, \ldots, \alpha_i)\). In this case, \(W_{\alpha}\) is said to be \(i\)-recursive (cf. [CF3, Definition 5.14]).

We begin with:
Lemma 1.1 ([CF2, Propositions 2.3, 2.6, and 2.7]). Let \( A, B \in M_n(\mathbb{C}), \tilde{A}, \tilde{B} \in M_{n+1}(\mathbb{C}) \) \((n \geq 1)\) be such that
\[
\tilde{A} = \begin{pmatrix} A & * \\ * & * \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} * & * \\ * & B \end{pmatrix}.
\]

Then we have:

(i) If \( A \geq 0 \) and \( \tilde{A} \) is a flat extension of \( A \) \(i.e.,\) \( \text{rank}(\tilde{A}) = \text{rank}(A) \) then \( \tilde{A} \geq 0 \);

(ii) If \( A \geq 0 \) and \( \tilde{A} \geq 0 \) then \( \text{det}(A) = 0 \) implies \( \text{det}(\tilde{A}) = 0 \);

(iii) If \( B \geq 0 \) and \( \tilde{B} \geq 0 \) then \( \text{det}(B) = 0 \) implies \( \text{det}(\tilde{B}) = 0 \).

Lemma 1.2. If \( \alpha = (\alpha_0, \cdots, \alpha_k)^{\wedge} \) then
\[(1.2.1) \quad W_\alpha \text{ is subnormal} \iff W_\alpha \text{ is } \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)-\text{hyponormal}.
\]

In the cases where \( W_\alpha \) is subnormal and \( i := \text{rank}(\alpha) \), we have \( \alpha = (\alpha_0, \cdots, \alpha_{2i-2})^{\wedge} \).

Proof. We only need to establish the sufficiency condition in (1.2.1). Let \( i := \text{rank}(\alpha) \). Since \( W_\alpha \) is \( i \)-recursive, [CF3, Proposition 5.15] implies that the subnormality of \( W_\alpha \) follows after we verify that \( A(0, i - 1) \geq 0 \) and \( A(1, i - 1) \geq 0 \). Now observe that \( i - 1 \leq \left\lfloor \frac{k}{2} \right\rfloor + 1 \) and
\[
A(j, \left\lfloor \frac{k}{2} \right\rfloor + 1) = \begin{pmatrix} A(j, i - 1) & * \\ * & * \end{pmatrix} \quad (j = 0, 1),
\]
so the positivity of \( A(0, i - 1) \) and \( A(1, i - 1) \) is a consequence of the positivity of the \( \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \)-hyponormality of \( W_\alpha \). For the second assertion, observe that \( \text{det}(A(n, i)) = 0 \) for all \( n \geq 0 \). By assumption \( A(n, i + 1) \geq 0 \), so by Lemma 1.1 (ii) we have \( \text{det}(A(n, i + 1)) = 0 \), which says that \( (\alpha_0, \cdots, \alpha_{2i-1}) \subset (\alpha_0, \cdots, \alpha_{2i-2})^{\wedge} \). By iteration we obtain \( (\alpha_0, \cdots, \alpha_k) \subset (\alpha_0, \cdots, \alpha_{2i-2})^{\wedge} \), and therefore \( (\alpha_0, \cdots, \alpha_k)^{\wedge} = (\alpha_0, \cdots, \alpha_{2i-2})^{\wedge} \). This proves the lemma.

In what follows, and for notational convenience, we shall set \( x_{-j} := \alpha_j \) \((0 \leq j \leq k)\).

