Hyponormality and Subnormality for Powers of Commuting Pairs of Subnormal Operators

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(joint work with S.H. Lee and J. Yoon, JFA(2007))

September 8, 2007

Notation and Preliminaries

$\mathcal{L}(\mathcal{H})$: algebra of bounded operators on a Hilbert space $\mathcal{H}$
$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- hyponormal if $T^*T \geq TT^*$
- For $S, T \in \mathcal{B}(\mathcal{H})$, $[S, T] := ST - TS$.
- An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, an operator T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

- An n-tuple $\mathbf{T}$ is normal if $\mathbf{T}$ is commuting and each T_i is normal
- $\mathbf{T}$ is subnormal if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace
- normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.
- (Bram-Halmos, $n = 1$)

$$\begin{aligned} T \text{ subnormal} \quad &\Leftrightarrow \quad T \text{ is } k\text{-hyponormal for all } k \geq 1 \\ &\Leftrightarrow \quad (T, T^2, ..., T^k) \text{ is hyponormal for all } k \geq 1. \end{aligned}$$

Problem

(Lifting Problem for Commuting Subnormals (LPCS)) Given a commuting pair $\mathbf{T} \equiv (T_1, T_2)$, find necessary and sufficient conditions on T_1 and T_2 to guarantee the subnormality of $\mathbf{T}$.

It is well known that the subnormality of each of T_1 and T_2 is necessary but not sufficient.

Conjecture

(RC-Muhly-Xia, 1988) Let $\mathbf{T} \equiv (T_1, T_2)$ be a pair of commuting subnormal operators. Then $\mathbf{T}$ is subnormal if and only if $\mathbf{T}$ is hyponormal.

- We now know how to build many different kinds of examples to disprove the conjecture; one of the constructions even allows for $T_1 \cong T_2$. We also know that for every $k \geq 1$ one can build a pair $\mathbf{T}$ which is k-hyponormal but not $(k+1)$-hyponormal, therefore not subnormal.
- To study LPCS, we let $\mathfrak{H}_0$ denote the class of commuting pairs of subnormal operators. More generally, we let $\mathfrak{H}_\infty$ denote the class of subnormal pairs, and for each integer $k \geq 1$, we let $\mathfrak{H}_k$ denote the class of k-hyponormal pairs in $\mathfrak{H}_0$.

- Clearly,

$$\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0;$$

these inclusions are all proper. Suitable examples have been found in the class $\mathcal{TC}$, a large class of 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2)$ for which hyponormality and subnormality are far apart. Briefly described, $\mathcal{TC}$ consists of hyponormal pairs $\mathbf{T}$ of subnormal operators such that

$$\mathbf{T}|_{c(\mathbf{T})} \cong (I \bigotimes W_\alpha, W_\beta \bigotimes I),$$

where $c(\mathbf{T})$ is the *c*ore of $\mathbf{T}$ (a large invariant subspace).

Weighted Shifts and Berger's Theorem

Given a bounded sequence of positive numbers (weights) $\alpha \equiv \alpha_0, \alpha_1, \alpha_2, ...$, the unilateral weighted shift on $\ell^2(Z_+)$ associated with α is

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0).$$

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot ... \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_k \leq \alpha_{k+1}$ (all $k \geq 0$)
- (Berger; Gellar-Wallen) W_α is subnormal iff there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \ \ (\text{all } k \geq 0).$$

ξ is the Berger measure of W_α.

- For $0 < a < 1$ we let $S_a := shift\{a, 1, 1, ...)$.
- The Berger measures of U_+ is δ_1.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

Multivariable Weighted Shifts

$$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^{\infty}(\mathbb{Z}_+^2), \quad \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$$

$$\ell^2(\mathbb{Z}_+^2) \cong \ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+).$$

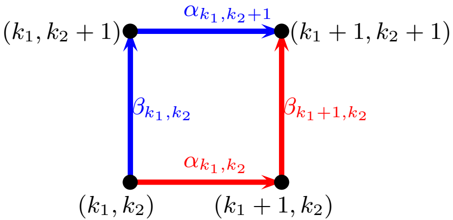
We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

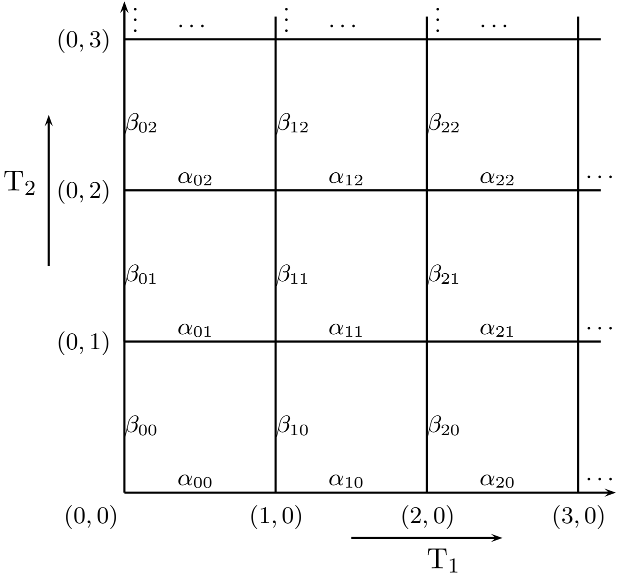
$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \quad \text{(all } \mathbf{k}\text{)}.$$




(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T2
T1
β02
β12
β22
α02
α12
α22
β01
β11
β21
α01
α11
α21
β00
β10
β20
α00
α10
α20

- Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$; in fact, using the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, one can let

$$T_1 \cong I \bigotimes W_\alpha$$

and

$$T_2 \cong W_\beta \bigotimes I.$$

In this case, $\mathbf{T}$ is doubly commuting, so $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2. For this reason, we only use these shifts when the above mentioned triviality is desirable or needed.

