Spherically quasinormal pairs of commuting operators

(joint work with Jasang Yoon)

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In memory of Professor Ronald G. Douglas

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Abstract

- We study spherically quasinormal commuting pairs of operators.
- These are the fixed points of the spherical Aluthge transform.
- For commuting 2-variable weighted shifts, we prove that spherically quasinormal pairs are directly related to spherical isometries.
- We show that each spherically quasinormal 2-variable weighted shift arises from a unilateral weighted shift.

Abstract, cont.

- We then focus on the case when this unilateral weighted shift is recursively generated.
- We show that in this case the 2-variable weighted shift is also recursively generated.
- To prove this, we use Riesz functionals and the functional calculus for the columns of the associated moment matrix.

Overview

Hyponormality and Subnormality

$\mathcal{L}(\mathcal{H})$: algebra of operators on a Hilbert space $\mathcal{H}$

$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$ (We say that N is a lifting of T, or an extension of T.
- hyponormal if $T^*T \geq TT^*$

normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal

For $S, T \in \mathcal{B}(\mathcal{H})$, $[S, T] := ST - TS$.

- An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, an operator T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

- (Bram-Halmos):

T subnormal $\Leftrightarrow$ T is k-hyponormal for all $k \geq 1$.

Unilateral Weighted Shifts

- $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^{\infty}(\mathbb{Z}_+)$, $\alpha_k > 0$ (all $k \geq 0$)
- $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0)$$

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift
- In general, $W_\alpha = U_+ D_\alpha \quad$ (polar decomposition)
- $\|W_\alpha\| = \sup_k \alpha_k$

  $W_\alpha^n e_k = \alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1} e_{k+n}$, so

$$W_\alpha^n \cong \bigoplus_{i=0}^{n-1} W_{\beta(i)},$$

Weighted Shifts and Berger's Theorem

Given a bounded sequence of positive numbers (weights) $\alpha \equiv \alpha_0, \alpha_1, \alpha_2, \ldots$, the unilateral weighted shift on $\ell^2(Z_+)$ associated with α is

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0).$$

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_k \leq \alpha_{k+1}$ (all $k \geq 0$)

Berger Measures

- (Berger; Gellar-Wallen) W_α is subnormal if and only if there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \ \ (\text{all } k \geq 0).$$

  ξ is the Berger measure of W_α.
- For $0 < a < 1$ we let $S_a := \text{shift}\,(a, 1, 1, \ldots)$.
- The Berger measure of U_+ is δ_1.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

(RC, 1990) W_α is k-hyponormal if and only if the following Hankel moment matrices are positive for $m = 0, 1, 2, \ldots$:

$$\begin{pmatrix} \gamma_m & \gamma_{m+1} & \gamma_{m+2} & \cdots & \gamma_{m+k} \\ \gamma_{m+1} & \gamma_{m+2} & & \cdots & \gamma_{m+k+1} \\ \gamma_{m+2} & \cdots & & \cdots & \gamma_{m+k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_{m+k} & \gamma_{m+k+1} & & \cdots & \gamma_{m+2k} \end{pmatrix} \geq 0.$$

(Thus, an operator matrix condition is replaced by a scalar matrix condition.)

Aluthge Transform

Let T be a Hilbert space operator, let $P := |T|$ be its positive part, and let $T = VP$ denote the canonical polar decomposition of T, with V a partial isometry and $\ker V = \ker T = \ker P$.

We define the Aluthge transform of T as

$$\hat{T} := \sqrt{P} V \sqrt{P}.$$

The iterates are

$$\hat{T}^{n+1} := \widehat{(\hat{T})^n} \ (n \geq 1).$$

The Aluthge transform has been extensively studied, in terms of algebraic, structural and spectral properties.

