A NEW APPROACH TO THE 2-VARIABLE SUBNORMAL COMPLETION PROBLEM

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Abstract. We study the Subnormal Completion Problem (SCP) for 2-variable weighted shifts. We use tools and techniques from the theory of truncated moment problems to give a general strategy to solve SCP. We then show that when all quadratic moments are known (equivalently, when the initial segment of weights consists of five independent data points), the natural necessary conditions for the existence of a subnormal completion are also sufficient. To calculate explicitly the associated Berger measure, we compute the algebraic variety of the associated truncated moment problem; it turns out that this algebraic variety is precisely the support of the Berger measure of the subnormal completion.

1. Introduction

We present a new approach to the Subnormal Completion Problem (SCP) for 2-variable weighted shifts. It employs the localizing matrices introduced and studied in [11] in the context of the truncated K-moment problem (K-TMP). This helps identify potential candidates for weights, and makes the problem more accessible.

We first give a general strategy to solve SCP, and we later apply it to solve the SCP with *quadratic data*. That is, given an initial set of weights Ω_1 consisting of five independent data points (α_{00}, β_{00}, α_{10}, α_{01} and β_{01}), we prove that the natural necessary condition for the existence of a subnormal completion is also sufficient. Concretely, associated to the five given weights is a 3×3 *moment matrix* $M(\Omega_1)$, whose positive semi-definiteness is a necessary condition for the existence of a subnormal completion; in symbols, $M(\Omega_1) := (\gamma_{\mathbf{u}+\mathbf{v}})_{\mathbf{u},\mathbf{v}\in\mathbb{Z}_+^2,|\mathbf{u}|,|\mathbf{v}|\leq 1}$, where $\gamma_{00} := 1$, $\gamma_{10} := \alpha_{00}^2$, $\gamma_{01} := \beta_{00}^2$, $\gamma_{20} := \alpha_{10}^2\alpha_{00}^2$, $\gamma_{11} := \alpha_{01}^2\beta_{00}^2$, and $\gamma_{02} := \beta_{01}^2\beta_{00}^2$. We prove that the necessary condition $M(\Omega_1) \geq 0$ turns out to be sufficient for the existence of a representing measure μ supported in $\mathbb{R}_+^2$ and satisfying the property supp $\mu \cap (0, +\infty)^2 \neq \varnothing$; the measure μ then gives rise to a subnormal completion of Ω_1. Once we know that a representing measure exists, we use techniques from the theory of truncated moment problems to find a concrete expression for it.

As a first step, we build *new* weights α_{20}, α_{11}, α_{02} and β_{02}, and we use them to construct the *localizing matrices* $M_x(\hat{\Omega}_3)$ and $M_y(\hat{\Omega}_3)$, where $\hat{\Omega}_3$ is a proposed extension of Ω_1. The positive semi-definiteness of $M(\Omega_1)$ is then used to establish that the localizing matrices $M_x(\hat{\Omega}_3)$ and $M_y(\hat{\Omega}_3)$ can be made positive semi-definite for suitable choices of the new weights α_{20}, α_{11}, α_{02} and β_{02}. That is, the condition $M(\Omega_1) \geq 0$ triggers the two conditions $M_x(\hat{\Omega}_3) \geq 0$ and $M_y(\hat{\Omega}_3) \geq 0$ for appropriate values of α_{20}, α_{11}, α_{02} and β_{02}. Once that happens, we prove that a *flat* (i.e., rank-preserving) extension $M(\hat{\Omega}_3)$ of $M(\Omega_1)$

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exists, thereby giving rise to a unique representing measure μ for $\hat{\Omega}_3$, which is the Berger measure of the subnormal completion. The explicit form of μ can be obtained by first determining the support of μ, which agrees with the algebraic variety of $\hat{\Omega}_3$.

In one variable, SCP was stated and solved in [7]:

Problem 1.1. *(One-Variable Subnormal Completion Problem) Given $m \geq 0$ and a finite collection of positive numbers $\Omega_m \equiv \{\alpha_k\}_{k=0}^m$, find necessary and sufficient conditions on Ω_m to guarantee the existence of a subnormal weighted shift whose initial weights are given by Ω_m.*

Since subnormality implies hyponormality, the condition $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$ is obviously necessary; moreover, it is easy to dispose of the case when $\alpha_k = \alpha_{k+1}$ for some $0 \leq k \leq m-1$, so one can always assume that $\alpha_0 < \alpha_1 < \cdots < \alpha_m$.

The cases $m = 0$ and $m = 1$ are straightforward, with canonical completions given by $\alpha_0, \alpha_0, \alpha_0, \cdots$ and $\alpha_0, \alpha_1, \alpha_1, \cdots$, respectively. The solution of the case $m = 2$ is based on the positivity of the moment matrix $H(1) := \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_2 \end{pmatrix}$ and of the localizing matrix $H_x(2) := \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix}$; the explicit calculation of the subnormal completion requires *recursively generated weighted shifts* [7, Example 3.12].

In the general (1-variable) case, the Subnormal Completion Criterion (SCC) [7, Theorem 3.5] states that a subnormal completion exists if and only if an ℓ-hyponormal completion exists, where $\ell := [\frac{m}{2}] + 1$.

Theorem 1.2. *(One-Variable Subnormal Completion Criterion; cf. [7, Theorem 3.5]) Let $\Omega_m \equiv \{\alpha_k\}_{k=0}^m$ be a finite collection of positive numbers, let $k := [\frac{m+1}{2}]$ and $\ell := [\frac{m}{2}] + 1$, and let $H(k) \equiv H(\Omega_m) := (\gamma_{i+j})_{0 \leq i,j \leq k}$, $H_x(\ell - 1) \equiv H_x(\Omega_m) := (\gamma_{i+j+1})_{0 \leq i,j \leq \ell-1}$ and $\mathbf{v}(i,j) := (\gamma_i \quad \gamma_{i+1} \quad \cdots \quad \gamma_{i+j})^T$. The following statements are equivalent.*
(i) Ω_m admits a subnormal completion;
(ii) Ω_m admits an ℓ-hyponormal completion;
(iii) $H(k) \geq 0$, $H_x(\ell - 1) \geq 0$, and $\mathbf{v}(k+1, k) \in Ran\ H(k)$ if m is even ($\mathbf{v}(\ell+1, \ell-1) \in Ran\ H_x(\ell-1)$ if m is odd);
(iv) $H(\Omega_m)$ admits a positive flat (i.e., rank-preserving) extension $H(\hat{\Omega}_{m+1})$ such that $H_x(\hat{\Omega}_{m+1}) \geq 0$.

We now formulate the 2-variable SCP:

Problem 1.3. *(2-variable Subnormal Completion Problem) Given $m \geq 0$ and a finite collection of pairs of positive numbers $\Omega_m \equiv \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}_{|\mathbf{k}| \leq m}$ satisfying (2.1) for all $|\mathbf{k}| \leq m$ (where $|\mathbf{k}| := k_1 + k_2$), find necessary and sufficient conditions to guarantee the existence of a subnormal 2-variable weighted shift whose initial weights are given by Ω_m.*

While the research in [7] provided a complete solution to SCP in one variable, the 2-variable version requires new tools and techniques. At present, no general solution exists, and the problem appears to be quite difficult. When $m = 0$, in one variable the canonical subnormal completion of α_0 is the weighted shift $\alpha_0, \alpha_0, \alpha_0, \ldots$, with *Berger measure* $\mu := \delta_{\alpha_0^2}$; in two variables, the canonical subnormal completion of $(\alpha_{00}, \beta_{00})$ is the 2-variable weighted with weight sequences $\alpha_{ij} := \alpha_{00}$ and $\beta_{ij} := \beta_{00}$ (all $i, j \geq 0$) and Berger measure $\mu := \delta_{\alpha_{00}^2} \times \delta_{\beta_{00}^2}$.

When $m = 1$, the 1-variable case is still straightforward; i.e., the canonical subnormal completion is $\alpha_0, \alpha_1, \alpha_1, \ldots$, with Berger measure $(1 - \frac{\alpha_0^2}{\alpha_1^2})\delta_0 + \frac{\alpha_0^2}{\alpha_1^2}\delta_{\alpha_1^2}$. In two variables, however, the problem becomes highly nontrivial. For the singular case, and using the results in [9, Section 6], C. Li gave in [21] a solution, which seems a bit ad hoc and unmotivated, with extensive calculations using *Mathematica*. The proof in [9, pages 39 and 40] establishes the existence of a representing measure μ for SCP with quadratic moment data (this is the case $m = 1$ in two variables); however, the ensuing statement that supp $\mu \subseteq \mathbb{R}_+^2$ is made without a proof, and it does not appear to follow easily from the comments preceding it. It is indeed true, as we show in the present paper using the tools and techniques from [11].