- (Jewell-Lubin)

$$W_\alpha \text{ is subnormal} \quad \Leftrightarrow \quad \gamma_{\mathbf{k}} := \prod_{i=0}^{k_1-1} \alpha^2_{(i,0)} \cdot \prod_{j=0}^{k_2-1} \beta^2_{(k_1-1,j)}$$

$$= \quad \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \;\; (\text{all } \mathbf{k} \geq \mathbf{0}).$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

A Measure-Theoretic Formulation of LPCS

- $\mathbf{T} \equiv (T_1, T_2)$ subnormal $\Rightarrow T_i$ subnormal for $i = 1, 2$. For instance,

$$T_1 \cong \bigoplus_{j=0}^{\infty} W_{\alpha^{(j)}},$$

where $\alpha_i^{(j)} := \alpha_{(i,j)}$ is the j-th row. If μ is the Berger measure of $\mathbf{T}$, and if

$$d\mu(t_1, t_2) \equiv d\Phi_{t_1}(t_2) d\eta(t_1)$$

is the canonical disintegration of μ by vertical slices, we prove that the Berger measure of $W_{\alpha^{(j)}}$ is

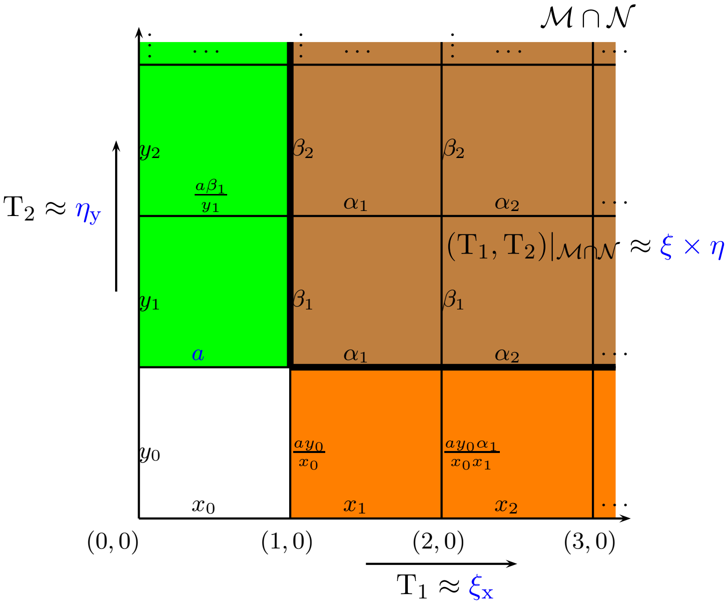
$$d\nu_j(t_1) := \frac{1}{\gamma_{(0,j)}} \int_{[0,a_2]} t_2^j d\Phi_{t_1}(t_2).$$

In terms of the marginal measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$, LPCS can be phrased as a

Problem (Reconstruction-of-Measure Problem (ROMP))

Under what conditions on the measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$ does there exist a 2-variable measure μ correctly interpolating all the powers $t_1^{k_1} t_2^{k_2}$ $(k_1, k_2 \geq 0)$.

Special Case (tensor core): Given $\xi, \eta, \xi_x, \eta_y, a$, find μ.



Definition

(i) The *core* of a 2-variable weighted shift $\mathbf{T}$ is $c(\mathbf{T}) := \mathbf{T}|_{\mathcal{M}\cap\mathcal{N}}$;
(ii) $\mathbf{T}$ is said to be *of tensor form* if $\mathbf{T} \cong (I \otimes W_\alpha, W_\beta \otimes I)$. (When $\mathbf{T}$ is subnormal, this is equivalent to requiring that the Berger measure be a Cartesian product $\xi \times \eta$);
(iii) $\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0 : c(\mathbf{T}) \text{ is of tensor form}\}$.

Problem

Let $\mathbf{T} \in \mathcal{TC}$ and assume $\mathbf{T}$ is hyponormal. Additionally, assume that $c(\mathbf{T})$ is subnormal, with Berger measure $\xi \times \eta$. Find necessary and sufficient conditions on ξ, η, ξ_x, η_y and a to guarantee the subnormality of $\mathbf{T}$.

