Truncated Moment Problems:
Existence, Uniqueness, and Localization of the Support of Representing Measures

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Abstract. In [CuFi1], [Fia1] we succeeded in obtaining a complete solution to the truncated moment problem in case the interpolating measure has compact support in the real line; our main contribution there consisted in bringing to light the notion of *recursiveness*, which was central to our analysis. As we move into several variables, the interpolating measure must be allowed to have support away from the line; one instance of particular interest, associated with 2-variable weighted shifts, is the case of compact support in the complex plane, which we label as the truncated complex moment problem (TCMP).

Let μ be a positive Borel measure on $\mathbf{C}$, assume that $\mathbf{C}[z,\bar{z}] \subseteq L^1(\mu)$ and define $\gamma_{ij} := \int \bar{z}^i z^j \, d\mu(z,\bar{z})$. Given $p \in \mathbf{C}[z,\bar{z}]$, $p(z,\bar{z}) = \sum_{ij} a_{ij} \bar{z}^i z^j$, we have

$$0 \leq \int \mid p(z,\bar{z}) \mid^2 \, d\mu(z,\bar{z}) = \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} d\mu(\bar{z},z) = \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell}\gamma_{i+\ell,j+k}. \tag{1}$$

Observe that $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma_{ij}}$ for all i,j. To understand the matricial positivity associated with $\gamma := \{\gamma_{ij}\}$, we introduce the following lexicographic order on the rows and columns of infinite matrices: $\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3, ...$, e.g., the first column is labeled $\mathit{1}$, the second column is labeled Z, the third $\bar{Z}$, the fourth Z^2, et cetera; this order corresponds to the graded homogeneous decomposition of $\mathbf{C}[z,\bar{z}]$. For $m,n \geq 0$ let $M[m,n]$ be the $(m+1)\times(n+1)$ block of Toeplitz form whose first row has entries given by $\gamma_{mn}, \gamma_{m+1,n-1}, ..., \gamma_{m+n,0}$ and whose first column has entries given by $\gamma_{mn}, \gamma_{m-1,n+1}, ..., \gamma_{0,n+m}$ (as a consequence, the lower right-hand corner of $M[m,n]$ is γ_{nm}). The matrix $M = M(\gamma)$ is then built as follows:

$$M := \begin{pmatrix} M[0,0] & M[0,1] & M[0,2] & ... \\ M[1,0] & M[1,1] & M[1,2] & ... \\ M[2,0] & M[2,1] & M[2,2] & ... \\ ... & ... & ... & ... \end{pmatrix}.$$

It is now not hard to see that the above mentioned positivity (1) is equivalent to the condition $M \geq 0$, as a quadratic form on $\mathbf{C}^\omega$. Suppose now that we are just given a double-indexed sequence $\gamma \equiv \{\gamma_{ij}\}$ subject to the constraints $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma_{ij}}$ for all i,j. The classical (complex) *full* moment problem asks for necessary and sufficient conditions on the sequence γ to guarantee the existence of a positive Borel measure μ which interpolates γ, i.e.,

$$\int \bar{z}^i z^j d\mu(z,\bar{z}) = \gamma_{ij} \qquad (i,j \geq 0). \tag{2}$$

An obvious necessary condition is then $M \geq 0$; this corresponds to the positivity of the Riesz functional $L(p) := \sum_{ij} a_{ij}\gamma_{ij}$ on the cone Σ^2 generated by polynomials of the form $p\bar{p}$. If K is a closed subset of $\mathbf{C}$, the Riesz-Haviland Criterion states that γ admits a representing measure supported on K if and only if $L(p) \geq 0$ for every polynomial p which is nonnegative on K.

With γ, L, M and K as above, suppose there exists a polynomial q such that $K = K_q := \{z \in \mathbf{C} : q(z,\bar{z}) \geq 0\}$. In the presence of a representing measure μ supported on K, the inequality $L_q(p\bar{p}) := L(qp\bar{p}) = \int_K qp\bar{p} \geq 0$ (all $p \in \mathbf{C}[z,\bar{z}]$) must hold, in addition to $L(p\bar{p}) \geq 0$ (all $p \in \mathbf{C}[z,\bar{z}]$). Therefore both conditions are necessary for the existence of a representing measure supported in K. K. Schmüdgen established in [Sch2, Theorem 1] that for K_q compact these two conditions are indeed sufficient, and this is the case also for compact sets K which are *semi-algebraic*, that is, obtained as the intersection of a finite family of K_q's. (For related results, see [Sch2, Corollary 3], [Atz], [PuVa], [StSz1], [Vas].)

