THE EXTENDED ALUTHGE TRANSFORM

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To the memory of Professor Ronald G. Douglas

ABSTRACT. Given a bounded linear operator T with canonical polar decomposition $T \equiv V\,|T|$, the Aluthge transform of T is the operator $\Delta(T) := \sqrt{|T|}V\sqrt{|T|}$. For P an arbitrary positive operator such that $VP = T$, we define the *extended* Aluthge transform of T associated with P by $\Delta_P(T) := \sqrt{P}V\sqrt{P}$. First, we establish some basic properties of Δ_P; second, we study the fixed points of the extended Aluthge transform; third, we consider the case when T is an idempotent; next, we discuss whether Δ_P leaves invariant the class of complex symmetric operators. We also study how Δ_P transforms the numerical radius and numerical range. As a key application, we prove that the spherical Aluthge transform of a commuting pair of operators corresponds to the extended Aluthge transform of a 2×2 operator matrix built from the pair; thus, the theory of extended Aluthge transforms yields results for spherical Aluthge transforms.

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1. Introduction

The Aluthge transform for a bounded operator T acting on a Hilbert space $\mathcal{H}$ was introduced by A. Aluthge in [1]. If $T \equiv V\,|T|$ is the canonical polar decomposition of T, the Aluthge transform $\Delta(T)$ is given as $\Delta(T) := \sqrt{|T|}V\sqrt{|T|}$. One of Aluthge's motivations was to use this transform in the study of p-hyponormal and log-hyponormal operators. Roughly speaking, the idea was to convert an operator, T, into another operator, $\Delta(T)$, which shares with the first one many spectral properties, but which is closer to being a normal operator.

Key words and phrases. extended Aluthge transform, polar decomposition, extended quasinormal operator.

The first named author was partially supported by Labex CEMPI (ANR-11-LABX-0007-01). The second named author was partially supported by NSF Grant DMS-1302666.

Over the last two decades, substantial and significant results about $\Delta(T)$, and how it relates to T, have been obtained by a long list of mathematicians who devoted considerable attention to this topic (see, for instance, [2], [4], [9], [12], [13], [14], [15], [23], [24], [25], [26], [28], [29], [30], [33], [34], [35], [36]). Aluthge transforms have been generalized to the case of powers of $|T|$ different from $\frac{1}{2}$ ([5], [7], [8], [31]) and to the case of commuting pairs of operators ([10], [11]).

In this paper, we set out to extend the Aluthge transform in a different direction. Starting with the canonical polar decomposition $T \equiv V\,|T|$, we consider the class of positive operators P such that $VP = V\,|T|$, that is, all positive operators P that mimic the action of $|T|$ in the canonical polar decomposition. For each such P we then define the *extended* Aluthge transform as $\Delta_P(T) := \sqrt{P}V\sqrt{P}$. Naturally, the classical Aluthge transform is simply $\Delta_{|T|}(T)$.

We first study the basic properties of this new operator transform, and how it relates to the classical Aluthge transform. We do this in Section 2. We then study, in Section 3, the fixed points of the extended Aluthge transform, in an effort to see what is the correct generalization of quasinormality to this new environment. Third, in Section 4 we consider the case when T is an idempotent. Next, we discuss whether Δ_P leaves invariant the class of complex symmetric operators (Section 7).

We also study how Δ_P transforms the numerical radius and numerical range; we do this in Section 8. As a key application, we prove that the spherical Aluthge transform of a commuting pair of operators (introduced in [10] and further studied in [11]) corresponds to the extended Aluthge transform of a 2×2 operator matrix built from the pair; thus, the theory of extended Aluthge transforms is well positioned to yield new results for spherical Aluthge transforms.

Along the way, we strive to maintain contact with the classical Aluthge transform, in an effort to shed light on how this new extended Aluthge transform can help unravel the relative position of $|T|$ within the equation $V\,|T| = T$. For instance, we prove in Section 2 that $|T|$ is the smallest positive solution of the equation $VP = T$.

2. The Extended Aluthge Transform

Let $\mathcal{H}$ denote a (complex, separable) Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, let $T \equiv V\,|T|$ be the canonical polar decomposition of T; that is, $|T| := (T^*T)^{\frac{1}{2}}$, V is a partial isometry, and $\ker V = \ker |T| = \ker T$. The Aluthge transform of T is the operator $\Delta(T) := |T|^{\frac{1}{2}}\,V\,|T|^{\frac{1}{2}}$.

Consider now an arbitrary positive operator $P \in \mathcal{B}(\mathcal{H})$ such that $VP = T$. The *extended Aluthge transform* of T associated with P is the operator

$$\Delta_P(T) := P^{\frac{1}{2}}VP^{\frac{1}{2}}.$$

Lemma 2.1. *For P as above,*

$$|T| \le P.$$

Proof.

$$|T|^2 = T^*T = PV^*VP \le P^2,$$

since V is a contraction. It follows that $|T| \le P$. □

Corollary 2.2. *For P as above,*

$$\ker P \subseteq \ker |T| .$$

Corollary 2.3. *For P as above,*

$$\overline{\operatorname{Ran} |T|} \subseteq \overline{\operatorname{Ran} P},$$

where $\overline{\mathcal{M}}$ denotes the closure of the linear space $\mathcal{M}$.