Extremal and Marginal Measures

Definition

μ probability measure on $X \times Y$, with $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} (also a probability measure) on $X \times Y$ is given by

$$d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\mu)}} d\mu(s,t).$$

Definition

For μ on $X \times Y$, the marginal measure μ^X is given by

$$\mu^X := \mu \circ \pi_X^{-1},$$

where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus,

$$\mu^X(E) = \mu(E \times Y),$$

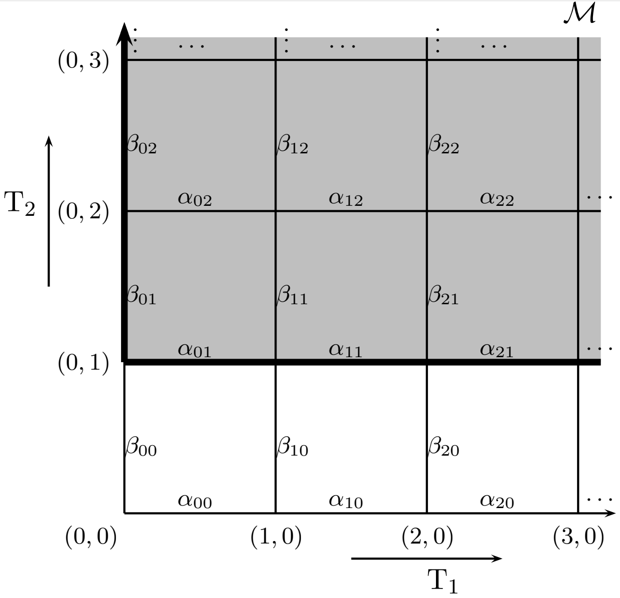
for every $E \subseteq X$. If μ is a probability measure, then so is μ^X.

Proposition (Special Case of ROMP)

(Subnormal backward extension for 2-variable weighted shifts)
$\mathcal{M}$: *subspace of* $\ell^2(Z_+^2)$ *associated to indices* $\mathbf{k}$ *with* $k_2 \geq 1$.
Assume $\mathbf{T}_{\mathcal{M}}$ *subnormal with measure* $\mu_{\mathcal{M}}$ *and*
$W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ *subnormal with measure* ν.
Then $\mathbf{T}$ *is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;

(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \nu$.
Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$, *then* $(\mu_{\mathcal{M}})_{ext}^X = \nu$.
When $\mathbf{T}$ *is subnormal, its Berger measure is*

$$\begin{aligned} d\mu(s,t) &= \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) \\ &\quad + (d\nu(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)) d\delta_0(t). \end{aligned}$$



$\mathcal{M}$
(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T_2
T_1
β_{02} β_{12} β_{22}
α_{02} α_{12} α_{22}
β_{01} β_{11} β_{21}
α_{01} α_{11} α_{21}
β_{00} β_{10} β_{20}
α_{00} α_{10} α_{20}

(i)

A Necessary Condition for the Existence of a Lifting

Theorem

Let μ be the Berger measure of a subnormal 2-variable weighted shift, and for $j \geq 0$ let ξ_j be the Berger measure of the associated j-th horizontal 1-variable weighted shift $W_{\alpha^{(j)}}$. Then

$$d\xi_j(s) = \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t)\} \, d\mu^X(s),$$

where $d\mu(s,t) = d\Phi_s(t) \, d\mu^X(s)$. A similar result holds for the Berger measure η_i of the associated i-th vertical 1-variable weighted shifts $W_{\beta^{(i)}}$ $(i \geq 0)$.

Theorem

Let μ, ξ_j *and* η_i *be as before. For every* $i, j \geq 0$ *we have*

$$\xi_{j+1} \ll \xi_j$$

and

$$\eta_{i+1} \ll \eta_i.$$

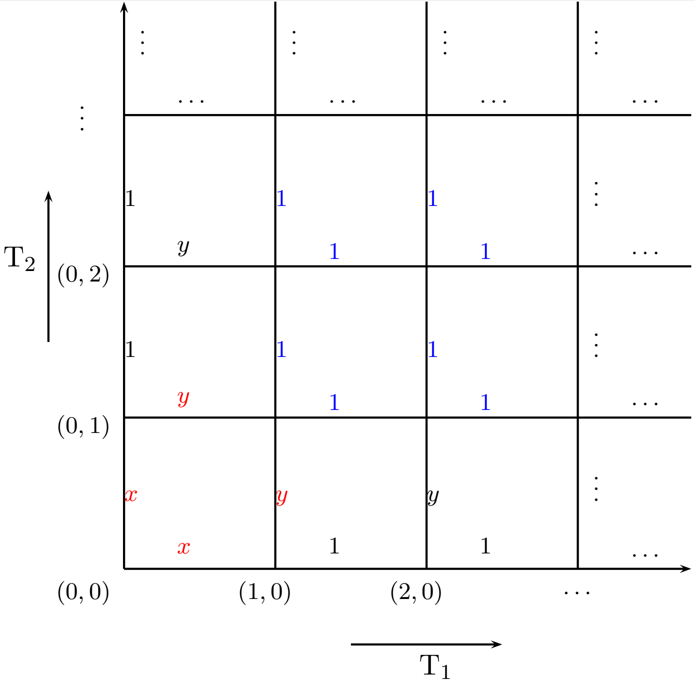
The Necessary Condition is Not Sufficient