Theorem 1.3 (Subnormality Criterion). If \( \alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^{\wedge} \) then
\[(1.3.1) \quad W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)-\text{hyponormal} & (n = 1) \\ W_\alpha \text{ is } \left(\left\lfloor \frac{k}{2} \right\rfloor + 2\right)-\text{hyponormal} & (n > 1). \end{cases}
\]

Furthermore, in the cases where the above equivalence holds, if \( \text{rank}(\alpha_0, \cdots, \alpha_k) = i \) then
\[(1.3.2) \quad W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } i-\text{hyponormal} & (n = 1) \\ W_\alpha \text{ is } (i + 1)-\text{hyponormal} & (n > 1). \end{cases}
\]

In fact,
\[
\begin{align*}
x_1 &= H_i(x_0, \cdots, x_{2-2i}) \\
x_2 &= H_i(x_1, \cdots, x_{3-2i}) \\
& \quad \cdots \\
x_{n-1} &= H_i(x_{n-2}, \cdots, x_{n-2i}) \\
x_n &= H_i(x_{n-1}, \cdots, x_{n-2i+1}),
\end{align*}
\]
where \( H_i \) is the modulus of \( i \)-hyponormality (cf. [CF3, Proposition 3.4 and (3.4)]) \(, i.e.,\)
\[
H_i(\alpha) := \sup\{x > 0 : W_{\alpha x} \text{ is } i-\text{hyponormal}\}.
\]
Therefore, \( W_\alpha = W_{x_n(x_{n-1}, \ldots, x_{n-k+1})} \).

**Proof.** Consider the \((k+1) \times (l+1)\) “Hankel” matrix \( A(n; k, l) \) by (cf. [CL1])

\[
A(n; k, l) := \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+l} \\
\gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+l+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+l+k}
\end{pmatrix} \quad (n \geq 0).
\]

**Case 1** \((\alpha : x_1, (\alpha_0, \cdots, \alpha_k)^\wedge)\): Let \( \hat{A}(n; k, l) \) and \( A(n; k, l) \) denote the Hankel matrices corresponding to the weight sequences \((\alpha_0, \cdots, \alpha_k)^\wedge \) and \( \alpha \), respectively. Suppose \( W_\alpha \) is \(((\frac{k+1}{2}) + 1)\)-hyponormal. Then by Lemma 1.2, \( W_{(\alpha_0, \cdots, \alpha_k)^\wedge} \) is subnormal. Observe that

\[
A(n+1; m, m) = x_1^2 \hat{A}(n; m, m) \quad \text{for all } n \geq 0 \text{ and all } m \geq 0.
\]

Thus it suffices to show that \( A(0; m, m) \geq 0 \) for all \( m \geq \lceil \frac{k+1}{2} \rceil + 2 \). Also observe that if \( \hat{B} \) denotes the \((k-1) \times k\) matrix obtained by eliminating the first row of a \( k \times k \) matrix \( B \) then

\[
\hat{A}(0; m, m) = x_1^2 \hat{A}(0; m-1, m) \quad \text{for all } m \geq \lceil \frac{k+1}{2} \rceil + 2.
\]

Therefore, for every \( m \geq \lceil \frac{k+1}{2} \rceil + 2 \), \( A(0; m, m) \) is a flat extension of \( A(0; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) \). This implies \( A(0; m, m) \geq 0 \) for all \( m \geq \lceil \frac{k+1}{2} \rceil + 2 \) and therefore \( W_\alpha \) is subnormal.

**Case 2** \((\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge)\): As in Case 1, let \( \hat{A}(n; k, l) \) and \( A(n; k, l) \) denote the Hankel matrices corresponding to the weight sequences \((\alpha_0, \cdots, \alpha_k)^\wedge \) and \( \alpha \), respectively. Observe that \( \det \hat{A}(n; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = 0 \) for all \( n \geq 0 \). Suppose \( W_\alpha \) is \(((\frac{k+1}{2}) + 2)\)-hyponormal. Observe that

\[
A(n+1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = x_1^2 \cdots x_n^2 \hat{A}(1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1),
\]

so that

\[
A(n+1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = x_1^2 \cdots x_n^2 \hat{A}(1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1),
\]

(1.3.3)

\[
\det A(n+1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = 0.
\]