In Section 5 below, we shall apply our general strategy to solve SCP to the case $m = 1$ and prove that a representing measure always exist if the associated moment matrix $M(1)$ is positive semi-definite. In Section 6 we shall calculate the Berger measure using canonical column relations in the flat extension $M(2)$ of $M(1)$. The reader will note how effective the theory of truncated moment problems can be in detecting the location of the atoms of the unique representing measure for $M(2)$; this is in sharp contrast with the ad hoc techniques and extensive symbolic manipulation present in [21].

The case $m = 2$, in full generality, remains open; however, in Example 4.4 below we solve SCP whenever the associated moment matrix $M(1)$ is singular. For $m \geq 3$, the results in [7] and [8] show that, in the 1-variable case, it is not always possible to build a subnormal completion; of course the same is true in two variables: indeed, if $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ is a collection of weights admitting no subnormal completion, it suffices to consider the 2-variable collection given by $\alpha_{\mathbf{k}} := \alpha_{k_1}$ and $\beta_{\mathbf{k}} := 1$ ($|\mathbf{k}| \leq 3$) in order to produce such an example.

Problem 1.3 is closely related to truncated moment problems. Given real numbers $\gamma \equiv \gamma^{(2n)} := \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \cdots, \gamma_{02n}, \cdots, \gamma_{2n0}$ with $\gamma_{00} > 0$, the *truncated real moment problem* for γ entails finding conditions for the existence of a positive Borel measure μ, supported in $\mathbb{R}^2$, such that

$$\gamma_{ij} = \int y^i x^j \, d\mu, \quad 0 \leq i + j \leq n.$$

Given $\gamma \equiv \gamma^{(2n)}$, we can build an associated *moment matrix* $M(n) \equiv M(n)(\gamma) := (M[i,j](\gamma))_{i,j=0}^n$, where

$$M[i,j](\gamma) := \begin{pmatrix} \gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \gamma_{j,i} \\ \gamma_{1,i+j-1} & \gamma_{2,i+j-2} & \cdots & \gamma_{j+1,i-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \end{pmatrix}.$$

We denote the successive rows and columns of $M(n)(\gamma)$ by

$$1, X, Y, X^2, YX, Y^2, \cdots, X^n, \cdots, Y^n.$$

Observe that each block $M[i,j](\gamma)$ is of *Hankel* form, i.e., constant in cross-diagonals. (For basic results about truncated moment problems we refer to [9] and [11].)

We conclude this section by stating a result from [14], which we will need in Section 3. Recall that a commuting pair (T_1, T_2) is 2-*hyponormal* if the 5-tuple $(T_1, T_2, T_1^2, T_1T_2, T_2^2)$ is hyponormal (cf. Section 2 below). For 2-variable weighted shifts, this is equivalent to the condition

$$M_{\mathbf{u}}(2) := (\gamma_{\mathbf{u}+(m,n)+(p,q)})_{\substack{0 \leq m+n \leq 2 \\ 0 \leq p+q \leq 2}} \geq 0 \text{ (all } \mathbf{u} \in \mathbb{Z}_+^2\text{) (cf. [14, Theorem 2.4])},$$

that is,

$$\begin{pmatrix} \gamma_{\mathbf{u}} & \gamma_{\mathbf{u}+(0,1)} & \gamma_{\mathbf{u}+(1,0)} & \gamma_{\mathbf{u}+(0,2)} & \gamma_{\mathbf{u}+(1,1)} & \gamma_{\mathbf{u}+(2,0)} \\ \gamma_{\mathbf{u}+(0,1)} & \gamma_{\mathbf{u}+(0,2)} & \gamma_{\mathbf{u}+(1,1)} & \gamma_{\mathbf{u}+(0,3)} & \gamma_{\mathbf{u}+(1,2)} & \gamma_{\mathbf{u}+(2,1)} \\ \gamma_{\mathbf{u}+(1,0)} & \gamma_{\mathbf{u}+(1,1)} & \gamma_{\mathbf{u}+(2,0)} & \gamma_{\mathbf{u}+(1,2)} & \gamma_{\mathbf{u}+(2,1)} & \gamma_{\mathbf{u}+(3,0)} \\ \gamma_{\mathbf{u}+(0,2)} & \gamma_{\mathbf{u}+(0,3)} & \gamma_{\mathbf{u}+(1,2)} & \gamma_{\mathbf{u}+(0,4)} & \gamma_{\mathbf{u}+(1,3)} & \gamma_{\mathbf{u}+(2,2)} \\ \gamma_{\mathbf{u}+(1,1)} & \gamma_{\mathbf{u}+(1,2)} & \gamma_{\mathbf{u}+(2,1)} & \gamma_{\mathbf{u}+(1,3)} & \gamma_{\mathbf{u}+(2,2)} & \gamma_{\mathbf{u}+(3,1)} \\ \gamma_{\mathbf{u}+(2,0)} & \gamma_{\mathbf{u}+(2,1)} & \gamma_{\mathbf{u}+(3,0)} & \gamma_{\mathbf{u}+(2,2)} & \gamma_{\mathbf{u}+(3,1)} & \gamma_{\mathbf{u}+(4,0)} \end{pmatrix} \geq 0 \text{ (all } \mathbf{u} \in \mathbb{Z}_+^2\text{)}. \tag{1.1}$$

An entirely similar formulation exists for ℓ-hyponormality ($\ell \geq 1$), i.e., one requires

$$M_{\mathbf{u}}(\ell) := (\gamma_{\mathbf{u}+(m,n)+(p,q)})_{\substack{0 \leq m+n \leq \ell \\ 0 \leq p+q \leq \ell}} \geq 0 \text{ (all } \mathbf{u} \in \mathbb{Z}_+^2\text{) (cf. [14, Theorem 2.4])}. \tag{1.2}$$

2. Notation and Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$. For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S,T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive semi-definite on the direct sum of n copies of $\mathcal{H}$ (cf. [1], [15]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.

The Bram-Halmos criterion for subnormality states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([3], [4]). Using Choleski's algorithm for operator matrices [22], it is easy to see that this is equivalent to asserting that the k-tuple $(T, T^2, \cdots, T^k)$ is hyponormal for all $k \geq 1$.

For $k \geq 1$, we say that a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is k-*hyponormal* if $\mathbf{T}(k) := (T_1, T_2, T_1^2, T_2T_1, T_2^2, \cdots, T_1^k, T_2T_1^{k-1}, \cdots, T_2^k)$ is hyponormal (cf. [14]). Clearly, subnormal $\Rightarrow$ $(k+1)$-hyponormal $\Rightarrow$ k-hyponormal for every $k \geq 1$, and of course 1-hyponormality agrees with the usual definition of joint hyponormality. The multivariable Bram-Halmos criterion was obtained in [14], and its formulation is essentially identical to the 1-variable one: $\mathbf{T}$ is subnormal if and only if $\mathbf{T}(k)$ is hyponormal for all $k \geq 1$.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha \equiv \{\alpha_{\mathbf{k}}\}, \beta \equiv \{\beta_{\mathbf{k}}\} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \ \ (\text{all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{2.1}$$

In an entirely similar way one can define multivariable weighted shifts.

Given $\mathbf{k} \in \mathbb{Z}_+^2$, the *moment* of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \left\{ \begin{array}{cc} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{array} \right\}.$$

We remark that, due to the commutativity condition (2.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1, k_2).

We also recall a well known characterization of subnormality for multivariable weighted shifts [20], due to C. Berger (and independently to R. Gellar and L.J. Wallen [19]) in the 1-variable case: $\mathbf{T} \equiv (T_1, \cdots, T_n)$ is subnormal if and only if there is a probability measure μ (called the Berger measure of $\mathbf{T}$) defined on the n-dimensional rectangle $R = [0, a_1] \times \cdots \times [0, a_n]$ where $a_i = \|T_i\|^2$ such that $\gamma_{\mathbf{k}} = \int_R \mathbf{t}^{\mathbf{k}} d\mu(t) := \int_R t_1^{k_1} \cdots t_n^{k_n} d\mu(\mathbf{t})$, for all $\mathbf{k} \in \mathbb{Z}_+^n$.

Consider now a subnormal 1-variable weighted shift W_α, with Berger measure ξ, and let $h \geq 1$. If we let

$$\mathcal{M}_h := \bigvee \{e_n : n \geq h\} \tag{2.2}$$

denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h d\xi(t)$.