Proposition

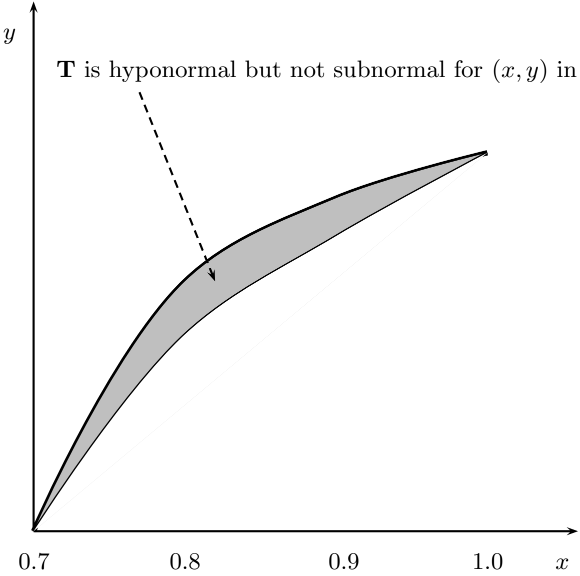
Let $\mathbf{T} \equiv (T_1, T_2)$ *be the following 2-variable weighted shift. Then*
(i) $\mathbf{T}$ *is hyponormal* $\Leftrightarrow 1 - 2x^2 + y^2 \geq 0$
(ii) $\mathbf{T}$ *is subnormal* $\Leftrightarrow 1 - 2x^2 + x^2y^2 \geq 0$.
As a consequence, for $(x, y) \in \mathbb{R}_+^2$ *such that*

$$1 - 2x^2 + x^2y^2 < 0 \leq 1 - 2x^2 + y^2,$$

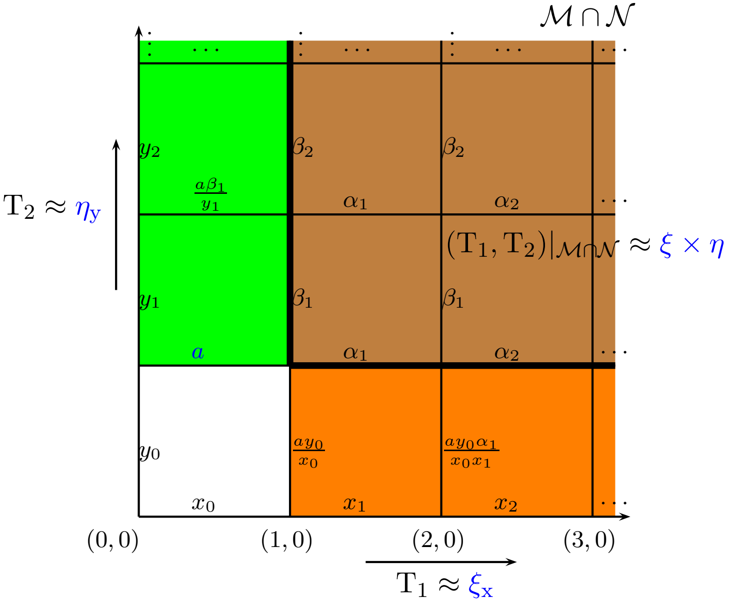
$\mathbf{T}$ *is hyponormal but not subnormal.*



T_2
$(0,2)$
$(0,1)$
$(0,0)$
$(1,0)$
$(2,0)$
$\cdots$
T_1
x
y
1


y
T is hyponormal but not subnormal for (x, y) in this region
0.7
0.8
0.9
1.0
x

2-variable Shifts Whose Cores are of Tensor Form



Assume that $c(\mathbf{T})$ is subnormal, with Berger measure $\xi \times \eta$, and let

$$\psi := (\eta_y)_1 - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \eta$$

and

$$\varphi := \xi_x - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 - \frac{a^2 y_0^2}{s} \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \xi,$$

where $(\eta_y)_1$ is the Berger measure of the subnormal shift $shift(y_1, y_2, \cdots)$. Trivially, ψ and φ are measures, but they may or may not be *positive* measures.

Theorem

Let $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ *be a 2-variable weighted shift whose core is of tensor form. Then* $\mathbf{T} \in \mathfrak{H}_\infty \iff \psi \geq 0$ *and* $\varphi \geq 0$.

Sketch of Proof.

($\Leftarrow$) It suffices to find a probability measure μ satisfying

$$\gamma_{\mathbf{k}}(\mathbf{T}) = \int t^{\mathbf{k}} d\mu(\mathbf{t}) := \int t_1^{k_1} t_2^{k_2} d\mu(\mathbf{t}), \text{ for all } \mathbf{k} \geq 0.$$