For instance,

- (i) $T = \hat{T} \Leftrightarrow T$ is quasinormal;
- (ii) (Aluthge, 1990) If $0 < p < \frac{1}{2}$ and T is p-hyponormal, then $\hat{T}$ is $(p + \frac{1}{2})$-hyponormal;
- (iii) (Jung, Ko & Pearcy, 2000) If $\hat{T}$ has a n.i.s., then T has a n.i.s.
- (iv) (Kim-Ko, 2005; Kimura, 2004) T has property (β) if and only if $\hat{T}$ has property (β); and
- (v) (Ando, 2005) $\left\|(T - \lambda)^{-1}\right\| \geq \left\|(\hat{T} - \lambda)^{-1}\right\|$ $(\lambda \notin \sigma(T))$.
- (vi) Observe that if $A := \sqrt{P}$ and $B := V\sqrt{P}$, then $\hat{T} = AB$ and $T = BA$, and therefore

$$\sigma(\hat{T}) \setminus \{0\} = \sigma(T) \setminus \{0\}.$$

On the other hand,

G. Exner (IWOTA 2006 Lecture): subnormality is not preserved under the Aluthge transform. Concretely, Exner proved that the Aluthge transform of the weighted shift in the following example is not subnormal.

Example

(RC, Y. Poon and J. Yoon, 2005) Let

$$\alpha \equiv \alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n + \frac{1}{2}}{2^n + 1}}, & \text{if } n \geq 1 \end{cases},$$

Then W_α is subnormal, with 3-atomic Berger measure

$$\mu = \frac{1}{3}(\delta_0 + \delta_{1/2} + \delta_1).$$

(S.H. Lee, W.Y. Lee and J. Yoon, 2012) For $k \geq 2$, the Aluthge transform, when acting on weighted shifts, need not preserve k-hyponormality.

Note that the Aluthge transform of a weighted shift is again a weighted shift.

Concretely, the weights of $\widehat{W_\alpha}$ are

$$\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \sqrt{\alpha_2\alpha_3}, \sqrt{\alpha_3\alpha_4}, \cdots .$$

Define

$$W_{\sqrt{\alpha}} := \text{shift } \left(\sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \cdots\right).$$

Then $\widehat{W_\alpha}$ is the Schur product of $W_{\sqrt{\alpha}}$ and its restriction to the subspace $\bigvee\{e_1, e_2, \cdots\}$. Thus, a sufficient condition for the subnormality of $\widehat{W_\alpha}$ is the subnormality of $W_{\sqrt{\alpha}}$.

Agler Shifts

For $j = 2, 3, \ldots$, the j-th Agler shift A_j is given by

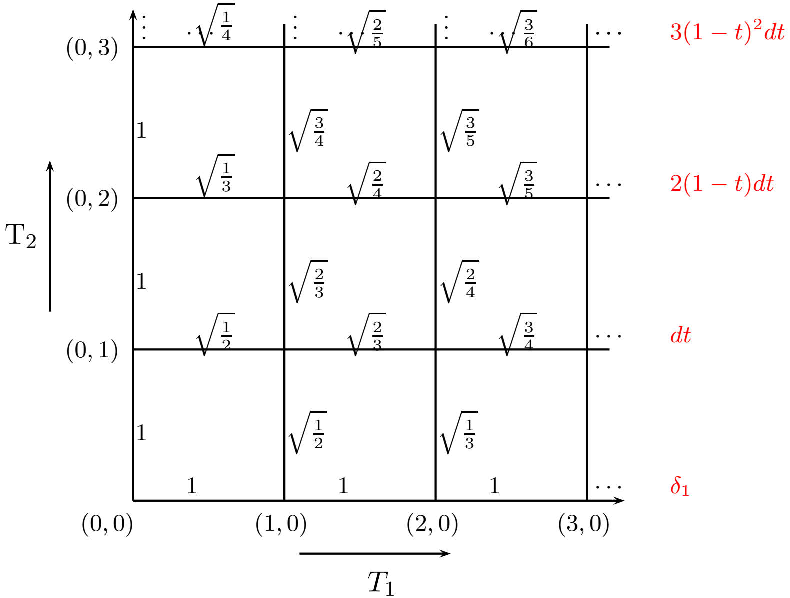
$$\alpha^{(j)} := \sqrt{\frac{1}{j}}, \sqrt{\frac{2}{j+1}}, \sqrt{\frac{3}{j+2}}, \ldots.$$

It is well known that A_j is subnormal, with Berger measure

$$d\mu^j(t) = (j-1)(1-t)^{j-2}dt.$$

Clearly, A_2 is the Bergman shift, and the remaining Agler shifts are the upper row shifts of the Drury-Arveson 2-variable weighted shift, which incidentally is a spherical complete hyperexpansion.