Lemma 2.4. *For P as above, $|T|$ commutes with P.*

Proof.

$$V |T| = V P \Longrightarrow |T| - P \in \ker V = \ker |T| .$$

It follows that

$$|T| (|T| - P) = 0 \Longrightarrow |T|^2 = |T| P \Longrightarrow |T|^2 = (|T|^2)^* = (|T| P)^* = P |T| \Longrightarrow |T| P = P |T| .$$

□

Lemma 2.5. *For P as above,*

$$P|_{\operatorname{Ran}|T|} = |T| \, |_{\operatorname{Ran}|T|}.$$

Proof. By the Proof of Lemma 2.4, we have

$$P |T| x = |T| Px = |T| \, |T| x \ \ (\text{all } x \in \mathcal{H}) .$$

It follows that P and $|T|$ agree on $\operatorname{Ran} |T|$. □

Lemma 2.6. *Write $\mathcal{H} = \overline{\operatorname{Ran} |T|} \oplus \ \ker T$. Then*

$$|T| = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A := |T| \, |_{\overline{\operatorname{Ran}|T|}}$ and $B := P|_{\ker T}$.

Proof. By Lemma 2.5, P leaves $\overline{\operatorname{Ran} |T|}$ invariant, so $\overline{\operatorname{Ran} |T|}$ is a reducing subspace for P. □

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Consider now the orthogonal decomposition

$$\mathcal{H} = \overline{\operatorname{Ran} |T|} \oplus (\overline{\operatorname{Ran} B} \oplus \ker P),$$

where the orthogonal sum in parentheses equals $\ker |T|$. Then

$$|T| = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Observe that $P = |T|$ if and only if $C = 0$. We wish to find the matrix for V. Recall that $\ker V = \ker |T|$. Therefore,

$$V = \begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix}.$$

Since V^*V is the projection onto $(\ker V)^\perp = \overline{\operatorname{Ran}|T|}$, we must have

$$X^*X + Y^*Y + Z^*Z = I_{\overline{\operatorname{Ran}|T|}}.$$

Since $VP = V\,|T|$, it follows that

$$\begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Also,

$$T = \begin{pmatrix} XA & 0 & 0 \\ YA & 0 & 0 \\ ZA & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Then

$$\Delta(T) = |T|^{\frac{1}{2}}\,V\,|T|^{\frac{1}{2}} = \begin{pmatrix} A^{\frac{1}{2}}XA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

while

$$\Delta_P(T) = P^{\frac{1}{2}}VP^{\frac{1}{2}} = \begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & C^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix}\begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & C^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}}XA^{\frac{1}{2}} & 0 & 0 \\ C^{\frac{1}{2}}YA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Therefore,

$$\Delta(T)^*\Delta(T) = \begin{pmatrix} A^{\frac{1}{2}}X^*AXA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Delta_P(T)^*\Delta_P(T) = \begin{pmatrix} A^{\frac{1}{2}}X^*AXA^{\frac{1}{2}} + A^{\frac{1}{2}}Y^*CYA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a consequence,

$$|\Delta_P(T)| \geq |\Delta(T)|$$

and

$$\|\Delta_P(T)\| \geq \|\Delta(T)\|.$$

As is well known, the Aluthge transform is homogeneous, that is $\Delta(\lambda T) = \lambda\Delta(T)$ for every $\lambda \in \mathbb{C}$. The following result shows what form of homogeneity holds for the extended Aluthge transform.

Proposition 2.7. *Let $T \equiv V|T|$ be the canonical polar decomposition of T, and let P be a positive operator such that $T = VP$. For $\lambda \in \mathbb{C}$ we have*

$$\Delta_{|\lambda|P}(\lambda T) = \lambda \Delta_P(T).$$

Proof. Without loss of generality, assume that $\lambda \neq 0$, and let $\lambda \equiv e^{i\theta}|\lambda|$ be its canonical polar decomposition. Then

$$\lambda T = e^{i\theta} V |\lambda| |T| = (e^{i\theta} V)(|\lambda| |T|)$$

is the canonical polar decomposition of λT. Moreover,

$$\lambda T = e^{i\theta} V |\lambda| P = (e^{i\theta} V)(|\lambda| P),$$

so that

$$\Delta_{|\lambda|P}(\lambda T) = (|\lambda|)^{1/2} P^{\frac{1}{2}} e^{i\theta} V (|\lambda|)^{1/2} P^{\frac{1}{2}} = \lambda P^{1/2} V P^{1/2} = \lambda \Delta_P(T). \quad \square$$

□

In an entirely similar way, one can establish the following two results.

Proposition 2.8. *Let $T \equiv V|T|$ be the canonical polar decomposition of T, and let P be a positive operator such that $T = VP$. Let U be a unitary operator on $\mathcal{H}$. Then*

$$\Delta_{UPU^*}(UTU^*) = U\Delta_P(T)U^*.$$

We now discuss an extension of the so-called *-Aluthge transform, used by P.Y. Wu [33] and T. Yamazaki [35] to prove [33, Theorem 1] (cf. Theorem 8.4). This transform is defined as $\Delta(T)^{(*)} := \sqrt{|T^*|} V \sqrt{|T^*|}$. It is not difficult to prove that $\Delta(T)^{(*)} = (\Delta(T^*))^*$; thus, $\Delta(T)^{(*)} = V\Delta(T)V^*$.