An important class of subnormal weighted shifts is obtained by considering measures μ with exactly two atoms t_0 and t_1. These shifts arise naturally in the Subnormal Completion Problem ([7], [8]) and in the theory of truncated moment problems (cf. [6], [9]). For $t_0, t_1 \in \mathbb{R}_+$, $t_0 < t_1$, and $\rho_0, \rho_1 > 0$, the moments of the 2-atomic measure $\mu := \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1}$ (here δ_p denotes the point-mass probability measure with support the singleton $\{p\}$) satisfy the 2-step recursive relation

$$\gamma_{n+2} = \varphi_0 \gamma_n + \varphi_1 \gamma_{n+1} \ (n \geq 0); \tag{2.3}$$

at the weight level, this can be written as $\alpha_{n+1}^2 = \frac{\varphi_0}{\alpha_n^2} + \varphi_1$ $(n \geq 0)$. The atoms t_0 and t_1 are the zeros of the generating function

$$g(t) := t^2 - \varphi_1 t - \varphi_0. \tag{2.4}$$

More generally, any finitely atomic Berger measure corresponds to a *recursively generated* subnormal weighted shift (i.e., one whose moments satisfy an r-step recursive relation); in fact, $r = \text{card supp } \mu$. In the special case of $r = 2$, the theory of recursively generated weighted shifts makes contact with the work of J. Stampfli in [23], in which he proved that given three positive numbers $\alpha_0 < \alpha_1 < \alpha_2$, it is always possible to find a subnormal weighted shift, denoted $W_{(\alpha_0, \alpha_1, \alpha_2)^\wedge}$, whose first three weights are α_0, α_1 and α_2. The shift $T \equiv W_{(\alpha_0, \alpha_1, \alpha_2)^\wedge}$ received special attention in [8], and has a 2-atomic Berger measure as above; letting $a := \alpha_0^2$, $b := \alpha_1^2$ and $c := \alpha_2^2$, we often refer to this shift as the *abc* shift. We will have occasion to use these shifts in Section 6.

3. Statement of the Subnormal Completion Problem

Definition 3.1. *Given $m \geq 0$ and a finite family of positive numbers $\Omega_m \equiv \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}_{|\mathbf{k}| \leq m}$, we say that a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ with weight sequences $\alpha_{\mathbf{k}}^{\mathbf{T}}$ and $\beta_{\mathbf{k}}^{\mathbf{T}}$ is a subnormal completion of Ω_m if (i) $\mathbf{T}$ is subnormal, and (ii) $(\alpha_{\mathbf{k}}^{\mathbf{T}}, \beta_{\mathbf{k}}^{\mathbf{T}}) = (\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$ whenever $|\mathbf{k}| \leq m$.*

Remark 3.2. Note that since a subnormal 2-variable weighted shift is necessarily commuting, Ω_m in Definition 3.1 satisfies the commutativity condition in (2.1). When a family of positive numbers has this property, we say that it is *commutative.*

Definition 3.3. *Given $m \geq 0$ and a finite family of positive numbers $\Omega_m \equiv \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}_{|\mathbf{k}| \leq m}$, we say that $\hat{\Omega}_{m+1} \equiv \{(\hat{\alpha}_{\mathbf{k}}, \hat{\beta}_{\mathbf{k}})\}_{|\mathbf{k}| \leq m+1}$ is an extension of Ω_m if $(\hat{\alpha}_{\mathbf{k}}, \hat{\beta}_{\mathbf{k}}) = (\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$ whenever $|\mathbf{k}| \leq m$. The degree of Ω_m, $\deg\ \Omega_m$, is $m+1$. When $m = 1$, we say that Ω_1 is quadratic. For $m = 2\ell + 1$, the moment matrix of Ω_m is*

$$M(\ell) \equiv M(\Omega_m) \equiv M_{\mathbf{0}}(\Omega_m) := (\gamma_{(i,j)+(p,q)})_{\substack{0 \leq i+j \leq m \\ 0 \leq p+q \leq m}}.$$

Observe that if $\hat{\Omega}_{m+1}$ is commutative, then so is Ω_m. For m odd, $M(\hat{\Omega}_{m+2})$ is an extension of $M(\Omega_m)$.

Notation 3.4. When $m = 1$, we shall let $a := \alpha_{00}^2$, $b := \beta_{00}^2$, $c := \alpha_{10}^2$, $d := \beta_{01}^2$, $e := \alpha_{01}^2$ and $f := \beta_{10}^2$. To be consistent with the commutativity of a 2-variable weighted shifts whose weight sequences satisfy (2.1), we shall always assume $af = be$. The moments of Ω_1 are

$$\begin{cases} \gamma_{00} := 1 \\ \gamma_{01} := a \quad \gamma_{10} := b \\ \gamma_{02} := ac \quad \gamma_{11} := be \quad \gamma_{20} := bd \end{cases},$$

and the associated moment matrix is

$$M(\Omega_1) := \begin{pmatrix} 1 & a & b \\ a & ac & be \\ b & be & bd \end{pmatrix}.$$

In this case, solving the SCP consists of finding a probability measure μ supported on $\mathbb{R}_+^2$ such that $\int_{\mathbb{R}_+^2} y^i x^j \, d\mu(x,y) = \gamma_{ij}$ $(i, j \geq 0,\ i+j \leq 2)$.

Associated with the measure μ of a subnormal completion is the moment matrix

$$M(2)[\mu] := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03}[\mu] & \gamma_{12}[\mu] & \gamma_{21}[\mu] \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12}[\mu] & \gamma_{21}[\mu] & \gamma_{30}[\mu] \\ \gamma_{02} & \gamma_{03}[\mu] & \gamma_{12}[\mu] & \gamma_{04}[\mu] & \gamma_{13}[\mu] & \gamma_{22}[\mu] \\ \gamma_{11} & \gamma_{12}[\mu] & \gamma_{21}[\mu] & \gamma_{13}[\mu] & \gamma_{22}[\mu] & \gamma_{31}[\mu] \\ \gamma_{20} & \gamma_{21}[\mu] & \gamma_{30}[\mu] & \gamma_{22}[\mu] & \gamma_{31}[\mu] & \gamma_{40}[\mu] \end{pmatrix} \quad \text{(cf. (1.1))}.$$

The (quartic) moments of μ give rise to an extension $\hat{\Omega}_3$ of Ω_1, so that $M(2)[\mu] = M(\hat{\Omega}_3)$. It is thus clear that a necessary condition for the existence of a measure μ is the positivity of $M(\hat{\Omega}_3)$, which in turn implies the positivity of $M(\Omega_1)$. If we now let $p := \hat{\alpha}_{20}^2$, $q := \hat{\alpha}_{11}^2$, $r := \hat{\alpha}_{02}^2$ and $s := \hat{\beta}_{02}^2$, we see that

$$M(\hat{\Omega}_3) := \begin{pmatrix} 1 & a & b & ac & be & bd \\ a & ac & be & acp & beq & bdr \\ b & be & bd & beq & bdr & bds \\ ac & acp & beq & \gamma_{04}[\mu] & \gamma_{13}[\mu] & \gamma_{22}[\mu] \\ be & beq & bdr & \gamma_{13}[\mu] & \gamma_{22}[\mu] & \gamma_{31}[\mu] \\ bd & bdr & bds & \gamma_{22}[\mu] & \gamma_{31}[\mu] & \gamma_{40}[\mu] \end{pmatrix}.$$

The localizing matrices $M_x(\hat{\Omega}_3)$ and $M_y(\hat{\Omega}_3)$ (cf. [11, Introduction]) are

$$M_x(\hat{\Omega}_3) = \begin{pmatrix} a & ac & be \\ ac & acp & beq \\ be & beq & bdr \end{pmatrix} \text{ and } M_y(\hat{\Omega}_3) = \begin{pmatrix} b & be & bd \\ be & beq & bdr \\ bd & bdr & bds \end{pmatrix}.$$

(The matrix $M_x(\hat{\Omega}_3)$ is the compression of $M(\hat{\Omega}_3)$ to the first three rows and to the columns indexed by monomials containing X, that is, X, X^2 and YX; the matrix $M_y(\hat{\Omega}_3)$ is defined similarly.) Observe that

$M_x(\hat{\Omega}_3) = M_{(0,1)}(1)$ and $M_y(\hat{\Omega}_3) = M_{(1,0)}(1)$ (cf. (1.1)). For the existence of a measure μ supported in $\mathbb{R}^2_+$, it is necessary to have $M_x(\hat{\Omega}_3) \geq 0$ and $M_y(\hat{\Omega}_3) \geq 0$.