Let

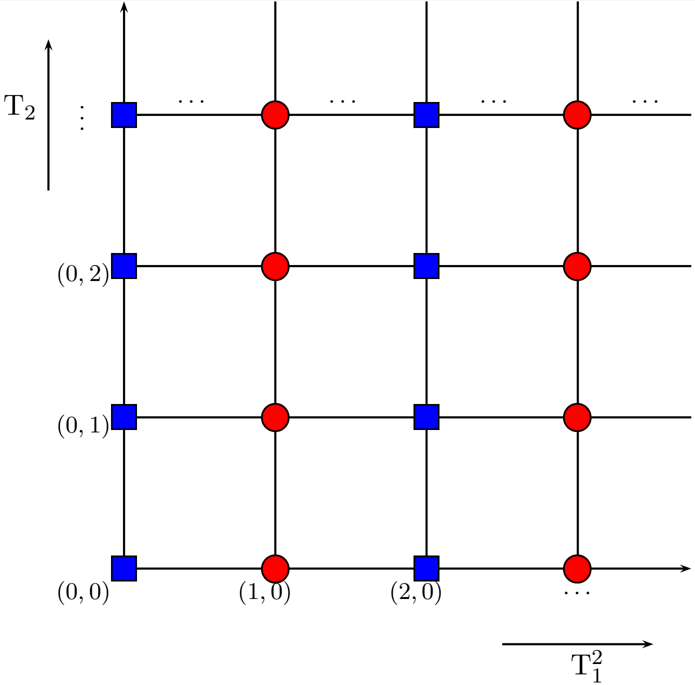
$$\mu := \varphi \times (\delta_0 - \widetilde{\eta}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \widetilde{\eta} \right) + \xi_x \times \widetilde{\eta},$$

where $d\widetilde{\eta}(t) := \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\eta)}} d\eta(t)$ and $d\widetilde{\psi}(t) := \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\psi)}} d\psi(t)$. □

Can Powers Detect Lifting?

J. Stampfli (1962)
(i) If T is hyponormal and T^n is normal for some $n \geq 1$, then T is normal.
(ii) There exists T hyponormal, non-subnormal, such that T^n is subnormal for all $n \geq 1$; e.g., a shift with weights $a, b, 1, 1, \cdots$, where $0 < a < b < 1$.
If W_α is a weighted shift, then W_α^2 is a direct sum of two shifts, with weights $\alpha_0\alpha_1, \alpha_2\alpha_3, \cdots$ and $\alpha_1\alpha_2, \alpha_3\alpha_4, \cdots$, resp.
In two variables, we consider (T_1^2, T_2) and (T_1, T_2^2), which can be split as orthogonal direct sums. For instance,

$$(T_1^2, T_2) \cong (W_{\alpha(2:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0}) \bigoplus (W_{\alpha(2:1)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_1}).$$



T_2
$(0,2)$
$(0,1)$
$(0,0)$
$(1,0)$
$(2,0)$
T_1^2

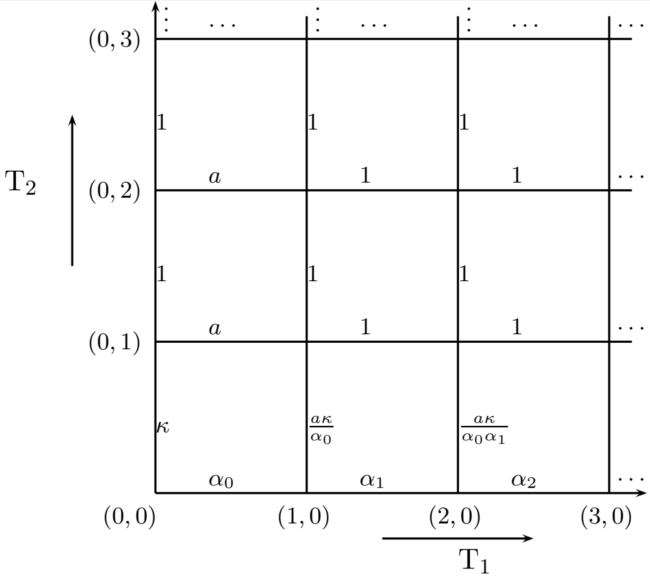
We give an example of $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_1$ such that $(T_1^2, T_2) \notin \mathfrak{H}_1$. For $0 < \kappa \leq 1$, let $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ be defined by

$$\alpha_n := \begin{cases} \kappa\sqrt{\frac{3}{4}} & \text{if } n = 0 \\ \frac{\sqrt{(n+1)(n+3)}}{(n+2)} & \text{if } n \geq 1. \end{cases} \tag{0.1}$$

W_α is subnormal, with Berger measure

$$d\xi_\alpha(s) := (1 - \kappa^2) d\delta_0(s) + \frac{\kappa^2}{2} ds + \frac{\kappa^2}{2} d\delta_1(s).$$

For $0 < a < 1$, consider the 2-variable weighted shift shown below, with $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ as above.


(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T2
T1
1
a
κ
aκ/α0
aκ/(α0α1)
α0
α1
α2