We will now obtain the proper analog for the extended Aluthge transform. Let $T = V|T| = VP$ where $T = V|T|$ is the canonical polar decomposition of T. It is well known that $T^* = V^*|T^*|$ is the canonical polar decomposition of T^*. Now observe that, with the notation from Section 2, we have

$$PV^* = \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X^* & Y^* & Z^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} AX^* & AY^* & AZ^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$V^*V = \begin{pmatrix} X^* & Y^* & Z^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix} = \begin{pmatrix} X^*X + Y^*Y + Z^*Z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that

$$(V^*V)PV^* = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} AX^* & AY^* & AZ^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} AX^* & AY^* & AZ^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = PV^*. \tag{2.4}$$

As a result, $V^*V\sqrt{P}V^* = \sqrt{P}V^*$.

To state the following result, we first recall that the canonical polar decomposition of T^* is $T^* = V^*|T^*|$. Since $T = VP$ we get $T^* = PV^*$, and using (2.4) we obtain $T^* = PV^* =$

$(V^*V)PV^* = V^*(VPV^*)$. Moreover, VPV^* is a positive operator, so we may consider the extended Aluthge transform of T^* associated with VPV^*.

Proposition 2.9. *With T, V and P as above, we have*

$$\Delta_{VPV^*}(T^*) = V\Delta_P(T)^*V^*.$$

Proof.

$$\begin{aligned} \Delta_{VPV^*}(T^*) &= \sqrt{VPV^*}V^*\sqrt{VPV^*} \\ &= (V\sqrt{P}V^*)V^*(V\sqrt{P}V^*) \\ &= V\sqrt{P}V^*V^*V\sqrt{P}V^* \\ &= V\sqrt{P}V^*(V^*V\sqrt{P}V^*) \\ &= V\sqrt{P}V^*\sqrt{P}V^* \\ &= V(\sqrt{P}V^*\sqrt{P})V^* \\ &= V\Delta_P(T)^*V^*. \end{aligned}$$

□

Corollary 2.10. *In the case when $P = |T|$ we have*

$$\Delta(T^*)^* = V\Delta(T)V^*.$$

Proof. We first observe that $V|T|V^* = |T^*|$, which is established as follows: $T = V|T|$ and $T^* = V^*|T^*|$ imply that $V^*|T^*| = |T|V^*$, and therefore $V|T|V^* = VV^*|T^*| = |T^*|$. Next, we use Proposition 2.9 to conclude that $\Delta(T^*) = V\Delta(T)^*V^*$. Finally, we take adjoints to get $\Delta(T^*)^* = V\Delta(T)V^*$. □

We end this section with a result about orthogonal direct sums.

Proposition 2.11. *Let $T_i \equiv V_i|T_i|$ be the canonical polar decomposition of T_i $(i = 1, 2)$, and let P_i be a positive operator such that $T_i = V_iP_i$ $(i = 1, 2)$. Then*

$$\Delta_{P_1 \oplus P_2}(T_1 \oplus T_2) = \Delta_{P1}(T_1) \oplus \Delta_{P_2}(T_2).$$

3. Fixed Points of the Extended Aluthge Transform

It is well known that the fixed points of the classical Aluthge transform are the quasinormal operators, that is, those operators $T = V|T|$ such that V and $|T|$ commute. In this section we study the class of operators which are fixed points for the extended Aluthge transform. In what follows, we frequently use the matricial decompositions introduced in Section 2. From (2.1) and (2.3), we easily see that

$$\Delta_P(T) = T \iff \begin{cases} A^{\frac{1}{2}}XA^{\frac{1}{2}} = XA \\ C^{\frac{1}{2}}YA^{\frac{1}{2}} = YA \\ 0 = ZA. \end{cases}$$

Recall that $\operatorname{Ran} A$ is dense in $\overline{\operatorname{Ran}|T|}$. Thus, $ZA = 0 \Rightarrow Z = 0$. Also,

$$(A^{\frac{1}{2}}X - XA^{\frac{1}{2}})A^{\frac{1}{2}} = A^{\frac{1}{2}}XA^{\frac{1}{2}} - XA = 0 \Longrightarrow A^{\frac{1}{2}}X = XA^{\frac{1}{2}}$$

and therefore

$$AX = A^{\frac{1}{2}}A^{\frac{1}{2}}X = A^{\frac{1}{2}}XA^{\frac{1}{2}} = XA^{\frac{1}{2}}A^{\frac{1}{2}} = XA.$$

It follows that A and X commute. Finally,

$$(C^{\frac{1}{2}}Y - YA^{\frac{1}{2}})A^{\frac{1}{2}} = C^{\frac{1}{2}}YA^{\frac{1}{2}} - YA = 0 \Longrightarrow C^{\frac{1}{2}}Y = YA^{\frac{1}{2}}$$

so that

$$CY = C^{\frac{1}{2}}(C^{\frac{1}{2}}Y) = C^{\frac{1}{2}}YA^{\frac{1}{2}} = YA^{\frac{1}{2}}A^{\frac{1}{2}} = YA.$$

We then have:

$$\begin{cases} AX &= XA \\ CY &= YA \\ Z &= 0, \end{cases}$$

which readily implies

$$V\,|T| = PV$$

and

$$VP = PV.$$

It follows that $\ker P$ reduces V.

We summarize the previous discussion in the following result.