Weight Diagram of the Drury-Arveson Shift



A Subnormal Completion Problem

Let $\frac{p}{q} < \frac{r}{s}$ be two rational numbers in the open interval $(0, 1)$.
Question: Does there exist a subnormal weighted shift W_α such that $\alpha_0 = \frac{p}{q}$ and $\alpha_1 = \frac{r}{s}$?

Answer: Yes. (C. Benhida, RC and G. Exner, 2018)

Proof: Given any two rational numbers $\frac{p}{q} < \frac{r}{s}$ in the open interval $(0, 1)$ there exists an Agler shift A_k having these two numbers as weights. Now consider the regular subshift of A_k whose first two weights are $\frac{p}{q}$ and $\frac{r}{s}$. Since A_k is subnormal, the regular subshift is also subnormal. Actually, both A_k and its regular subshift are infinitely divisible, i.e., their p-th powers are subnormal for all $p > 0$.

Multivariable Weighted Shifts

$$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2), \quad \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$$

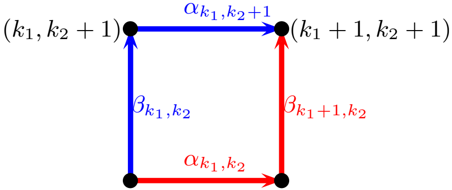
$$\ell^2(\mathbb{Z}_+^2) \cong \ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+).$$

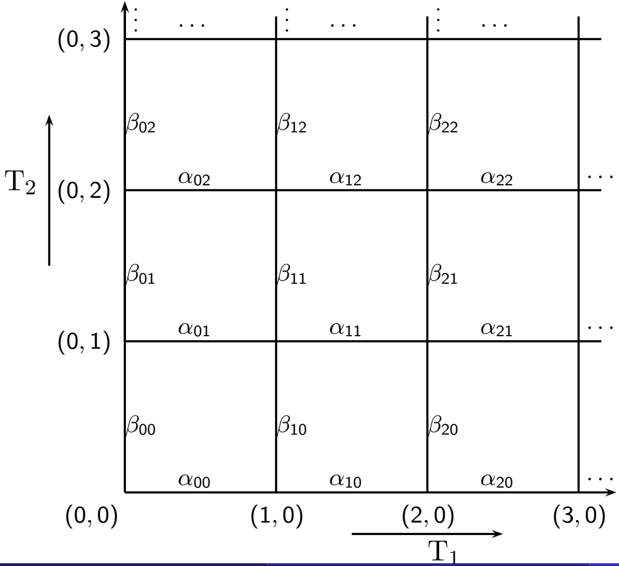
We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \iff \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \ \text{ (all } \mathbf{k}\text{)}.$$




(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T2
T1
β02
β12
β22
β01
β11
β21
β00
β10
β20
α02
α12
α22
α01
α11
α21
α00
α10
α20

To detect hyponormality, there is a simple criterion:

Theorem

(RC, 1988) (Six-point Test) Let $\mathbf{T} \equiv (T_1, T_2)$ *be a 2-variable weighted shift, with weight sequences* α *and* β*. Then*

$$\mathbf{T} \text{ is hyponormal } \Leftrightarrow \begin{pmatrix} \alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geq 0$$

(all $\mathbf{k} \in \mathbb{Z}^2_+$*).*

We now recall the notion of moment of order $\mathbf{k}$ for a commuting pair (α, β). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is $\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta)$

$$:= \begin{cases} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

By commutativity, $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).