The *truncated* complex moment problem (TCMP) corresponds to the case when only an *initial segment* of γ is known. Our approach to TCMP follows the strategy we employed to solve the *real* TMP [CuFi1]. Indeed, part of the overall strategy can still be carried out, and concrete conditions can be found in a number of fundamental cases.

What we believe must be used now is a combination of a few revealing examples (cf. [CuFi2, Chapter 6], [CuFi4, Sections 2, 3, 4, and Appendix], [CuFi5]) and the interplay between $M(n)$ and $M(n)_q$, a new associated matrix we have introduced in [CuFi5].

Theorem 1. Let $M(n) \geq 0$ and suppose $\deg q = 2k$ or $2k-1$. There exists $\mathrm{rank} M(n)$-atomic representing measure supported in K_q if and only if there is some flat extension $M(n+1)$ for which $M_q(n+k) \geq 0$. In this case, there exists such a representing measure having exactly $\mathrm{rank} M(n) -$ $\mathrm{rank} M_q(n+k)$ atoms in $\mathcal{Z}(q) := \{z \in \mathbf{C} : q(z, \bar{z}) = 0\}$.

M_q keeps track of the location of the support, and this in turn can be used to establish additional constraints when searching for representing measures. In what follows, we list four open problems to be discussed by the research group.

Quadratures and Cubatures. A disc of center a and radius r can be thought of as the *quadrature domain* completely determined by the moments $\gamma_{00} = \pi r^2$, $\gamma_{01} = \pi a r^2$ and $\gamma_{11} = \pi r^2(\frac{r^2}{2} + |a|^2)$, or equivalently, by the moment matrix

$$M(1) = \begin{pmatrix} 1 & a & \bar{a} \\ \bar{a} & \frac{r^2}{2} + |a|^2 & \bar{a}^2 \\ a & a^2 & \frac{r^2}{2} + |a|^2 \end{pmatrix}.$$

Quadrature domains have received ample attention recently, in view of a natural connection with the theory of hyponormal operators with rank-one self-commutator, and with rationally cyclic subnormal operators [GuPu], [McCYa], [Put1], [Put2].

Problem 1. Does the moment matrix $M(n)$ associated with a quadrature domain admit a flat extension, thereby giving rise to a $rank M(n)$-atomic representing measure?

The study of *minimal* representing measures (those with exactly $rank M(n)$ atoms) is intimately connected with quadrature problems. For K a closed subset of $\mathbf{R}^n$, w a positive weight function, and d a nonnegative integer, the K-quadrature problem for w of precision d entails finding nodes $x_0, ..., x_{M-1} \in K$ and densities $\rho_0, ..., \rho_{M-1}$ such that $\int p(x) w(x) dx = \sum_{k=0}^{M-1} \rho_k p(x_k)$ for every polynomial p of total degree d. In [Cur], [CuFi4], and [Fia2], we have applied techniques derived from TCMP to obtain minimal-node solutions for various compact sets in $\mathbf{R}^2$. The problem of *explicitly* computing the nodes and densities of minimal quadrature rules, however, remains largely unsolved, except in special cases (squares and discs, and small values of d; cf. [CoRa]. Our methods circumvent the theory of orthogonal polynomials and considerations of symmetry; instead, the search for $x_0, ..., x_{M-1}$ gets restricted to a suitable algebraic variety.

Problem 2. Find minimal quadrature rules of precision $2n$ ($n > 3$) for the unit square, unit disc, or equilateral triangle, by building a flat extension of the associated $M(n)$.

Problem 3. Let $M(2)$ be a positive moment matrix. Does $M(2)$ always admit a representing measure?

Our solution of TCMP for flat data was based on the following

Theorem 2. ([CuFi2, Theorem 4.7]) Let M be a finite-rank positive infinite moment matrix. Then M has a unique representing measure, whose support consists of $\mathrm{rank} M$ atoms, obtained as the zeros of an associated analytic polynomial.

Theorem 3. ([CuFi2, Theorem 5.4]) Let $M(n)$ be a flat positive moment matrix (i.e., $\mathrm{rank} M(n) = \mathrm{rank} M(n-1)$). Then $M(n)$ admits a unique flat extension $M(n+1)$.