Theorem

Let $0 < a \leq \sqrt{\frac{1}{2}}$. *Then*

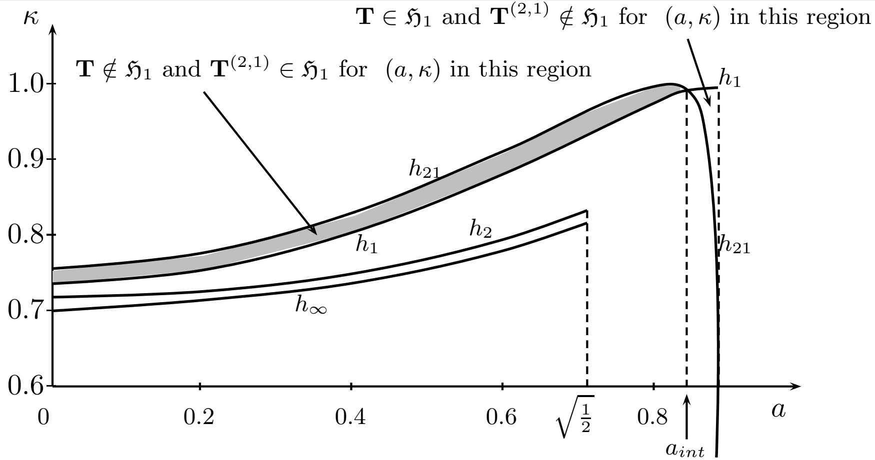
(i) T_1 *and* T_2 *are subnormal;*

(ii) $\mathbf{T} \in \mathfrak{H}_1$ *if and only if* $0 < \kappa \leq h_1(a) := \sqrt{\frac{32-48a^4}{59-72a^2}}$;

(iii) $\mathbf{T} \in \mathfrak{H}_2$ *if and only if* $0 < \kappa \leq h_2(a) := \sqrt{\frac{81-144a^2}{157-360a^2+144a^4}}$;

(iv) $\mathbf{T} \in \mathfrak{H}_\infty$ *if and only if* $0 < \kappa \leq h_\infty(a) := \frac{1}{\sqrt{2-a^2}}$;

(v) *Then* (T_1^2, T_2) *is hyponormal if and only if* $0 < \kappa \leq h_{21}(a) := 3\sqrt{\frac{3-5a^4}{47-60a^2}}$.


$\mathbf{T} \in \mathfrak{H}_1$ and $\mathbf{T}^{(2,1)} \notin \mathfrak{H}_1$ for (a, κ) in this region
$\mathbf{T} \notin \mathfrak{H}_1$ and $\mathbf{T}^{(2,1)} \in \mathfrak{H}_1$ for (a, κ) in this region
κ
1.0
0.9
0.8
0.7
0.6
h_1
h_{21}
h_1
h_2
h_∞
h_{21}
0
0.2
0.4
0.6
$\sqrt{\frac{1}{2}}$
0.8
a_{int}
a

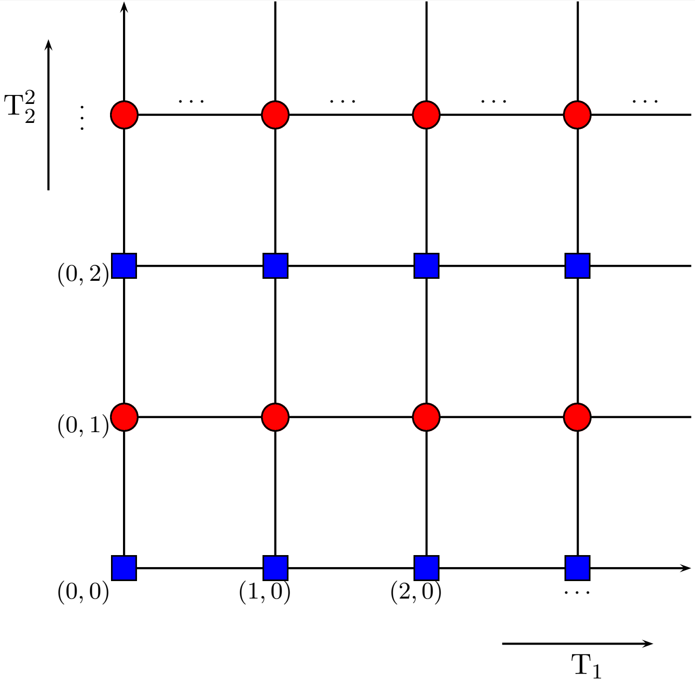
A Large Class for Which $(T_1^2, T_2) \in \mathfrak{H}_\infty \iff (T_1, T_2^2) \in \mathfrak{H}_\infty \iff (T_1, T_2) \in \mathfrak{H}_\infty$

We recall that (T_1, T_2^2) can be regarded as the orthogonal direct sum of two 2-variable weighted shifts:

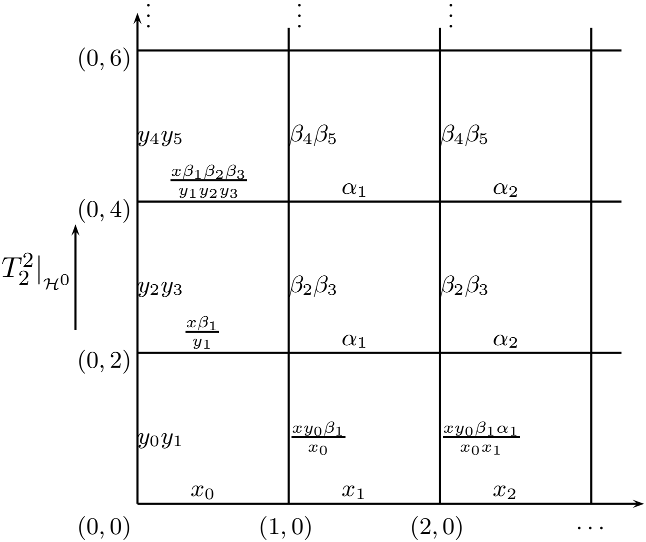
$$\mathcal{H}^0 := \bigvee \{e_{(0,0)}, e_{(0,2)}, e_{(0,4)}, \cdots, e_{(1,0)} e_{(1,2)} e_{(1,4)}, \cdots\}$$