In this paper we prove that starting with the positivity of $M(\Omega_1)$ alone, it is possible to choose new weights p, q, r and s to ensure the positivity of $M_x(\hat{\Omega}_3)$ and $M_y(\hat{\Omega}_3)$. We can do this while simultaneously building a positive flat moment matrix extension $M(\hat{\Omega}_3)$ of $M(\Omega_1)$. Once we establish the simultaneous positivity of $M(\hat{\Omega}_3)$, $M_x(\hat{\Omega}_3)$ and $M_y(\hat{\Omega}_3)$, the existence of a representing measure μ follows from the main result in [11]. We prove this in Section 5. In Section 6 we give a concrete description of μ in terms of the initial data a, b, c, d and e and the new weights p, q, r and s. First, we present in Section 4 an abstract solution to SCP, which uses our new approach, involving localizing matrices and the results in [11].

While the flat extension approach is successful in the case $m = 1$, it will not lead to a solution of SCP in all cases. Indeed, it is possible to build a moment matrix $M(2) \equiv M(\Omega_3)$ admitting a representing measure, but with no flat extension $M(3)$ (cf. Section 7 below). This shows that our approach, while very general, will not yield subnormal completions merely by one-step flat extension techniques. In many instances, solving SCP will require a finite sequence of rank-increasing extensions followed by a flat extension; this is despite the fact that for SCP one looks for a measure with support in the nonnegative quarter-plane. As a matter of fact, the "translation of support" technique we use in Section 7 shows that solving SCP is equivalent to solving K-TMP, where K is a compact set satisfying $K \subseteq \mathbb{R}^2_+$ and $K \cap (0, +\infty)^2 \neq \varnothing$. Thus, SCP is a special case of K-TMP, and it is natural to expect that qualitative aspects of TMP theory will be appropriately reflected in SCP.

4. Abstract Solution of SCP

In this section we will give an abstract solution of Problem 1.3. We first consider the main theorem in [11]. Although [11, Theorem 1.6] deals with truncated complex moment problems, there is an entirely equivalent version for the case of two real variables, which we now state.

Theorem 4.1. *Let $\mathcal{P} \equiv \{p_1, \ldots, p_N\} \subseteq \mathbb{C}[x, y]$ and define k_i by* deg $p_i = 2k_i$ *or* deg $p_i = 2k_i - 1$ $(1 \leq i \leq N)$*. There exists a* rank $M(n)$*-atomic representing measure for $\gamma^{(2n)}$ supported in $K_{\mathcal{P}} := \{(x, y) \in \mathbb{R}^2 : p_i(x, y) \geq 0,\ 1 \leq i \leq N\}$ if and only if $M(n) \geq 0$ and there is some flat extension $M(n+1)$ for which $M_{p_i}(n + k_i) \geq 0$ $(1 \leq i \leq N)$. In this case, the representing measure for $M(n+1)$ is* rank $M(n)$*-atomic, supported in $K_{\mathcal{P}}$, and with precisely* rank $M(n)$ − rank $M_{p_i}(n + k_i)$ *atoms in $\mathcal{Z}(p_i) := \{(x, y) \in \mathbb{R}^2 : p_i(x, y) = 0\}$ $(1 \leq i \leq N)$.*

With the aid of Theorem 4.1, we can now state and prove a result which gives a sufficient condition for the solubility of SCP in two variables. Our version does not completely match the conditions listed on Theorem 1.2, and we now explain why. In one variable, building a flat moment matrix extension of a Hankel matrix entails adding an extra row and an extra column, and checking that the rank is preserved. This entails checking the range condition in Theorem 1.2(iii) and ensuring that the new lower right-hand corner entry satisfies the requirement in Smul'jan's Lemma [22]:

Lemma 4.2. *(cf.* [9, Proposition 2.2]*) Consider the 2×2 block matrix $D := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Then*

$$D \geq 0 \iff A \geq 0,\ B = AW \text{ for some } W, \text{ and } C \geq W^*AW.$$

In two variables, what one adds is not a row and a column but instead a *block* of rows and a *block* of columns; while it is still possible to preserve the range condition, the new lower right-hand corner is not a number but a square matrix, which must necessarily be Hankel for the extension to be a moment matrix. One easily finds out that ℓ-hyponormality (cf. Theorem 1.2(ii)), while necessary, is no longer sufficient to prove the hankelicity of the new lower right-hand block. Thus, our result avoids mention of ℓ-hyponormality. Moreover, solving the SCP admits two structurally different cases: m odd and m even. In the former case, deg Ω_m $(= m+1)$ is even, so we have enough moments to build the moment matrix $M(\Omega_m)$.

The same is not true, however, when $m = 2k$, since we have moments up to degree $2k+1$, and this does not allow us to build a complete moment matrix. In the terminology of Lemma 4.2, we have $A := M(\Omega_{m-1})$, and also the B block (consisting of moments up to degree $m+1$), but no C block. Since we are seeking a moment matrix $M(\Omega_{m+1})$, with moments up to degree $2m+2$, we can certainly require that Ran $B \subseteq$ Ran $A \equiv$ Ran $M(\Omega_{m-1})$, but that in itself does not generate the additional moments. One could attempt to define the C block as W^*AW (where W solves the equation $AW = B$), but this in general does not produce a Hankel block C, as has been observed in [12]. Therefore, it becomes necessary to postulate the existence of moments of degree $m+1$ that, together with the initial data Ω_m, allows us to build a moment matrix, which we will call $M(\Omega_{m+1})$.

Theorem 4.3. *Let $\Omega_m := \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}) : |\mathbf{k}| \le m\}$ be an initial set of positive weights satisfying the commutativity condition $\beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}}$ (all $\mathbf{k} \in \mathbb{Z}_+^2$ with $|\mathbf{k}+\varepsilon_i| \le m$ $(i = 1, 2)$), and let $\tilde{m} := 2\left[\frac{m}{2}\right] + 1$; thus $\tilde{m} = m$ if m is odd and $\tilde{m} = m+1$ if m is even. Assume that $M(\Omega_{\tilde{m}}) \ge 0$, and that $\Omega_{\tilde{m}}$ admits a commutative extension $\hat{\Omega}_{\tilde{m}+2}$ such that the moment matrix $M(\hat{\Omega}_{\tilde{m}+2})$ is a flat (i.e., rank-preserving) extension of $M(\Omega_{\tilde{m}})$, with $M_x(\hat{\Omega}_{\tilde{m}+2}) \ge 0$ and $M_y(\hat{\Omega}_{\tilde{m}+2}) \ge 0$. Then there exists a* rank $M(\Omega_{\tilde{m}})$*-atomic representing measure μ supported in $\mathbb{R}_+^2$, with precisely* rank $M(\Omega_{\tilde{m}})$ − rank $M_x(\hat{\Omega}_{\tilde{m}+2})$ *atoms in $\{0\} \times \mathbb{R}_+$ (resp.* rank $M(\Omega_{\tilde{m}})$ − rank $M_y(\hat{\Omega}_{\tilde{m}+2})$ *atoms in $\mathbb{R}_+ \times \{0\}$). The measure μ is the Berger measure of a subnormal completion $\hat{\Omega}_\infty$ of Ω_m, provided at least one atom of μ lies inside the positive quadrant in $\mathbb{R}^2$.*

Proof. In the case at hand, the polynomials p_i are $p_1(x, y) := x$ and $p_2(x, y) := y$; thus, $k_1 = k_2 = 1$. It follows that $K_{\mathcal{P}} = \mathbb{R}_+^2$ and that $M_{p_1}(n + k_1) = M_x(n+1)$ and $M_{p_2}(n + k_2) = M_y(n+1)$. Our result now follows from a straightforward application of Theorem 4.1. □

Despite its simplicity, Theorem 4.3 is quite useful, as we will see in the next section. We conclude this section by showing how the additional moments required in case m is even are sometimes determined by $M(\Omega_{m-1})$.