Also observe that

\[
A(n-1; [\frac{k+1}{2}] + 2, [\frac{k+1}{2}] + 2) = \begin{pmatrix}
x_1^2 \cdots x_n^2 & A(n+1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1)
\end{pmatrix}.
\]

Since \( W_\alpha \) is \(((\frac{k+1}{2}) + 1)\)-hyponormal, it follows from Lemma 1.1 (iii) and (1.3.3) that \( \det A(n-1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = 0 \). Note that

\[
A(n-1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = x_1^2 \cdots x_n^2 \begin{pmatrix}
\frac{1}{x_1} & \gamma_0 & \cdots & \hat{\gamma}_{\frac{k+1}{2}+1} \\
\gamma_0 & \gamma_1 & \cdots & \hat{\gamma}_{\frac{k+1}{2}+2} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\gamma}_{\frac{k+1}{2}+1} & \hat{\gamma}_{\frac{k+1}{2}+2} & \cdots & \gamma_2^{\frac{k+1}{2}+2}
\end{pmatrix},
\]

where \( \hat{\gamma}_j \) denotes the moments corresponding to the weight sequence \((\alpha_0, \cdots, \alpha_k)^\wedge \). Therefore \( x_1 \) is determined uniquely by \( \{\alpha_0, \cdots, \alpha_k\} \) such that \((x_1, \alpha_0, \cdots, \alpha_{k-1})^\wedge = x_1, (\alpha_0, \cdots, \alpha_k)^\wedge \): more precisely, if \( i := \text{rank}(\alpha) \) and \( \varphi_0, \cdots, \varphi_i \) denote the coefficients of recursion in \((\alpha_0, \cdots, \alpha_k)^\wedge \) then

\[
x_1 = H_i[(\alpha_0, \cdots, \alpha_k)^\wedge] = \left[ \frac{\varphi_0}{\hat{\gamma}_i - \varphi_{i-1} \hat{\gamma}_{i-2} - \cdots - \varphi_1 \hat{\gamma}_0} \right]^\frac{1}{2}
\]
and $W(x_{n-1}, \cdots, x_{n-1-k})^\wedge$ is subnormal. Therefore, after $(n-1)$ steps, Case 2 reduces to Case 1. This completes the proof of the first assertion. For the second assertion, note that if \( \text{rank}(\alpha_0, \cdots, \alpha_k) = i \) then

$$\det \hat{A}(n, i, i) = 0.$$  

Now applying the above argument with $i$ in place of $k$ gives that $x_1, \cdots, x_{n-1}$ are determined uniquely by $\alpha_0, \cdots, \alpha_{2i-2}$ such that $W(x_{n-1}, \cdots, x_{n-1-k})^\wedge$ is subnormal. Thus the second assertion immediately follows. Finally, observe that the preceding argument also establish the remaining assertions. 

**Remark 1.4.** (a) From Theorem 1.3 we note that the subnormality of an extension of a recursive shift is independent of its length if the length is bigger than 1.

(b) In Theorem 1.3, “$$\frac{k+1}{2}$$” can not be relaxed to “$$\frac{k}{2}$$”. For example consider the following weight sequences:

1. $\alpha : \sqrt{2}, (\sqrt{3}, \sqrt{10/3}, \sqrt{17/5})^\wedge$ with $\varphi_0 = 0$;
2. $\alpha' : \sqrt{1/2}, (\sqrt{3}, \sqrt{10/3}, \sqrt{17/5})^\wedge$.

Observe that $\alpha$ equals $\alpha'$. Then a straightforward calculation shows that $W_\alpha$ (and hence $W_{\alpha'}$) is 2-hyponormal but not 3-hyponormal (and hence, not subnormal). Note that $k = 3$ and $n = 1$ in (i) and $k = 2$ and $n = 2$ in (ii).