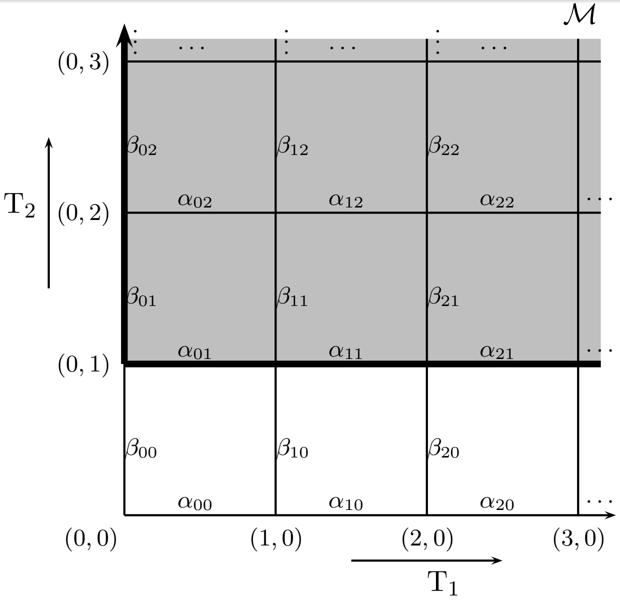
and

$$\mathcal{H}^1 := \bigvee \{e_{(0,1)}, e_{(0,3)}, e_{(0,5)}, \cdots, e_{(1,1)} e_{(1,3)} e_{(1,5)}, \cdots\}.$$


T_2^2
$(0,2)$
$(0,1)$
$(0,0)$
$(1,0)$
$(2,0)$
$\cdots$
T_1

Thus, (T_1, T_2^2) is subnormal if and only if each of $(T_1, T_2^2)|_{\mathcal{H}^0}$ and $(T_1, T_2^2)|_{\mathcal{H}^1}$ is subnormal. The weight diagrams of $(T_1, T_2^2)|_{\mathcal{H}^0}$ is shown below.





M
(0, 3)
(0, 2)
(0, 1)
T2
β02
β12
β22
α02
α12
α22
β01
β11
β21
α01
α11
α21
β00
β10
β20
α00
α10
α20
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T1

(i)

Proposition

Let $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$. *Then* $(T_1, T_2^2)|_{\mathcal{H}_1}$ *is subnormal if and only if* $\mathbf{T}|_{\mathcal{M}}$ *is subnormal.*

Theorem

Let $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$. *Then*

$$(T_1, T_2^2) \in \mathfrak{H}_\infty \iff (T_1^2, T_2) \in \mathfrak{H}_\infty \iff (T_1, T_2) \in \mathfrak{H}_\infty.$$

Sketch of Proof. Clearly, it is enough to show that $(T_1, T_2^2) \in \mathfrak{H}_\infty \Rightarrow (T_1, T_2) \in \mathfrak{H}_\infty$. Since $(T_1, T_2^2) \in \mathfrak{H}_\infty \Rightarrow (T_1, T_2^2)|_{\mathcal{H}^0} \in \mathfrak{H}_\infty$, our strategy consists of first characterizing the subnormality of $\mathbf{T}$ and of $(T_1, T_2^2)|_{\mathcal{H}^0}$ in terms of the given parameters (y_0, ν, etc), and then establishing the desired implication at the parameter level.

First, the Subnormal Backward Extension Theorem helps us characterize the subnormality of $\mathbf{T}$, with Berger measure given by

$$\mu_{\mathcal{M}} = x^2 \widetilde{\xi} \times \eta + \delta_0 \times ((\eta_y)_1 - x^2 r \eta).$$

$$(T_1, T_2) \text{ is subnormal} \quad \Leftrightarrow \quad y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \nu$$

$$\Leftrightarrow \quad y_0^2 \left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} \varphi \leq \mu_X.$$

We then use reconstruction-of-measure techniques to prove that

$$(T_1, T_2^2)|_{\mathcal{H}^0} \text{ is subnormal } \Leftrightarrow y_0^2 \left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} \varphi \leq \mu_X.$$

We thus have a characterization of the subnormality of $(T_1, T_2^2)|_{\mathcal{H}^0}$. It now follows that the subnormality of (T_1, T_2^2) implies the subnormality of (T_1, T_2). □