Example 4.4. Let $m = 2$ and assume that $A := M(\Omega_1) \ge 0$ and $\det A = 0$. Then there exist moments $\gamma_{i,j}$ $(i + j = 4)$ such that $M(\Omega_3) \ge 0$ is a flat extension of A. The case when rank $A = 1$ is easily disposed of, so without loss of generality we focus on the case $Y = a1 + bX$ in the column space of A. We are assuming that $A \ge 0$, $M_x(\Omega_3) \ge 0$, $M_y(\Omega_3) \ge 0$ and Ran $B \subseteq$ Ran A. (Observe that $M_x(\Omega_3)$ and $M_y(\Omega_3)$ include moments up to degree 3, so building them requires no new moments.) The equation det $A = 0$ uniquely determines γ_{02}, from which we obtain at once the weight

$$\beta_{01} = \frac{\alpha_{00}^2\beta_{00}^2\alpha_{10}^2 - 2\alpha_{00}^2\beta_{00}^2\alpha_{01}^2 + \beta_{00}^2\alpha_{01}^4}{\alpha_{00}^2(\alpha_{10}^2 - \alpha_{00}^2)}.$$

Since Ran $B \subseteq$ Ran A, each column in B must be a linear combination of the columns 1 and X, and straightforward calculations using *Mathematica* yield unique values for α_{20}, α_{11} and α_{02}. Concretely,

$$\begin{aligned}
\alpha_{20}^2 &= \frac{\alpha_{00}^2\alpha_{10}^4 - \alpha_{00}^2\alpha_{10}^2\alpha_{01}^2 + \alpha_{00}^2\alpha_{01}^2\alpha_{11}^2 - \alpha_{10}^2\alpha_{01}^2\alpha_{11}^2}{\alpha_{10}^2(\alpha_{00}^2 - \alpha_{01}^2)} \\
\alpha_{11}^2 &= \frac{\alpha_{00}^2\alpha_{10}^2\alpha_{01}^2 - \alpha_{00}^2\alpha_{01}^4 - \alpha_{00}^2\alpha_{10}^2\alpha_{02}^2 + 2\alpha_{00}^2\alpha_{01}^2\alpha_{02}^2 - \alpha_{01}^4\alpha_{02}^2}{\alpha_{01}^2(\alpha_{00}^2 - \alpha_{01}^2)} \\
\alpha_{02}^2 &= \frac{\alpha_{00}^2(\beta_{00}^2\alpha_{10}^2 - \beta_{00}^2\alpha_{01}^2 + \alpha_{00}^2\beta_{02}^2 - \alpha_{10}^2\beta_{02}^2)}{\beta_{00}^2(\alpha_{00}^2 - \alpha_{01}^2)}.
\end{aligned}$$

With this information at our disposal, it is now straightforward to check that the C block, defined as $C := mW^*AW$ (where $AW = B$) is Hankel. Thus, $M(\Omega_3) := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is a moment matrix extension of A, and moreover rank $M(\Omega_3) =$ rank $A = 2$. It is now clear that SCP admits a solution in this particular case.

One might wish to extend the above reasoning to the case rank $A = 3$, as follows. Let $W := A^{-1}B$ and let $C := W^*AW$. It is well known that C is in general not Hankel, and that one can make it Hankel by adding a rank-one positive matrix P. Thus, $M(\Omega_3) := \begin{pmatrix} A & B \\ B^* & C+P \end{pmatrix}$ is a positive moment matrix, and rank $M(\Omega_3) = 4$. The solution of the Quartic Moment Problem [12] now says that there exists a flat extension $M(\Omega_5)$ of $M(\Omega_3)$. Unfortunately, we can't tell whether the support of the representing measure for $M(\Omega_5)$ is contained in the first quadrant in $\mathbb{R}^2$. This would require verifying that the localizing matrices $M_x(\Omega_5)$ and $M_y(\Omega_5)$ are positive. If we knew that they are flat extensions of $M_x(\Omega_3)$ and $M_y(\Omega_3)$, resp., then of course we would be done. This fact is false in general, but it might be true in the context of SCP; however, we have not been able to prove it for SCP.

5. Localizing Matrices as Flat Extension Builders

We now specialize to the case $m = 1$ in two variables, and show that the condition $M(\Omega_1) \geq 0$ is sufficient for the existence of a subnormal completion.

Theorem 5.1. *Let Ω_1 be a quadratic, commutative, initial set of positive weights, and assume $M(\Omega_1) \geq 0$. Then there always exists a quartic commutative extension $\hat{\Omega}_3$ of Ω_1 such that $M(\hat{\Omega}_3)$ is a flat extension of $M(\Omega_1)$, and $M_x(\hat{\Omega}_3) \geq 0$ and $M_y(\hat{\Omega}_3) \geq 0$. As a consequence, Ω_1 admits a subnormal completion $\mathbf{T}_{\hat{\Omega}_\infty}$.*

Proof. Since $m = 1$, we have $\ell = 1$. By Theorem 4.3, we first need to show that six new weights, $\hat{\alpha}_{20}, \hat{\beta}_{20}, \hat{\alpha}_{11}, \hat{\beta}_{11}, \hat{\alpha}_{02}$ and $\hat{\beta}_{02}$ can be chosen in such a way that $M_x(\hat{\Omega}_3) \geq 0$ and $M_y(\hat{\Omega}_3) \geq 0$. Once we prove this, we shall employ techniques from truncated moment problems to establish the existence of a flat extension $M(\hat{\Omega}_3)$ of $M(\Omega_1)$. We will then appeal to the main result in [11]; the existence of a flat extension will readily imply the existence of a representing measure μ for $M(1)$, and the positivity of the localizing matrices $M_x(2)$ and $M_y(2)$ means that supp $\mu \subseteq \mathbb{R}_+^2$. Thus, μ will be the Berger measure of a subnormal 2-variable weighted shift $\mathbf{T}_{\Omega_\infty}$, which will be the desired subnormal completion of Ω_1.

We now build $M(2)$. To simplify the calculations, we let

$$\left\{ \begin{array}{ll} a := \alpha_{00}^2 & b := \beta_{00}^2 \\ c := \alpha_{10}^2 & d := \beta_{01}^2 \\ e := \alpha_{01}^2 & f := \beta_{10}^2 \end{array} \right. .$$

(The family Ω_1 is shown in Figure 1.)

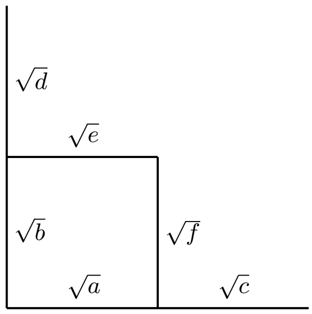

FIGURE 1. The initial family of weights Ω_1

Thus,

$$M(1) = \begin{pmatrix} 1 & a & b \\ a & ac & be \\ b & be & bd \end{pmatrix}. \tag{5.1}$$

Since $M(1) \geq 0$, it follows that det $\begin{pmatrix} ac & be \\ be & bd \end{pmatrix} \geq 0$, i.e.,

$$acd \geq be^2. \tag{5.2}$$

By the commutativity of Ω_1, we have

$$af = be, \tag{5.3}$$

and therefore

$$cd \geq ef. \tag{5.4}$$

A straightforward calculation shows that

$$\det\ M(1) = acbd - b^2e^2 - a^2bd + 2ab^2e - b^2ac$$

and that

$$\det\ M(1) > 0 \Longrightarrow cd - ef > 0; \tag{5.5}$$

for, if $cd - ef = 0$ then the rank of the 2×2 lower right-hand corner of $M(1)$ is 1, and then $M(1)$ cannot be invertible. Inspection of (5.4) reveals that we must have $c \geq e$ or $d \geq f$. Without loss of generality, we shall assume that $c \geq e$. We also assume that $a < c$, since otherwise a trivial solution exists. (In fact, if $a = c$ in (5.1), the positivity of $M(1)$ implies that $a = e$ and $b = f \leq d$; when $b = d$ (resp. $b < d$), the point mass $\delta_{(a,b)}$ is the Berger measure of the subnormal completion (resp. $(1 - \frac{b}{d})\delta_{(a,0)} + \frac{b}{d}\delta_{(a,d)}$). Thus, in what follows we shall always assume $c \geq e$ and $a < c$.

To build $M(2) \equiv M(\hat{\Omega}_3)$, we first need six new weights (the quadratic weights), namely $\hat{\alpha}_{20}$, $\hat{\beta}_{20}$, $\hat{\alpha}_{11}$, $\hat{\beta}_{11}$, $\hat{\alpha}_{02}$ and $\hat{\beta}_{02}$. Since the extension $\hat{\Omega}_3$ will also be commutative, two of these weights will be expressible in terms of other weights. We thus denote $\hat{\alpha}_{20}$ by $\sqrt{p}$, $\hat{\alpha}_{11}$ by $\sqrt{q}$, $\hat{\alpha}_{02}$ by $\sqrt{r}$, and $\hat{\beta}_{02}$ by $\sqrt{s}$

($\hat{\beta}_{20}$ and $\hat{\beta}_{11}$ can be written in terms of the other four new weights). It follows that

$$M(2) = \begin{pmatrix} 1 & a & b & ac & be & bd \\ a & ac & be & acp & beq & bdr \\ b & be & bd & beq & bdr & bds \\ ac & acp & beq & & & \\ be & beq & bdr & & & \\ bd & bdr & bds & & & \end{pmatrix} \tag{5.6}$$

(with the lower right-hand 3×3 corner yet undetermined) and

$$M_x(2) = \begin{pmatrix} a & ac & be \\ ac & acp & beq \\ be & beq & bdr \end{pmatrix} \text{ and } M_y(2) = \begin{pmatrix} b & be & bd \\ be & beq & bdr \\ bd & bdr & bds \end{pmatrix}.$$

Now, since the zero-th row of a subnormal completion of Ω_1 will be a subnormal completion of the zero-th row of Ω_1, which is given by the weights $a \le c$, we let $p := c$. By one of the main results in [18], having $\alpha_{10} = \hat{\alpha}_{20}$ immediately implies that $\hat{\alpha}_{11} = \sqrt{c}$, that is, $q := c$. Thus,

$$M_x(2) = \begin{pmatrix} a & ac & be \\ ac & ac^2 & bce \\ be & bce & bdr \end{pmatrix}.$$

By Choleski's Algorithm [2], $M_x(2) \ge 0$ if and only if $bdr \ge \frac{(be)^2}{a}$, so that we need $r \ge \frac{ef}{d}$. Thus, provided we take $r \ge \frac{ef}{d}$, the positivity of $M_x(2)$ is guaranteed. It remains to show that we can choose s in such a way that $s \ge d$ and $M_y(2) \equiv M_y(2)(s) \ge 0$. We consider two cases.