(c) Note that the second assertion of Theorem 1.3 does not imply that if $\text{rank}(\alpha_0, \cdots, \alpha_k) = i$ then (1.3.2) holds in general. Theorem 1.3 says only that when $W_\alpha$ is $((k+1)/2)$-hyponormal (i.e., $i$-hyponormality and subnormality coincide), and that when $W_\alpha$ is $((k+1)/2)$-hyponormal (i.e., $(i+1)$-hyponormality and subnormality coincide. For example consider the weight sequence

$$\hat{\alpha} = (\sqrt{2}, \sqrt{3}, \sqrt{10/3}, \sqrt{17/5}, 2)^\wedge$$

with $\varphi_0 = 0$ (here $\varphi_1 = 0$ also).

Since $(\sqrt{2}, \sqrt{3}, \sqrt{10/3}, \sqrt{17/5}) \subset (\sqrt{2}, \sqrt{3}, \sqrt{10/3}, 2)^\wedge$, we can see that $\text{rank}(\alpha) = 2$. Put

$$\beta = 1, (\sqrt{2}, \sqrt{3}, \sqrt{10/3}, \sqrt{17/5}, 2)^\wedge.$$  

If (1.3.2) held true without assuming (1.3.1), then 2-hyponormality would imply subnormality for $W_\beta$. However, a straightforward calculation shows that $W_\beta$ is 2-hyponormal but not 3-hyponormal (and hence not subnormal): in fact, $\det A(n, 2) = 0$ for all $n \geq 0$ except for $n = 2$ and $\det A(2, 2) = 160 > 0$, while since

$$\varphi_3 = \frac{\alpha_3^2 \alpha_4^2 (\alpha_2^2 - \alpha_3^2)}{\alpha_4^2 - \alpha_3^2} = -102$$

and

$$\varphi_4 = \frac{\alpha_4^2 (\alpha_2^2 - \alpha_3^2)}{\alpha_4^2 - \alpha_3^2} = 34$$

(so that $\alpha_6 = \sqrt{\varphi_4 - \frac{\alpha_4^2}{\alpha_6^2}} = \sqrt{17/2}$), we have that

$$\det A(1, 3) = \det \begin{pmatrix} 1 & 2 & 6 & 20 \\ 2 & 6 & 20 & 68 \\ 6 & 20 & 68 & 272 \\ 20 & 68 & 272 & 2312 \end{pmatrix} = -3200 < 0.$$
(d) On the other hand, Theorem 1.3 does show that if $\alpha \equiv (\alpha_0, \cdots, \alpha_k)$ is such that $\text{rank}(\alpha) = i$ and $W_\alpha$ is subnormal with associated Berger measure $\mu$, then $W_\alpha$ has an $n$-step $(i+1)$-hyponormal extension $W_{x_n, \cdots, x_1, \alpha}$ $(n \geq 2)$ if and only if $\frac{1}{n} \in L^1(\mu)$,

$$x_{j+1} = \left[ \frac{\varphi_0}{\gamma_{i-1} - \varphi_{i-1} \gamma_{i-2} - \cdots - \varphi_{1} \gamma_0} \right]^\frac{1}{2} (0 \leq j \leq n-2),$$

and

$$x_n \leq \left[ \frac{\varphi_0}{\gamma_{i-1} - \varphi_{i-1} \gamma_{i-2} - \cdots - \varphi_{1} \gamma_0} \right]^\frac{1}{2},$$

where $\varphi_0, \cdots, \varphi_{1-i}$ denote the coefficients of recursion in $(\alpha_0, \cdots, \alpha_{2i-2})^\wedge$ and $\gamma_m^j (0 \leq m \leq i-1)$ are the moments corresponding to the weight sequence $(x_j, \cdots, x_1, \alpha_0, \cdots, \alpha_{k-j})^\wedge$ with $\gamma_m^0 = \gamma_m$.