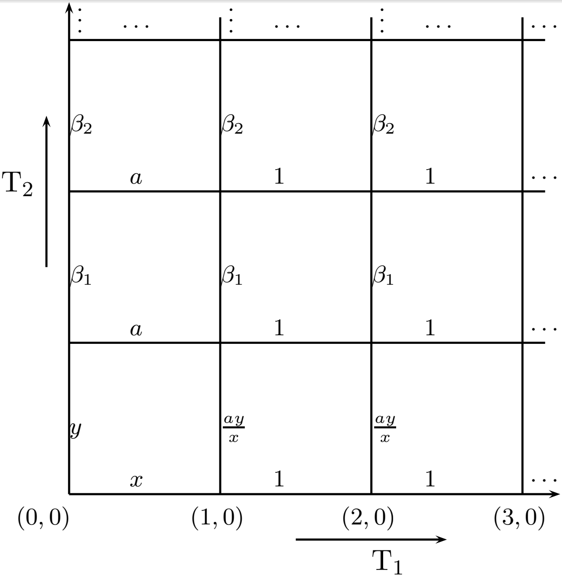
Subnormality for Powers of Hyponormal Pairs

We study the connection between the joint subnormality of pairs $(T_1, T_2) \in \mathfrak{H}_1$ and the subnormality of the associated monomials $T_1^m T_2^n$ $(m, n \geq 1)$. Our results further exhibit the large gap between the classes $\mathfrak{H}_\infty$ (subnormal pairs) and $\mathfrak{H}_0$ (commuting pairs of subnormal operators).

Proposition

Let $\mathbf{T} \equiv (T_1, T_2)$ *be hyponormal, and assume that* (T_1^m, T_2^n) *is normal for some* $m \geq 1$ *and* $n \geq 1$*. Then* (T_1, T_2) *is normal.*

In view of this result, one might conjecture that if (T_1, T_2) is hyponormal and $T_1^m T_2^n$ is normal for some $m \geq 1$ and $n \geq 1$, then (T_1, T_2) is normal. But this is not true even if we assume that (T_1, T_2) is subnormal and $T_1^m T_2^n$ is normal for all $m \geq 1$ and $n \geq 1$.



T_2
T_1
β_2 β_2 β_2
a 1 1
β_1 β_1 β_1
a 1 1
y $\frac{ay}{x}$ $\frac{ay}{x}$
x 1 1
$(0,0)$ $(1,0)$ $(2,0)$ $(3,0)$

Theorem

For the above mentioned 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$*, the following are equivalent.*
(i) $T_1^m T_2^n$ *is subnormal for all* $m \geq 1$ *and* $n \geq 1$*;*
(ii) $T_1 T_2^n$ *is subnormal for all* $n \geq 1$*;*
(iii) $y \leq \frac{x}{a} \cdot \frac{1}{\Pi_{j=1}^{n-1}\beta_j}\sqrt{\left\|\frac{1}{t}\right\|_{L^1(\eta^{(n)})}^{-1}}$ *for all* $n \geq 1$*, where*
$d\eta^{(n)}(t) := \frac{t^{1-\frac{1}{n}}}{\beta_1^2 \cdots \beta_{n-1}^2} d\eta(t^{\frac{1}{n}})$.

For a concrete example, let $d\eta(t) := dt$ on $[\frac{1}{2}, \frac{3}{2}]$. Then $\gamma_{n-1} = \beta_1^2 \beta_2^2 \cdots \beta_{n-1}^2 = \int_{\frac{1}{2}}^{\frac{3}{2}} t^{n-1} d\eta(t) = \frac{1}{n}(\frac{3^n-1}{2^n})$ and $\gamma_{2n-1} = \frac{1}{2n}(\frac{3^{2n}-1}{2^{2n}})$, we can give explicit expressions for the Berger measures of $shift(\beta_n, \beta_{n+1}, \cdots)$ and $shift(\Pi_{j=n}^{2n-1}\beta_j, \Pi_{j=2n}^{3n-1}\beta_j, \cdots)$.

We conclude that

(i) T_1 is subnormal if $0 < a < x < 1$;

(ii) T_2 is subnormal $\Leftrightarrow y \leq \sqrt{\frac{1}{\ln 3}}$;

(iii) $(T_1, T_2) \in \mathfrak{H}_1 \Leftrightarrow y \leq m := \min\{\frac{x\sqrt{1-x^2}}{\sqrt{x^2+a^4-2a^2x^2}}, \sqrt{\frac{1}{\ln 3}}\}$;

(iv) $(T_1, T_2) \in \mathfrak{H}_\infty \Leftrightarrow y \leq s := \sqrt{\frac{1}{\ln 3}\frac{1-x^2}{1-a^2}}$.

Example

For $s < y \leq m$ and $0 < a < x < 1$, we have

(i) $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_1$;

(ii) $\mathbf{T} \equiv (T_1, T_2) \notin \mathfrak{H}_\infty$;

(iii) $T_1^m T_2^n$ is subnormal for all $m \geq 1, n \geq 1$.

Remark

Recall that E. Franks proved that $\mathbf{T}$ is subnormal if and only if $p(\mathbf{T})$ is subnormal for all poly's p with degree at most 5.

Summary

- LPCS has abstract solution (Multivariable Bram-Halmos), but we seek concrete solution for multivariable weighted shifts.
- ROMP is a measure-theoretic formulation of LPCS for Berger measures.
- For a large collection of 2-variable weighted shifts (those whose core is of tensor form), we find a complete solution.
- Solution entails checking the positivity of two 1-variable measures, naturally associated to the initial data.
- We address the question: Can the powers (T_1^2, T_2) and (T_1, T_2^2) detect the existence of a lifting?

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