Case 1: $e = c$. By (5.4) we have $d \ge f$, so we can take $r := c$ and guarantee that $M_x(2) \ge 0$. We also let $s := d$. We then have

$$M_y(2) = \begin{pmatrix} b & bc & bd \\ bc & bc^2 & bcd \\ bd & bcd & bd^2 \end{pmatrix}.$$

It follows at once that rank $M_y(2) = 1$, and therefore $M_y(2) \ge 0$ (and of course $s \ge d$).

Case 2: $e < c$. We define r by this extremal value, i.e., $r := \frac{ef}{d}$. This immediately implies that $\hat{\beta}_{11} := \sqrt{f}$, and by propagation, $\hat{\beta}_{1j} := \sqrt{f}$ (all $j \ge 2$) in any subnormal completion. The resulting weight diagram is shown in Figure 2.

It remains to define s, in such a way that $s \ge d$ and $M_y(2) \ge 0$. Since

$$M_y(2) \equiv M_y(2)(s) = \begin{pmatrix} b & be & bd \\ be & bce & bef \\ bd & bef & bds \end{pmatrix}$$

and the 2×2 upper left-hand corner of $M_y(2)$ is invertible, we see that $M_y(2) \ge 0$ if and only if det $M_y(2)(s) \ge 0$. Since det $M_y(2)(s)$ is linear in s, we pick for s the unique value that makes det $M_y(2)(s) = 0$. A straightforward calculation shows that

$$s = \frac{a^2cd^2 - 2abde^2 + b^2e^3}{a^2d(c-e)}.$$

We then have

$$s - d = \frac{e(ad - be)^2}{a^2d(c-e)} \ge 0.$$

Thus, this particular choice of s guarantees both $s \ge d$ and $M_y(2) \ge 0$.

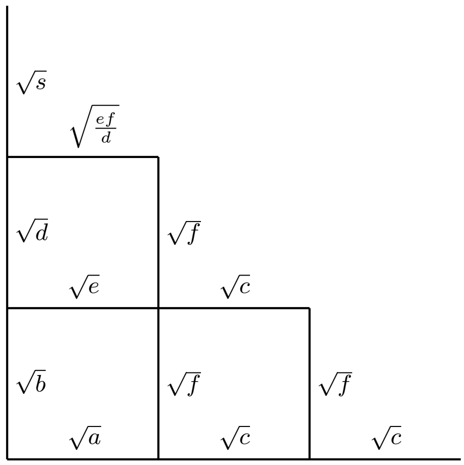

FIGURE 2. The family Ω_1 augmented with the inclusion of the quadratic weights

To complete the proof, we need to define the 3×3 lower right-hand corner of $M(2)$, and then show that $M(2)$ is a flat extension of $M(1)$, and therefore $M(2) \geq 0$. We consider the following two cases.

(i) rank $M(1) = 2$: Without loss of generality, we may assume that $a < c$, so that the columns 1 and X of $M(1)$ are linearly independent. The column Y must then be a linear combination of 1 and X, and that allows us to define YX and Y^2 in $M(2)$. Moreover, since the zero-th row of $\mathbf{T}_{\hat{\Omega}_\infty}$ is given by the weights $\sqrt{a}, \sqrt{c}, \sqrt{c}, \cdots$, whose Berger measure is $\xi_x = (1 - \frac{a}{c})\delta_0 + \frac{a}{c}\delta_c$ (and thus supported in the two-point set $\{0, c\}$), it is natural to let $X^2 := cX$ in the column space of $M(2)$. With these definitions, one easily verifies that the truncations to the first three rows of X^2, YX and Y^2 agree with the 3×3 upper right-hand corner of the matrix $M(2)$ in (5.6). It is clear that the matrix $M(2)$ thus defined is positive semi-definite, but one needs to verify that $M(2)$ is a moment matrix. This amounts to checking that the $(4, 6)$ and $(5, 5)$ entries are equal. Now, a straightforward calculation shows that in the column space of $M(1)$ we have

$$Y = \frac{b(c-e)}{c-a} \cdot 1 + \frac{f-b}{c-a} X, \tag{5.7}$$

so that

$$\begin{aligned} M(2)_{46} &= \left\langle Y^2, X^2 \right\rangle = \left\langle Y^2, cX \right\rangle \\ &= c \left\langle Y, YX \right\rangle = c \left\langle \frac{b(c-e)}{c-a} \cdot 1 + \frac{f-b}{c-a} X, YX \right\rangle \\ &= c\frac{b(c-e)}{c-a} be + c\frac{f-b}{c-a} bce \\ &= bce\frac{cf-be}{c-a} = bcef. \end{aligned}$$

On the other hand, using (5.7) we define $YX := \frac{b(c-e)}{c-a}X + \frac{f-b}{c-a}X^2$, so that

$$\begin{aligned} M(2)_{55} &= \langle YX, YX \rangle \\ &= \left\langle \frac{b(c-e)}{c-a}X + \frac{f-b}{c-a}X^2, YX \right\rangle \\ &= \frac{b(c-e)}{c-a}bce + \frac{f-b}{c-a}\langle cX, YX \rangle \\ &= \frac{b(c-e)}{c-a}bce + c\frac{f-b}{c-a}bce \\ &= bce\frac{cf-be}{c-a} = bcef. \end{aligned}$$

It follows that $M(2)_{46} = M(2)_{55}$, as desired. In this case, the representing measure is supported in the two-point set $\{(0, y_0), (c, y_c)\}$, where

$$y_0 := \frac{b(c-e)}{c-a} \tag{5.8}$$

and

$$y_c := \frac{b(c-e)}{c-a} + \frac{f-b}{c-a}c = \frac{cf-be}{c-a} = f. \tag{5.9}$$

(iii) rank $M(1) = 3$: We let B denote the upper right-hand corner of $M(2)$, that is,

$$B := \begin{pmatrix} ac & be & bd \\ acp & beq & bdr \\ beq & bdr & bds \end{pmatrix} = \begin{pmatrix} ac & be & bd \\ ac^2 & bce & bdr \\ bce & bdr & bds \end{pmatrix}.$$

We also let C denote the lower right-hand corner of $M(2)$. Since we want rank $M(2) =$ rank $M(1) = 3$, we must define $C := B^T M(1)^{-1} B$. Again, we need to verify that $M(2)_{46} = M(2)_{55}$, i.e., $C_{13} = C_{22}$. A straightforward calculation shows that

$$C_{13} = bcdr.$$

When $c > e$, we have $r = \frac{ef}{d}$, and another calculation shows that

$$C_{22} = \frac{b^2ce^2}{a};$$

it is then immediate that $C_{13} = C_{22}$. When $c = e$, we have $r = c$, and in this case $C_{13} = C_{22} = bc^2d$, as desired.

The proof of the Theorem is now complete. □

6. Description of the Representing Measure

In this section we provide a concrete description of the Berger measure for the subnormal completion in Theorem 5.1. We have already observed that when rank $M(1) = 1$, the representing measure is $\mu = \delta_{(a,b)}$. When rank $M(1) = 2$ (and the columns 1 and X linearly independent), there is a 2-atomic representing measure, with atoms $(0, y_0)$ and (c, y_c) given by (5.8) and (5.9); thus, $\mu = \rho_{(0,y_0)}\delta_{(0,y_0)} + \rho_{(c,y_c)}\delta_{(c,y_c)}$. To find the densities $\rho_{(0,y_0)}$ and $\rho_{(c,y_c)}$, we use the first two moments: $\int d\mu = \rho_{(0,y_0)} + \rho_{(c,y_c)} = 1$ and $\int s\, d\mu = c\rho_{(c,y_c)} = a$. It follows that the densities are $\rho_{(0,y_0)} = 1 - \frac{a}{c}$ and $\rho_{(c,y_c)} = \frac{a}{c}$. Thus, $\mu = (1 - \frac{a}{c})\delta_{(0,y_0)} + \frac{a}{c}\delta_{(c,y_c)}$.