We now observe that the determination of $k$-hyponormality and subnormality for canonical rank-one perturbations of recursive shifts falls within the scope of the theory of extensions.

**Corollary 1.5.** Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty = (\alpha_0, \cdots, \alpha_k)^\wedge$. If $W_{\alpha'}$ is a perturbation of $W_\alpha$ at the $j$-th weight then

$$W_{\alpha'} \text{ is subnormal } \iff \begin{cases} W_{\alpha'} \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 1)\text{-hyponormal} & (j = 0) \\ W_{\alpha'} \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 2)\text{-hyponormal} & (j \geq 1). \end{cases}$$

**Proof.** Observe that if $j = 0$ then $\alpha' = x, (\alpha_1, \cdots, \alpha_{k+1})^\wedge$ and if instead $j \geq 1$ then $\alpha' = (\alpha_0, \cdots, \alpha_{j-1}, x, (\alpha_{j+1}, \cdots, \alpha_{j+k+1})^\wedge$. Thus the result immediately follows from Theorem 1.3. \qed

## 2. Extremality of Recursively Generated Shifts

In Corollary 1.5, we showed that if $\alpha(x)$ is a canonical rank-one perturbation of a recursive weight sequence then subnormality and $k$-hyponormality for the corresponding shift eventually coincide. In this section we consider a converse.

**Problem 2.1 (Extremality Problem).** Let $\alpha(x)$ be a canonical rank-one perturbation of a weight sequence $\alpha$. If there exists $k \geq 1$ such that $(k+1)$-hyponormality and $k$-hyponormality for the corresponding shift $W_{\alpha(x)}$ coincide, does it follow that $\alpha(x)$ is recursively generated?

In [CF3], the following extremality criterion was established.

**Lemma 2.2 (Extremality Criterion)[CF3; Theorem 5.12, Proposition 5.13].** Let $\alpha$ be a weight sequence and let $k \geq 1$.

(i) If $W_\alpha$ is $k$-extremal (i.e., $\det A(j,k) = 0$ for all $j \geq 0$) then $W_\alpha$ is recursive subnormal.

(ii) If $W_\alpha$ is $k$-hyponormal and if $\det A(i_0, j_0) = 0$ for some $i_0 \geq 0$ and some $j_0 < k$ then $W_\alpha$ is recursive subnormal.

In particular, Lemma 2.2 (ii) shows that if $W_\alpha$ is subnormal and if $\det A(i_0, j_0) = 0$ for some $i \geq 0$ and some $j \geq 0$ then $W_\alpha$ is recursive subnormal.

We now answer Problem 2.1 affirmatively.
**Theorem 2.3.** Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence and let $\alpha_j(x)$ be a canonical perturbation of $\alpha$ in the $j$-th weight. Write

$$S_k := \{x \in \mathbb{R}^+ : W_{\alpha_j(x)} \text{ is k-hyponormal}\}.$$ 

If $S_k = S_{k+1}$ for some $k \geq 1$, and if $x \in S_k$, then $\alpha_j(x)$ is recursively generated, i.e., $W_{\alpha_j(x)}$ is recursive subnormal.

Proof. Suppose $S_k = S_{k+1}$ and let $H_k := \sup S_k$. To avoid triviality we assume $\alpha_{j-1} < x < \alpha_{j+1}$.

Case 1 ($j = 0$): In this case, clearly $H_k^2$ is the nonzero root of the equation $\det A(0, k) = 0$ and for $x \in (0, H_k)$, $W_{\alpha(0)}(x)$ is k-hyponormal. By assumption $H_k = H_{k+1}$, so $W_{\alpha(0)(H_k+1)}$ is $(k+1)$-hyponormal. The result now follows from Lemma 2.2 (ii).