A dilation-theoretic approach to cubature. Given a measure ν, a cubature formula of degree $2s-1$ for ν can be thought of as a finitely atomic measure μ such that $\int p \, d\mu = \int p \, d\nu$ for every polynomial p in $\mathcal{P}_{2s-1} := \{p \in \mathbf{C}[x_1, ..., x_d] : \deg p \leq 2s-1\}$. If π_{s-1} is the orthogonal projection of $L^2(\nu)$ onto $\mathcal{P}_{s-1}$, and M is the commutative d-tuple of multiplication operators by the coordinates x_j, the compression $\pi_{s-1} M \pi_{s-1}$ is a d-tuple of self-adjoint operators acting on a finite dimensional Hilbert space. In [Put3], [Put4], M. Putinar has obtained the following result.

Theorem 4. ([Put4, Theorem 2.3]) There exists a bijective correspondence between all finite-rank, cyclic and commutative dilations N of the self-adjoint d-tuple $\pi_{s-1} M \pi_{s-1}$ and triples (m, V, A) consisting of (i) an integer $m \geq s$; (ii) a vector subspace $V \subseteq \mathcal{P}_m$ satisfying $\mathcal{P}_{s-1} \perp (V \cap \mathcal{P}_s)$, $\dim(\mathcal{P}_{m-1}/(V \cap \mathcal{P}_{m-1}) < \dim(\mathcal{P}_m/V)$, and $\mathcal{P}_k V + S = \mathcal{P}_{m+k}$ $(0 \leq k \leq \max(2, m))$, where S is a vector complement of V in $\mathcal{P}_m$; and (iii) a positive operator A on S satisfying $\pi_{s-1} A Q_1 \pi_s = \pi_{s-1}$, where Q_1 is the parallel projection of $\mathcal{P}_{m+1}$ onto S.

Given a dilation N with cyclic vector $\mathbf{1}$, Putinar builds m and V by looking at the kernel $\mathcal{I}$ of the map $p \mapsto p(N)\mathbf{1}$. Since $\mathcal{I}$ is finite codimensional (because N is finite-rank), the theory of the Hilbert-Samuel polynomial implies that $\mathcal{I}$ is uniquely determined by a positive degree, m, and a subspace V. Consideration of the orthogonal differences $\mathcal{P}_s \ominus (\mathcal{I} \cap \mathcal{P}_s)$ then leads to the operator A in Theorem 4. As an application, one can then obtain an abstract parameterization of all cubature formulas of degree $2s-1$ for a given measure ν. It turns out that such parameterization can be formulated abstractly in terms of certain operator equations, as follows. Let $\mathcal{H}_0, \mathcal{H}_1$ be two finite dimensional Hilbert spaces, let $A, B, C : \mathcal{H}_0 \to \mathcal{H}_1$ be linear operators, and let X_1, Y_1, D be self-adjoint operators acting on $\mathcal{H}_1$. Find and describe all self-adjoint dilations (X, Y) of (X_1, Y_1), acting on the Hilbert space $H = H_1 \oplus H_2$ which satisfy

$$2i[X, Y] = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$X \begin{pmatrix} A \\ 0 \end{pmatrix} + Y \begin{pmatrix} B \\ 0 \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix}.$$

The above construction (which describes the space of cubature formulas for a given measure) and the construction in [CuFi2, Chapter 4] (which gives an *existence* criterion for solutions to TCMP) have intriguing similarities, which we wish to unravel.

Problem 4. Establish direct links among: (i) the results in [CuFi2, Chapter 4], (ii) the above operator equations, and (iii) the 3-term recurrence relations (associated with Jacobi matrices) studied by Y. Xu [Xu].

For the resolution of many of the above problems, some of the tools and techniques that we propose to utilize are derived from our previous work on *joint hyponormality*, which helped us establish the existence of polynomially hyponormal weighted shifts which are not subnormal [CuPu]. For the new problems at hand, we propose to consider suitable combinations of four basic notions:

- positivity for square matrices;
- extendibility of matrices obtained by adding a prescribed number of rows and columns;
- recursiveness; and
- the structure of the real or complex algebraic variety associated to the given moment matrix.

When these four basic ingredients interact in appropriate ways, aided by symbolic manipulation, the result is the construction of concrete algorithms that often describe in detail the space of *all* possible representing measures.

One fundamental idea in the basic construction used in [CuPu] was to extend the intrinsic connection between subnormal operators and classical moment problems in the positive real axis to classes of nearly subnormal operators and moment problems for certain linear functionals not necessarily represented by measures. These techniques also allowed us to obtain a simplification of the main result in [Cas] for the power moment problem in two dimensions. We would like to further exploit this circle of ideas to obtain further connections between operator theory and classical analysis (cf. [PuVa]).

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