Theorem 3.1. *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator such that $VP = T$, and assume that $\Delta_P(T) = T$. Then T commutes with P, V commutes with P, and $\ker P$ reduces T and V.*

Corollary 3.2. *In Theorem 3.1, assume that $P = |T|$, so that $\Delta_P(T) = \Delta(T) = T$. Then T is quasinormal (i.e., $|T|$ commutes with V, or equivalently, $|T|$ commutes with T).*

4. The Case of T Idempotent

In this section we consider the case when T is an idempotent, that is, $T^2 = T$. From (2.1) it easily follows that

$$\begin{cases} XAXA &= XA \\ YAXA &= YA \\ ZAXA &= ZA. \end{cases}$$

Since $\operatorname{Ran} A$ is dense in $\overline{\operatorname{Ran}|T|}$, we have

$$\begin{cases} XAX &= X \\ YAX &= Y \\ ZAX &= Z, \end{cases}$$

and therefore

$$\begin{cases} X^*XAX &= X^*X \\ Y^*YAX &= Y^*Y \\ Z^*ZAX &= Z^*Z. \end{cases}$$

Since $X^*X + Y^*Y + Z^*Z = I_{\overline{\operatorname{Ran}|T|}}$, we readily obtain

$$AX = I_{\overline{\operatorname{Ran}|T|}}.$$

As a result,

$$A^{\frac{1}{2}} X A^{\frac{1}{2}} = I_{\overline{\operatorname{Ran}|T|}}.$$

(For, given $x \in \operatorname{Ran} A^{\frac{1}{2}}$ one has

$$\left\langle (A^{\frac{1}{2}} X A^{\frac{1}{2}}) A^{\frac{1}{2}} x, A^{\frac{1}{2}} x \right\rangle = \langle AXAx, x \rangle = \langle Ax, x \rangle = \left\langle A^{\frac{1}{2}} x, A^{\frac{1}{2}} x \right\rangle,$$

and it follows that $A^{\frac{1}{2}} X A^{\frac{1}{2}} = I$ on $\overline{\operatorname{Ran} A^{\frac{1}{2}}} = \overline{\operatorname{Ran}|T|}$.)

Using (2.2) we readily see that $\Delta(T)$ is the projection from $\mathcal{H}$ onto $\overline{\operatorname{Ran}|T|}$; using (2.3) we see that

$$\Delta_P(T) = \begin{pmatrix} I & 0 & 0 \\ C^{\frac{1}{2}} Y A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $AX = I_{\overline{\operatorname{Ran}|T|}}$, we know that A is right invertible on $\overline{\operatorname{Ran}|T|}$, therefore invertible (as an operator on $\overline{\operatorname{Ran}|T|}$).

We summarize the previous discussion in the following result.

Theorem 4.1. *Let T be an idempotent. Then*

$$\Delta(T) = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and

$$\Delta_P(T) = \begin{pmatrix} I & 0 & 0 \\ C^{\frac{1}{2}} Y A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with A invertible.

The information we have gathered is somewhat optimal, as the following example shows.

Example 4.2. Given $a, b \in \mathbb{R}$, let

$$T := \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \in M_3(\mathcal{C}).$$

Let $\delta := \sqrt{1 + a^2 + b^2}$. It is straightforward to verify that $T^2 = T$, with canonical polar decomposition

$$T = \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \equiv V|T| = \begin{pmatrix} \frac{1}{\delta} & 0 & 0 \\ \frac{a}{\delta} & 0 & 0 \\ \frac{b}{\delta} & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $f > 0$ let

$$P \equiv P_{\delta, f} := \begin{pmatrix} \delta & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\Delta(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Delta_P(T) = \begin{pmatrix} 1 & 0 & 0 \\ a\sqrt{\frac{f}{\delta}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice in particular that V does not commute with P, so T is not a fixed point for Δ_P. Moreover, $\Delta(\Delta_P(T)) \neq \Delta_P(T)$; however, $\Delta(\Delta(\Delta_P(T))) = \Delta(\Delta_P(T))$. Also, as expected, $\Delta(T)$ is a projection, while $\Delta_P(T)$ is again an idempotent. Therefore, it makes sense to repeat this construction (with a new δ and a new f) to obtain the iterate $\Delta_{P_2}(\Delta_P(T))$, which is again an idempotent. One can then study the asymptotic behavior of these iterates, in a manner resembling the results in [3], [12], [25] and [32] for the classical Aluthge transform. We plan to report on the behavior of the iterates of the extended Aluthge transform in a forthcoming paper. □

5. Some Useful Identities

We devote this section to the proof of some identities involving T, its classical Aluthge transform $\Delta(T)$ and the extended Aluthge transform $\Delta_P(T)$. First, recall from (2.2) and (2.3) that

$$\Delta(T) = \begin{pmatrix} A^{\frac{1}{2}}XA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Delta_P(T) = \begin{pmatrix} A^{\frac{1}{2}}XA^{\frac{1}{2}} & 0 & 0 \\ C^{\frac{1}{2}}YA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Direct matrix calculation shows that

$$\Delta(T)P = \Delta(T)\left|T\right|,$$

consistent with Lemma 2.4. Similarly, one obtains the following result.