We now focus on the case rank $M(1) = 3$. Since $M(1)$ is invertible, the last three columns of the flat extension $M(2)$ can be written in terms of the first three columns; that is, the columns labeled X^2, YX and Y^2 are linear combinations of 1, X and Y. Each of these column relations is associated with a quadratic polynomial in x and y, whose zero sets give rise to the so-called *algebraic variety* of $\hat{\Omega}_3$ [13]; concretely, $\mathcal{V}(\hat{\Omega}_3) := \bigcap_{p(X,Y)=0,\ \deg\ p\leq 2} \mathcal{Z}(p)$, where $\mathcal{Z}(p)$ denotes the zero set of p. In our case, the three column relations are

$$\begin{aligned} X^2 &= cX \\ YX &= fX \\ Y^2 &= \frac{be(f-d)}{a(c-e)}X + \frac{cd-ef}{c-e}Y. \end{aligned}$$

The associated zero sets are

$$\begin{aligned} \{(x,y) &: x = 0 \text{ or } x = c\} \\ \{(x,y) &: x = 0 \text{ or } y = f\} \\ \{(x,y) &: y^2 = \frac{be(f-d)}{a(c-e)}x + \frac{cd-ef}{c-e}y\}. \end{aligned}$$

Let $z := \frac{cd-ef}{c-e}$ and observe that $z > 0$ by (5.5). The algebraic variety of $\hat{\Omega}_3$ is then $\mathcal{V}(\hat{\Omega}_3) = \{(0,0),(0,z),(c,f)\}$ and these are the three atoms of the unique representing measure for $M(2)$. To find the densities, we use the first three moments, γ_{00}, γ_{01} and γ_{10}:

$$\begin{cases} \rho_{(0,0)} + \rho_{(0,z)} + \rho_{(c,f)} &= 1 \\ \rho_{(c,f)}c &= a \\ \rho_{(0,z)}z + \rho_{(c,f)}f &= b. \end{cases}$$

We obtain

$$\begin{aligned} \rho_{(c,f)} &= \frac{a}{c} \\ \rho_{(0,z)} &= \frac{1}{z}(b - \frac{a}{c}f) = \frac{b(c-e)^2}{c(cd-ef)} \\ \rho_{(0,0)} &= 1 - \rho_{(0,z)} - \rho_{(c,f)} = \frac{1}{ab(cd-ef)} \det\ M(1). \end{aligned}$$

Thus, the representing measure is

$$\mu = \frac{\det\ M(1)}{ab(cd-ef)}\delta_{(0,0)} + \frac{b(c-e)^2}{c(cd-ef)}\delta_{(0,z)} + \frac{a}{c}\delta_{(c,f)}. \tag{6.1}$$

Direct calculation shows that $\int s^2\, d\mu(s,t) = \frac{a}{c}c^2 = ac$, $\int st\, d\mu(s,t) = \frac{a}{c}cf = af = be$, and

$$\begin{aligned} \int t^2\, d\mu(s,t) &= \frac{b(c-e)^2}{c(cd-ef)}z^2 + \frac{a}{c}f^2 = \frac{b(c-e)^2}{c(cd-ef)}(\frac{cd-ef}{c-e})^2 + \frac{bef}{c} \\ &= \frac{b(cd-ef)}{c} + \frac{bef}{c} = bd, \end{aligned}$$

so that μ correctly interpolates Ω_1.

Recall now that the marginal measures ν^X and ν^Y associated to a Borel measure ν on the Cartesian product $X \times Y$ are given by $\nu^X(E) := \nu(E \times Y)$ and $\nu^Y(F) := \nu(X \times F)$, for E and F Borel sets. In the specific case of the measure μ in (6.1), observe that the marginal measures μ^X and μ^Y are $(1-\frac{a}{c})\delta_0 + \frac{a}{c}\delta_c$ and $\frac{\det\ M(1)}{ab(cd-ef)}\delta_0 + \frac{b(c-e)^2}{c(cd-ef)}\delta_z + \frac{a}{c}\delta_f$, respectively. While μ^X is always 2-atomic, μ^Y is 3-atomic if and only if $z \neq f$. When μ^Y is 3-atomic, its moments (which are also the moments of an associated unilateral weighted shift W_η) satisfy the recursive relation $\gamma_{n+2} = -fz\gamma_n + (f+z)\gamma_{n+1}$ (all $n \geq 1$), with $\gamma_0 = 1$, $\gamma_1 = b$ and $\gamma_2 = bd$. It is easy to see that the restriction of W_η to the invariant subspace M_1 defined in

(2.2) has Berger measure $\frac{1}{\gamma_1} t\, d\mu^Y(t) = \frac{\rho_{(0,z)} z}{b} \delta_z(t) + \frac{\rho_{(c,f)} f}{b} \delta_f(t)$, whose recursive coefficients are $-zf$ and $z+f$, respectively.

On the other hand, it is indeed possible to have $z = f$, which occurs precisely when $d = f$. In that case, the three atoms of μ are $(0,0)$, $(0,f)$ and (c,f), and the unilateral weighted shift associated with μ^Y is $W_{(b,d,d,\cdots)}$. The reader will note that the location of these atoms can also be predicted by Theorem 4.3, once we observe that rank $M_x(2) = 1$ and rank $M_y(2) = 2$.

7. Flat Extensions May Not Exist

We now present an example of a set Ω_3 for which the associated moment matrix $M(2)$ admits a representing measure, but such that $M(2)$ has no flat extension $M(3)$. Thus, while Theorem 4.3 provides a general sufficient condition for solving SCP, not all SCP will fit that framework, and their associated moment matrices $M(\Omega_{\tilde{m}})$ will require a sequence of moment matrix extensions $M(\Omega_{\tilde{m}+2}), \cdots, M(\Omega_{\tilde{m}+2k})$, with $M(\Omega_{\tilde{m}+2k})$ admitting a flat extension $M(\Omega_{\tilde{m}+2(k+1)})$.

The example is motivated by the construction in [12, Examples 1.13 and 5.6], and also by [12, Proposition 1.12], which states that a TMP and its image under a degree-one transformation of the base space are equivalent as moment problems. In particular, the qualitative aspects of TMP are preserved under degree-one transformations; our idea is therefore to "translate" [12, Example 1.13] three units to the right and four units up, so that the support of the 6-atomic representing measure in [12, Example 1.13] will land in the positive quadrant. (We note that to produce a valid representing measure for SCP, it suffices to have all atoms in the nonnegative quadrant, and at least one atom in the positive quadrant.) To effectuate the above mentioned translation, we recall the definition of the Riesz functional L_γ associated to a TMP. The linear functional L_γ acts on polynomials by $L(y^i x^j) := \gamma_{ij}$. Given the moments γ_{ij}, one can translate the TMP by h units in the horizontal direction and k units in the vertical direction by letting $\tilde{\gamma}_{ij} \equiv \tilde{L}(v^i u^j) := L_\gamma((v+4)^i (u+3)^j)$. The associated moments of degree 4 are:

$$
\begin{array}{lllll}
\gamma_{00} = 1 & & & & \\
\gamma_{01} = 1 & \gamma_{10} = 1 & & & \\
\gamma_{02} = 2 & \gamma_{11} = 0 & \gamma_{20} = 3 & & \\
\gamma_{03} = 4 & \gamma_{12} = 0 & \gamma_{21} = 0 & \gamma_{30} = 9 & \\
\gamma_{04} = 9 & \gamma_{13} = 0 & \gamma_{22} = 0 & \gamma_{31} = 0 & \gamma_{40} = 28
\end{array}
\qquad
\begin{array}{lllll}
\tilde{\gamma}_{00} = 1 & & & & \\
\tilde{\gamma}_{01} = 4 & \tilde{\gamma}_{10} = 5 & & & \\
\tilde{\gamma}_{02} = 17 & \tilde{\gamma}_{11} = 19 & \tilde{\gamma}_{20} = 27 & & \\
\tilde{\gamma}_{03} = 76 & \tilde{\gamma}_{12} = 77 & \tilde{\gamma}_{21} = 97 & \tilde{\gamma}_{30} = 157 & \\
\tilde{\gamma}_{04} = 354 & \tilde{\gamma}_{13} = 331 & \tilde{\gamma}_{22} = 371 & \tilde{\gamma}_{31} = 535 & \tilde{\gamma}_{40} = 972.
\end{array}
$$