Case 2 ($j \geq 1$): Let $A_x(n, k)$ denote the Hankel matrix corresponding to $\alpha_j(x)$. Since $W_{\alpha_j(x)}$ is $(k+1)$-hyponormal for $x \in S_k$, we have that $A_x(n, k+1) \geq 0$ for all $n \geq 0$ and all $x \in S_k$. Observe that if $n \geq j + 1$ then

$$A_x(n, k) = \alpha_0^2 \cdots \alpha_{j-1}^2 x^2 \begin{pmatrix} \tilde{\gamma}_{n-j-1} & \cdots & \tilde{\gamma}_{n-j-k+1} \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n-j-k+1} & \cdots & \tilde{\gamma}_{n-j-2k} \end{pmatrix},$$

where $\tilde{\gamma}_x$ denotes the moments corresponding to the subsequence $\alpha_{j+1}, \alpha_{j+2}, \ldots$. Therefore for $n \geq j + 1$, the positivity of $A_x(n, k)$ is independent of the values of $x > 0$. This gives

$$W_{\alpha_j(x)} \text{ is k-hyponormal} \iff A_x(n, k) \geq 0 \text{ for all } n \leq j.$$ 

Write

$$S_k(i) := \left\{ x : \det A_x(i, k) \geq 0 \text{ and } \alpha_{j-1} < x < \alpha_{j+1} \right\} \quad (0 \leq i \leq j)$$

and

$$H_k(i) = \sup_{x} S_k(i) \quad (0 \leq i \leq j).$$

Since $\det A_x(i, k)$ is a polynomial in $x$, we have $\det A_{H_k(i)}(i, k) = 0$. Observe that

$$\bigcap_{i=0}^{j} S_k(i) = S_k \quad \text{and} \quad \max_{0 \leq i \leq j} H_k(i) = H_k.$$ 

Since $S_k$ is a closed interval, by [CL2, Theorem 2.11], it follows that $H_k \in S_k$, say $H_k = H_k(p)$ for some $0 \leq p \leq j$. Then $\det A_{H_k(p)}(p, k) = 0$ and $W_{\alpha(H_k(p))}$ is $(k+1)$-hyponormal. Therefore it follows from Lemma 2.2 (ii) that $W_{\alpha}$ is recursive subnormal. This completes the proof.

We conclude this section with two corollaries of independent interest.

**Corollary 2.4.** With the notations in Theorem 2.3, if $j \geq 1$ and $S_k = S_{k+1}$ for some $k$, then $S_k$ is a singleton set.

Proof. By [CL2, Theorem 2.2],

$$S_{\infty} := \{ x \in \mathbb{R}^+ : W_{\alpha_j(x)} \text{ is subnormal} \}$$

is a singleton set. By Theorem 2.3, we have that $S_k = S_{\infty}$.

**Corollary 2.5.** If $W_{\alpha}$ is a nonrecursive shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and if $\alpha(x)$ is a canonical rank-one perturbation of $\alpha$, then for every $k \geq 1$ there always exists a gap between $k$-hyponormality and $(k+1)$-hyponormality for $W_{\alpha(x)}$. More concretely, if we let

$$S_k := \{ x : W_{\alpha(x)} \text{ is k-hyponormal} \},$$

then $\{S_k\}_{k=1}^{\infty}$ is a strictly decreasing nested sequence of closed intervals in $(0, \infty)$ except when the perturbation occurs in the first weight. In that case, the intervals are of the form $(0, H_k]$.

Proof. Straightforward from Theorem 2.3.
3. Some Revealing Examples

We now illustrate our results with two examples. Consider \( \alpha(y, x) : \sqrt[y]{x}, \sqrt{x}, (\sqrt[y]{a}, \sqrt{b}, \sqrt{c})^n \), where \( a < b < c \). Without loss of generality, we assume \( a = 1 \). Observe that

\[
H_2(1, \sqrt{b}, \sqrt{c}) = \sqrt{\frac{bc - b^2}{1 + bc - 2b}} \quad \text{and} \quad \left( H_2(\sqrt[x]{a}, 1, \sqrt{b}) \right)^2 = \frac{x(b - 1)}{(x - 1)^2 + (b - 1)} := f(x).
\]