Proposition 5.1. *(Intertwining Property) For T, P, $\Delta(T)$ and $\Delta_P(T)$ as in Section 2, we have:*

$$\left|T\right|^{\frac{1}{2}}\Delta_P(T)P^{\frac{1}{2}} = P^{\frac{1}{2}}\Delta(T)\left|T\right|^{\frac{1}{2}}.$$

We briefly pause to recall an important feature of the class $\mathcal{C}_2$ of Hilbert-Schmidt operators on $\mathcal{H}$. Recall that the inner product of two Hilbert-Schmidt operators S and T is given by

$$\langle S, T\rangle_{\mathcal{C}_2} := \operatorname{Tr}(T^*S).$$

The class $\mathcal{C}_2$ is a Hilbert space, with norm $\left\|S\right\|_2 := (\langle S, S\rangle_{\mathcal{C}_2})^{\frac{1}{2}} = (\operatorname{Tr}(S^*S))^{\frac{1}{2}}$. For E and F in the class $\mathcal{C}_2$, consider the operator matrix

$$\begin{pmatrix} E & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\left\| \begin{pmatrix} E & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\|_2^2 = \left\langle \begin{pmatrix} E & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} E & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle_{\mathcal{C}_2} = \operatorname{Tr}(E^*E + F^*F) = \|E\|_2^2 + \|F\|_2^2.$$

A direct consequence of this calculation is the following result.

Theorem 5.2. *Let T be a Hilbert-Schmidt operator. Then*

$$\|\Delta_P(T)\|_2^2 = \|\Delta(T)\|_2^2 + \left\| C^{\frac{1}{2}} Y A^{\frac{1}{2}} \right\|_2^2.$$

6. An Application: The Spherical Aluthge Transform

In this section we will show how the spherical Aluthge transform (introduced in [10] and [11]) can be obtained as a particular case of the extended Aluthge transform, for a suitable positive operator P. Given a commuting pair $\mathcal{T} \equiv (T_1, T_2)$ of operators acting on $\mathcal{H}$, let $Q := (T_1^* T_1 + T_2^* T_2)^{\frac{1}{2}}$. Clearly, $\ker Q = \ker T_1 \bigcap \ker T_2$. For $x \in \ker Q$, let $V_i x := 0$ $(i = 1, 2)$; for $y \in \operatorname{Ran} Q$, say $y = Qx$, let $V_i y := T_i x$ $(i = 1, 2)$. It is easy to see that V_1 and V_2 are well defined, and extend continuously to $\overline{\operatorname{Ran} Q}$. We then have

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 Q \\ V_2 Q \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} Q, \tag{6.1}$$

as operators from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$. Moreover, this is the canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. It follows that $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $(\ker Q)^\perp$ onto $\overline{\operatorname{Ran} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}}$.

The spherical Aluthge transform of $\mathcal{T}$ is $\widehat{\mathcal{T}} \equiv (\widehat{T}_1, \widehat{T}_2)$, where

$$\widehat{T}_i := Q^{\frac{1}{2}} V_i Q^{\frac{1}{2}} \ (i = 1, 2) \ \text{(cf. [10],[11])}.$$

Lemma 6.1. *(cf. [11]) $\widehat{\mathcal{T}}$ is commutative.*

We now let

$$\Phi(\mathcal{T}) := \begin{pmatrix} T_1 & 0 \\ T_2 & 0 \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{H}).$$

It is clear that

$$|\Phi(\mathcal{T})| = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}.$$

We also let $\mathbf{V} := (V_1, V_2)$. (Notice that $\mathbf{V}$ is not necessarily commuting.) Finally, let

$$\Phi(\mathbf{V}) := \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \end{pmatrix}.$$

Lemma 6.2. *With $\mathcal{T}$ and $\mathbf{V}$ as above, $\Phi(\mathcal{T}) = \Phi(V)\,|\Phi(\mathcal{T})|$ is the canonical polar decomposition of $\Phi(\mathcal{T})$.*

Proof. This is straightforward from the fact that (6.1) is the canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. $\square$

Consider now the positive operator

$$P \equiv P(Q) := \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

Then

$$\Phi(\mathbf{V})P = \Phi(\mathbf{V})\,|\Phi(\mathcal{T})| = \Phi(\mathcal{T}).$$

We wish to study the extended Aluthge transform $\Delta_P(\Phi(\mathcal{T}))$.

Theorem 6.3. *With $\mathcal{T}$ and P as above,*

$$\Delta_P(\Phi(\mathcal{T})) = \Phi(\widehat{\mathcal{T}}).$$

Proof.

$$\begin{aligned} \Phi(\widehat{\mathcal{T}}) = \begin{pmatrix} \widehat{T}_1 & 0 \\ \widehat{T}_2 & 0 \end{pmatrix} &= \begin{pmatrix} Q^{\frac{1}{2}}V_1Q^{\frac{1}{2}} & 0 \\ Q^{\frac{1}{2}}V_2Q^{\frac{1}{2}} & 0 \end{pmatrix} = \begin{pmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & Q^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \end{pmatrix} \begin{pmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & Q^{\frac{1}{2}} \end{pmatrix} \\ &= P^{\frac{1}{2}}\Phi(\mathbf{V})P^{\frac{1}{2}} = \Delta_P(\Phi(\mathcal{T})). \end{aligned}$$

$\square$

Remark 6.4. Proposition 6.3 shows that the spherical Aluthge transform can be expressed in terms of the extended Aluthge transform of the 2×2-operator matrix $\Phi(\mathcal{T})$. $\square$

Observation 6.5. For the spherical Aluthge transform, the operator P is uniquely determined by the commuting pair $\mathcal{T}$; that is, the pair $\mathcal{T}$ determines Q, which in turn determines P. $\square$

7. Extended Aluthge Transforms of Complex Symmetric Operators

Recall that a *conjugation* C on a Hilbert space $\mathcal{H}$ is an antilinear map satisfying: (i) $C^2 = I$; and (ii) $\langle Cx, Cy \rangle = \langle y, x \rangle$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C such that $T^* = CTC$.