For example,

$$
\begin{aligned}
\tilde{\gamma}_{21} &= L_\gamma((v+4)^2(u+3)) = L_\gamma((v^2+8v+16)(u+3)) \\
&= L_\gamma(v^2u + 8vu + 3v^2 + 16u + 24v + 48) = \gamma_{21} + 8\gamma_{11} + 3\gamma_{20} + 16\gamma_{01} + 24\gamma_{10} + 48\gamma_{00} \\
&= 0 + 8 \cdot 0 + 3 \cdot 3 + 16 \cdot 1 + 24 \cdot 1 + 48 \cdot 1 = 97.
\end{aligned}
$$

With the new moments at hand, we form the matrix $M(2)$. The corresponding weights are:

$$
\begin{array}{llll}
\alpha_{03} = \frac{\sqrt{535}}{\sqrt{157}} & & & \\
\alpha_{02} = \frac{\sqrt{97}}{3\sqrt{3}} & \alpha_{12} = \frac{\sqrt{371}}{\sqrt{97}} & & \\
\alpha_{01} = \frac{\sqrt{19}}{\sqrt{5}} & \alpha_{11} = \frac{\sqrt{77}}{\sqrt{19}} & \alpha_{21} = \frac{\sqrt{331}}{\sqrt{77}} & \\
\alpha_{00} = 2 & \alpha_{10} = \frac{\sqrt{17}}{2} & \alpha_{20} = \frac{2\sqrt{19}}{\sqrt{17}} & \alpha_{30} = \frac{\sqrt{17}}{\sqrt{38}}
\end{array}
\tag{7.1}
$$

$$
\begin{array}{llll}
\beta_{03} = \frac{18\sqrt{3}}{\sqrt{157}} & & & \\
\beta_{02} = \frac{\sqrt{157}}{3\sqrt{3}} & \beta_{12} = \frac{\sqrt{535}}{\sqrt{97}} & & \\
\beta_{01} = \frac{3\sqrt{3}}{\sqrt{5}} & \beta_{11} = \frac{\sqrt{97}}{\sqrt{19}} & \beta_{21} = \frac{\sqrt{371}}{\sqrt{97}} & \\
\beta_{00} = \sqrt{5} & \beta_{10} = \frac{\sqrt{19}}{2} & \beta_{20} = \frac{\sqrt{77}}{\sqrt{17}} & \beta_{30} = \frac{\sqrt{331}}{2\sqrt{19}}.
\end{array}
$$

Example 7.1. Let Ω_3 be given by (7.1) and let $M(2) \equiv M(2)(\Omega_3)$ the its associated moment matrix, with entries built from the data $\tilde{\gamma}_{ij}$. Let $M(3)$ be a positive semi-definite, recursively generated, moment matrix extension of $M(2)$. Then rank $M(3) >$ rank $M(2)$. As a consequence, $M(2)$ admits no flat extension $M(3)$. For, consider a moment matrix extension

$$
M(3) := \begin{pmatrix}
1 & 4 & 5 & 17 & 19 & 27 & 76 & 77 & 97 & 157 \\
4 & 17 & 19 & 76 & 77 & 97 & 354 & 331 & 371 & 535 \\
5 & 19 & 27 & 77 & 97 & 157 & 331 & 371 & 535 & 972 \\
17 & 76 & 77 & 354 & 331 & 371 & \tilde{\gamma}_{05} & \tilde{\gamma}_{14} & \tilde{\gamma}_{23} & \tilde{\gamma}_{32} \\
19 & 77 & 97 & 331 & 371 & 535 & \tilde{\gamma}_{14} & \tilde{\gamma}_{23} & \tilde{\gamma}_{32} & \tilde{\gamma}_{41} \\
27 & 97 & 157 & 371 & 535 & 972 & \tilde{\gamma}_{23} & \tilde{\gamma}_{32} & \tilde{\gamma}_{41} & \tilde{\gamma}_{50} \\
76 & 354 & 331 & \tilde{\gamma}_{05} & \tilde{\gamma}_{14} & \tilde{\gamma}_{23} & \tilde{\gamma}_{06} & \tilde{\gamma}_{15} & \tilde{\gamma}_{24} & \tilde{\gamma}_{33} \\
77 & 331 & 371 & \tilde{\gamma}_{14} & \tilde{\gamma}_{23} & \tilde{\gamma}_{32} & \tilde{\gamma}_{15} & \tilde{\gamma}_{24} & \tilde{\gamma}_{33} & \tilde{\gamma}_{42} \\
97 & 371 & 535 & \tilde{\gamma}_{23} & \tilde{\gamma}_{32} & \tilde{\gamma}_{41} & \tilde{\gamma}_{24} & \tilde{\gamma}_{33} & \tilde{\gamma}_{42} & \tilde{\gamma}_{51} \\
157 & 535 & 972 & \tilde{\gamma}_{32} & \tilde{\gamma}_{41} & \tilde{\gamma}_{50} & \tilde{\gamma}_{33} & \tilde{\gamma}_{42} & \tilde{\gamma}_{51} & \tilde{\gamma}_{60}
\end{pmatrix},
$$

where the moments of degree 5 and 6 are new. A direct computation shows that rank $M(2) = 5$, and that $(X-3)(Y-4) = 0$, that is, $YX = 4X + 3Y - 12$. In any positive semi-definite, recursively generated, extension $M(3)$ this column relation would still be valid, and it would also give rise to two new column relations, namely $YX^2 = 4X^2 + 3YX - 12X$ and $Y^2X = 4YX + 3Y^2 - 12Y$. These three identities lead at once to the values $\tilde{\gamma}_{14} = 1497$, $\tilde{\gamma}_{23} = 1513$, $\tilde{\gamma}_{32} = 1925$, $\tilde{\gamma}_{41} = 3172$, $\tilde{\gamma}_{15} = 243 + 4\tilde{\gamma}_{05}$, $\tilde{\gamma}_{24} = 6555$, $\tilde{\gamma}_{33} = 7375$, $\tilde{\gamma}_{42} = 10796$, and $\tilde{\gamma}_{51} = 1024 + 3\tilde{\gamma}_{50}$. Now, since the compression of $M(2)$ to the rows and columns indexed by 1, X, Y, X^2 and Y^2 is invertible, we can find coefficients A_1, A_X, A_Y, A_{X^2} and A_{Y^2} such that

$$
A_1[1]_{\mathcal{B}} + A_X[X]_{\mathcal{B}} + A_Y[Y]_{\mathcal{B}} + A_{X^2}[X^2]_{\mathcal{B}} + A_{Y^2}[Y^2]_{\mathcal{B}} = [X^3]_{\mathcal{B}}, \tag{7.2}
$$

where $[\cdot]_{\mathcal{B}}$ denotes the compression of a column to $\mathcal{B} := \{1, X, Y, X^2, Y^2\}$. A calculation using *Mathematica* [24] reveals that $A_1 = -25513 + 15\tilde{\gamma}_{05}$, $A_X = 13587 - 8\tilde{\gamma}_{05}$, $A_Y = 1$, $A_{X^2} = -1692 + \tilde{\gamma}_{05}$ and $A_{Y^2} = 0$. If $M(3)$ were a flat extension of $M(2)$, an identity similar to (7.2) should hold for the last row in $M(3)$, that is,

$$
A_1[1]_{\{Y^3\}} + A_X[X]_{\{Y^3\}} + A_Y[Y]_{\{Y^3\}} + A_{X^2}[X^2]_{\{Y^3\}} + A_{Y^2}[Y^2]_{\{Y^3\}} = [X^3]_{\{Y^3\}}. \tag{7.3}
$$

Using *Mathematica* again, it is easy to check that $A_1[1]_{\{Y^3\}} + A_X[X]_{\{Y^3\}} + A_Y[Y]_{\{Y^3\}} + A_{X^2}[X^2]_{\{Y^3\}} + A_{Y^2}[Y^2]_{\{Y^3\}} = 7376$, while $[X^3]_{\{Y^3\}} = \tilde{\gamma}_{33} = 7375$. It follows that $M(3)$ cannot be a flat extension of $M(2)$.

Remark 7.2. The SCP in Example 7.1 does admit a solution, and the subnormal completion has a 6-atomic Berger measure. We see this after we observe that the positive semi-definite moment matrix extension $M(3)$, while not a flat extension of $M(2)$, does admit a flat extension $M(4)$. Rather than showing the details here, we refer the reader to [12, Proposition 5.5 and Example 5.6]; the representing measure constructed there must be translated three units to the right and four units up to give rise to the Berger measure that solves SCP in Example 7.1.

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