Thus \( W_{\alpha(y, x)} \) is 2-hyponormal if and only if \( 0 < x \leq \frac{bc - b^2}{1 + bc - 2b} \) and \( 0 < y \leq f(x) \). To completely describe the region \( \mathcal{R} := \{(x, y) : W_{\alpha(y, x)} \) is 2-hyponormal\}, we study the graph of \( f \). Observe that

\[
f'(x) = \frac{(b - 1)(1 - x^2)}{(b - 2x + x^2)^2} > 0 \quad \text{and} \quad f''(x) = \frac{2(b - 1)(2b - 3bx + x^3)}{(b - 2x + x^2)^3}.
\]

Note that \( b - 2x + x^2 = (b - 1) + (1 - x)^2 > 0 \) and \( f'(\sqrt{b}) = 0 \). To consider the sign of \( f'' \), we let \( g(x) := 2b - 3bx + x^3 \). Then \( g'(\sqrt{b}) = 0 \), \( g(0) = 2b > 0 \), \( g(1) = -b + 1 < 0 \), and \( g''(x) > 0 \) \( (x > 0) \). Hence there exists \( x_0 \in (0, 1) \) such that \( f''(x_0) = 0 \), \( f''(x) > 0 \) on \( 0 < x < x_0 \), and \( f''(x) < 0 \) on \( x_0 < x \leq 1 \). We investigate which of the two values \( x_0 \) or \( \bar{H} := H_2(1, \sqrt{b}, \sqrt{c})^2 \) is bigger. By a simple calculation, we have

\[
g(\bar{H}) = \frac{(-1 + b)b \cdot g_1(b, c)}{(1 - 2b + bc)^3},
\]

where

\[
g_1(b, c) = -(2 - 10b + 17b^2 - 11b^3 + b^4 + 3bc - 9b^2c + 9b^3c - 3b^3c^2 + b^2c^3).
\]

For notational convenience we let \( b := 1 + h \), \( c := 1 + h + k \). Then

\[
g_1(b, c) = 2h^5 + (3h^3 + 3h^4)k + (-1 - 2h - h^2)k^3.
\]

If \( h \) is sufficiently small (i.e., \( b \) is sufficiently close to \( 1 \)), then \( g_1 < 0 \), i.e., \( \bar{H} > x_0 \). If \( k \) is sufficiently small (i.e., \( b \) is sufficiently close to \( c \)), then \( g_1 > 0 \), i.e., \( \bar{H} < x_0 \). Thus, if \( \bar{H} > x_0 \), then \( f \) is concave up on \( x \leq \bar{H} \). If \( \bar{H} < x_0 \), then \( (x_0, f(x_0)) \) is an inflection point. Thus, \( f \) is concave up on \( 0 < x < x_0 \) and concave down on \( x_0 < x \leq \bar{H} \). Moreover, \( W_{\alpha(y, x)} \) is 2-hyponormal if and only if \( (x, y) \in \{(x, y)| 0 \leq y \leq f(x), 0 < x \leq \bar{H}\} \), and \( W_{\alpha(y, x)} \) is \( k \)-hyponormal \( (k \geq 3) \) if and only if \( x = \bar{H} \) and \( 0 \leq y \leq f(\bar{H}) \).

**Example 3.1** \((b = 2, c = 3) \).

\[
f(x) = \frac{x}{1 + (1 - x)^2}.
\]

The graph of \( \mathcal{R} \) is given in Figure 1; notice that \( f \) is concave up in this case.
Example 3.2 \((b = \frac{11}{10}, c = 10)\).

\[
f(x) = \frac{x}{11 - 20x + 10x^2}.
\]

The graph of \(R\) is given in Figure 2; in this case, \(f\) has an inflection point at \(x_0 \approx 0.85821\).

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