If $T = U\,|T|$ is the canonical polar decomposition of T, one may use a generalization of a theorem of Godič and Lucenko to write $U = CJ$ where J is a partial conjugation supported on $\overline{\mathrm{Ran}(|T|)}$ such that $J|T| = |T|J$, where T is a C-complex symmetric operator (see [17, Theorem 2]); as a result, $T = CJ\,|T|$. (For additional results, see [6] and [7].)

Of course, J can be extended to a conjugation $\tilde{J}$ (which, with minor abuse of notation, we will usually denote again by J (cf. [5])) acting on the whole space $\mathcal{H}$, without affecting the equation $T = CJ|T|$. In this case, it was proven in [15] that the Aluthge transform $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is also complex symmetric, with the conjugation $\tilde{J}$.

Theorem 7.1 ([15]). *Let T be a complex symmetric operator. Then $\Delta(T)$ is complex symmetric.*

We will now establish that the extended Aluthge transform does not preserve the property of being complex symmetric. To this end, we will focus attention on a special class of finite rank operators. Let n be an integer and assume that $n \geq 2$. For a finite family of complex numbers $\lambda \equiv \lambda_1, \cdots, \lambda_{n-1}$ consider the operator

$$T(\lambda) := \sum_{i=1}^{n-1} \lambda_i e_{i+1} \otimes e_i,$$

where $e_1, \cdots, e_n, \cdots$ are elements of an orthonormal basis for $\mathcal{H}$, and for vectors $x, y \in \mathcal{H}$ we denote by $x \otimes y$ the rank-one operator $(x \otimes y)(z) := \langle z, y \rangle \, x \ (z \in \mathcal{H})$. Without loss of generality we may assume that $\lambda_i > 0$ for all $i = 1, \cdots, n-1$. The results, however, will be stated for complex λ_i's. It is easy to see that T admits the following matricial representation:

$$T(\lambda) = \begin{pmatrix} 0 & 0 & 0 & & \dots & 0 \\ \lambda_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & \dots & \lambda_{n-1} & 0 \end{pmatrix}.$$

It is straightforward to see that

$$V(\lambda) = \begin{pmatrix} 0 & 0 & 0 & & \dots & 0 \\ 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & \dots & 1 & 0 \end{pmatrix}$$

and

$$|T(\lambda)| = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & 0 \end{pmatrix}$$

are the factors in the canonical polar decomposition of $T(\lambda)$. Let γ be a given positive real number, and consider a positive operator $P_\gamma(\lambda)$ of the form

$$P_\gamma(\lambda) := \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & \gamma \end{pmatrix} \tag{7.1}$$

We now recall the following result, proved in [37, Theorem 3.1], appropriately adjusted to our situation.

Proposition 7.2. *Let $T(\lambda) \equiv \sum_{i=1}^{n-1} \lambda_i e_{i+1} \otimes e_i$ be as above. Then T is complex symmetric if and only if $|\lambda_i| = |\lambda_{n-i}|$ $(1 \le i \le n-1)$.*

For the class of operators $T(\lambda)$ we now determine which λ's and γ's give rise to a complex symmetric extended Aluthge transform $\Delta_{P_\gamma(\lambda)}$.

Theorem 7.3. *Let $T(\lambda)$ and γ be as above, and let $\Delta_{P_\gamma(\lambda)}(T(\lambda))$ be the associated extended Aluthge transform. Then $\Delta_{P_\gamma(\lambda)}(T(\lambda))$ is complex symmetric if and only if $\gamma\,|\lambda_{n-1}| = |\lambda_1\lambda_2|$ and $|\lambda_i\lambda_{i+1}| = |\lambda_{n-i}\lambda_{n-i+1}|$ for every $2 \le i \le n-2$.*

Proof. We have

$$\begin{aligned}
\Delta_{P_\gamma(\lambda)}(T(\lambda)) &= \sqrt{P_\gamma(\lambda)}V\sqrt{P_\gamma(\lambda)} \\
&= \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \ddots & & \\ & & \sqrt{\lambda_{n-1}} & \\ & & & \sqrt{\gamma} \end{pmatrix} \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \ddots & & \\ & & \sqrt{\lambda_{n-1}} & \\ & & & \sqrt{\gamma} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & & \cdots & 0 \\ \sqrt{\lambda_1\lambda_2} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \sqrt{\lambda_2\lambda_3} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \sqrt{\lambda_{n-2}\lambda_{n-1}} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{\gamma\lambda_{n-1}} & 0 \end{pmatrix}
\end{aligned}$$

From Proposition 7.2 we see that for $\Delta_{P_\gamma(\lambda)}$ to be complex symmetric one needs

$$\begin{cases} |\lambda_1\lambda_2| = |\gamma\lambda_{n-1}| \\ |\lambda_i\lambda_{i+1}| = |\lambda_{n-i}\lambda_{n-i+1}| \;\; for \;2 \le i \le n-2, \end{cases}$$

as desired. $\square$

Corollary 7.4. *Let $T(\lambda)$ and γ be as above. Then $T(\lambda)$ and its extended Aluthge transform $\Delta_{P_\gamma(\lambda)}(T(\lambda))$ are both complex symmetric if and only if*

(i) *(when n is odd)* $\gamma = |\lambda_1| = |\lambda_2| = \cdots = |\lambda_{n-1}|$;

(ii) *(when n is even)*

$$\begin{cases} |\lambda_1| = |\lambda_3| = \cdots = |\lambda_{n-1}| \\ \gamma = |\lambda_2| = |\lambda_4| = \cdots = |\lambda_{n-2}|\,. \end{cases}$$

Remark 7.5. (i) For $t \in [0,1]$, one may define the *generalized* extended Aluthge Transform as follows:

$$\Delta_{P_\gamma(\lambda)}(T(\lambda);t) := P_\gamma(\lambda)^t V P_\gamma(\lambda)^{1-t}.$$

As in the classical case, $\Delta_{P_\gamma(\lambda)}(T(\lambda);\frac{1}{2}) = \Delta_{P_\gamma(\lambda)}(T(\lambda))$ is the extended Aluthge transform, and $\Delta_{P_\gamma(\lambda)}(T(\lambda);0) = T$; also, $\Delta_{P_\gamma(\lambda)}(T(\lambda);1)$ is the analog of the so called Duggal transform.

(ii) As in the classical case, the generalized extended Aluthge transform of a complex symmetric operator may fail to be complex symmetric (except for the cases $t = 0$ and $t = \frac{1}{2}$ (see [37] and [5]). $\square$

Let us consider the same class of operators $T(\lambda)$, this time looking at the generalized extended Aluthge transforms. As before, P_γ is given by (7.1).

$$\Delta_{P_\gamma(\lambda)}(T(\lambda);t) := \big(P_\gamma(\lambda)\big)^t V \big(P_\gamma(\lambda)\big)^{1-t}$$

$$= \begin{pmatrix} \lambda_1^t & & & & \\ & \lambda_2^t & & & \\ & & \ddots & & \\ & & & \lambda_{n-1}^t & \\ & & & & \gamma^t \end{pmatrix} \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{1-t} & & & & \\ & \lambda_2^{1-t} & & & \\ & & \ddots & & \\ & & & \lambda_{n-1}^{1-t} & \\ & & & & \gamma^{1-t} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & & \cdots & 0 \\ \lambda_2^t\lambda_1^{1-t} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_3^t\lambda_2^{1-t} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \lambda_{n-1}^t\lambda_{n-2}^{1-t} & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \gamma^t\lambda_{n-1}^{1-t} & 0 \end{pmatrix}.$$

Using again Proposition 7.2, we obtain the following result.

Theorem 7.6. *Let $T(\lambda)$ and γ be as above. Then the generalized extended Aluthge transform $\Delta_{P_\gamma(\lambda)}(T(\lambda);t)$ is complex symmetric if and only if $\gamma^t\left|\lambda_{n-1}\right|^{1-t} = \left|\lambda_2\right|^t\left|\lambda_1\right|^{1-t}$ and $\left|\lambda_{i+1}\right|^t\left|\lambda_i\right|^{1-t} = \left|\lambda_{n-i+1}\right|^t\left|\lambda_{n-i}\right|^{1-t}$ for every $2 \le i \le n-2$.*

Rather surprisingly, for the case $\gamma > 0$, the generalized extended Aluthge transforms allows one to simplify the conditions describing complex symmetry, in the sense that if we start with $T(\lambda)$ complex symmetric, there is *one* condition that ensures that *all* $\Delta_{P_\gamma(\lambda)}(T(\lambda);t)$ are complex symmetric operators. Namely, we have the following result.

Corollary 7.7. *Let $T(\lambda)$ and γ be as above. Then $T(\lambda)$ and its generalized extended Aluthge transforms $\Delta_{P_\gamma(\lambda)}(T(\lambda);t)$ $(0 \le t \le 1)$ are all complex symmetric if and only if*

(i) *<u>(when n is odd)</u>* $\gamma = |\lambda_1| = |\lambda_2| = \cdots = |\lambda_{n-1}|$;

(ii) *<u>(when n is even)</u>*

$$\begin{cases} |\lambda_1| = |\lambda_3| = \cdots = |\lambda_{n-1}| \\ \gamma = |\lambda_2| = |\lambda_4| = \cdots = |\lambda_{n-2}|. \end{cases}$$

8. The numerical range and the extended Aluthge transform

For an operator $A \in \mathcal{B}(\mathcal{H})$, recall that the numerical range $W(A)$ of A is defined as

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ with } \|x\| = 1\}$$

The following lemmas are interesting and useful; they appear in [33] and the references therein.

Lemma 8.1. *Let A and B be operators on $\mathcal{H}$ such that $A = X^*BX$ for some contraction X. Then $W(A) \subseteq co(W(B) \cup \{0\})^\wedge$. If, in addition, X is a coisometry, then we also have $W(B) \subseteq W(A)$.*

Lemma 8.2. *(***Heinz inequality***) Let A, X and B be operators on $\mathcal{H}$, and assume that A and B are positive. Then the following inequalities hold:*

(i) $\|A^r X B^r\| \le \|AXB\|^r \|X\|^{1-r}$ *for* $r \in [0,1]$.
(ii) $\|A^r X B^r\| \ge \|AXB\|^r \|X\|^{1-r}$ *for* $r > 1$.

From the last result we can derive the following lemma (see [33]).

Lemma 8.3. *Let A and X be operators on $\mathcal{H}$, and assume that A is positive. Then*

$$\|A^r X A^{1-r} - zI\| \le \|AX - zI\|^r \|XA - zI\|^{1-r}, \quad \textit{for all } \ r \in [0,1] \textit{ and } \ z \in \mathbb{C}$$

The previous results were used to prove the following inclusion.

Theorem 8.4 ([33], Theorem 1)**.** *Let T be an operator on $\mathcal{H}$. Then*

$$\overline{W(\Delta(T))} \subseteq \overline{W(T)}$$

We now turn our attention to the extended Aluthge transform.

8.1. Numerical range for extended Aluthge transforms.

We begin with a natural question.

Question 8.5. *Is Theorem 8.4 still true for the extended Aluthge transform?*

We'll show here that Theorem 8.4 is not true for all extended Aluthge transforms.

However we have a relationship connecting the numerical ranges.
Recall that we have following decompositions

$$T = \begin{pmatrix} XA & 0 & 0 \\ YA & 0 & 0 \\ ZA & 0 & 0 \end{pmatrix} \quad \Delta(T) = \begin{pmatrix} \sqrt{A}X\sqrt{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Delta_P(T) = \begin{pmatrix} \sqrt{A}X\sqrt{A} & 0 & 0 \\ \sqrt{C}Y\sqrt{A} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, we have

$$\overline{W(\Delta(T))} \subseteq \overline{W(\Delta_P(T))} \quad \text{and} \quad \overline{W(\Delta(T))} \subseteq \overline{W(T)}.$$

It has been shown in [27, Theorem 4.1] that the numerical range of an upper triangular matrix of the form

$$A = \begin{pmatrix} 0 & 0 & 0 \\ x & p & 0 \\ y & z & p \end{pmatrix}$$

when $xyz = 0$ is the closed disc centered at p and with radius $\frac{1}{2}\sqrt{|x|^2 + |y|^2 + |z|^2}$.

So, if

$$T = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix},$$

and we recall that

$$P_\gamma = \begin{pmatrix} |\alpha| & 0 & 0 \\ 0 & |\beta| & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

it follows that

(i) $W(T) = \bar{\mathbf{D}}\big(0, \frac{1}{2}\sqrt{|\alpha|^2 + |\beta|^2}\big)$.

(ii) $W(\Delta(T)) = \bar{\mathbf{D}}\big(0, \frac{1}{2}\sqrt{|\alpha||\beta|}\big)$.

(iii) $W(\Delta_{P_\gamma}(T)) = \bar{\mathbf{D}}\Big(0, \frac{1}{2}\sqrt{\gamma|\alpha| + |\alpha||\beta|}\Big)$.

As expected, we have

$$\begin{cases} W(\Delta(T)) \subseteq W(T) \\ W(\Delta(T)) \subseteq W(\Delta_{P_\gamma}(T)). \end{cases}$$

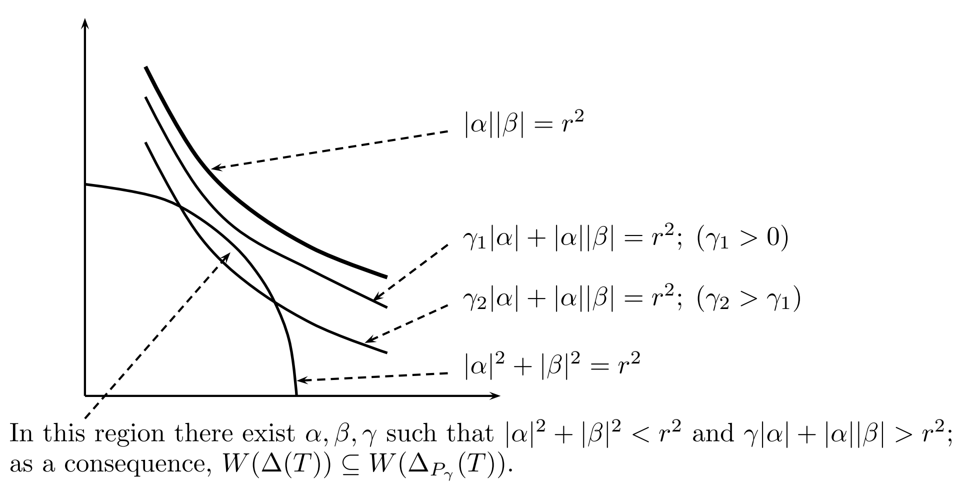

Figure 1. Graphs of radii in Remark 8.6

Remark 8.6. (i) We may choose α, β and γ in the previous example such that the inclusions above are strict; see Figure 1.

(ii) $\overline{W(\Delta_{P_\gamma}(T))}$ and $\overline{W(T)}$ are not comparable in general, unless we impose restrictions on γ. For example, observe that if $\gamma \leq |\beta|$ in the previous discussion, then $W(\Delta_{P_\gamma}(T)) \subseteq W(T)$, with equality holding if and only if $\gamma = |\alpha| = |\beta|$. $\square$

Acknowledgment. The authors are grateful to the referee for some helpful suggestions that improved the presentation.

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