- (Jewell-Lubin)

$$W_\alpha \text{ is subnormal} \quad \Leftrightarrow \quad \gamma_{\mathbf{k}} := \prod_{i=0}^{k_1-1} \alpha_{(i,0)}^2 \cdot \prod_{j=0}^{k_2-1} \beta_{(k_1-1,j)}^2$$

$$= \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \ \text{ (all } \mathbf{k} \geq \mathbf{0}).$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

Following A. Athavale-S. Poddar and J. Gleason, we say that $\mathbf{T}$ is **(jointly) quasinormal** if T_i commutes with $T_j^* T_j$ for all $i, j = 1, 2$; and **spherically quasinormal** if T_i commutes with

$$P := T_1^* T_1 + T_2^* T_2$$

for $i = 1, 2$. One has

$$\begin{aligned} \text{normal} \quad &\Longrightarrow \quad \text{(jointly) quasinormal} \Longrightarrow \text{spherically quasinormal} \\ &\Longrightarrow \quad \text{subnormal} \Longrightarrow k\text{-hyponormal.} \Longrightarrow \text{hyponormal.} \qquad (1) \end{aligned}$$

On the other hand, results of RC-S.H. Lee-J. Yoon and J. Gleason show that the reverse implications in (1) do not necessarily hold.

Spherical Aluthge Transform

Consider a (joint) polar decomposition of the form

$$(T_1, T_2) \equiv (V_1 P, V_2 P),$$

or equivalently,

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P,$$

as operators from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$. Moreover, this is the unique canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. It follows that $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $(\ker P)^\perp$ onto $\overline{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}}$. where $P := \sqrt{T_1^* T_1 + T_2^* T_2}$.

Now let

$$\widehat{\mathbf{T}} := \left(\sqrt{P}V_1\sqrt{P}, \sqrt{P}V_2\sqrt{P}\right). \tag{2}$$

One can prove that $V_1^*V_1 + V_2^*V_2$ is a (joint) partial isometry, and that $\widehat{\mathbf{T}}$ is commutative whenever $\mathbf{T}$ is commutative.

Spherically quasinormal pairs

Recall that the spherical Aluthge transform preserves commutativity for 2-variable weighted shifts. Also, we say that $\mathbf{T}$ is spherically quasinormal if T_i commutes with P, for $i = 1, 2$. Equivalently, $\mathbf{T}$ is spherically quasinormal if $\widehat{\mathbf{T}} = \mathbf{T}$.

Lemma

Assume P injective. Then $\mathbf{T}$ is spherically quasinormal if and only if $T_i P = P T_i$ $(i = 1, 2)$ if and only if $V_i P = P V_i$ $(i = 1, 2)$. As a consequence, if $\mathbf{T}$ is spherically quasinormal then (V_1, V_2) is commuting.

Proposition

(RC-J. Yoon; 2015) A 2-variable weighted shift $\mathbf{T}$ is spherically quasinormal if and only if there exists $C > 0$ such that $\frac{1}{C}\mathbf{T}$ is a spherical isometry, that is, $T_1^* T_1 + T_2^* T_2 = I$.

Definition

A commuting pair **T** is a spherical isometry if $T_1^* T_1 + T_2^* T_2 = I$.

Lemma

(RC-J. Yoon; 2015) $W_{\alpha,\beta}$ *is a spherical isometry if and only if*

$\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1$ *for all* $\mathbf{k} \in \mathbb{Z}_+^{\mathbf{2}}$.

Theorem

(Athavale; JOT, 1990) A spherical isometry is always subnormal.

Corollary

(RC-J. Yoon; 2016) A spherically quasinormal 2-variable weighted shift is subnormal.

Corollary

(RC-J. Yoon; 2016) Let $\mathbf{T}$ *be a spherically quasinormal pair, and assume that* P *is injective. Then* $\mathbf{T}$ *is hyponormal.*

Theorem

(A. Athavale - S. Poddar; 2015 and S. Chavan - V. Sholapurkar; 2013) Let $\mathbf{T}$ *be a spherically quasinormal pair. Then* $\mathbf{T}$ *is subnormal.*

Theorem

(V. Müller - M. Ptak; 1999) Spherical isometries are hyporeflexive.

Theorem

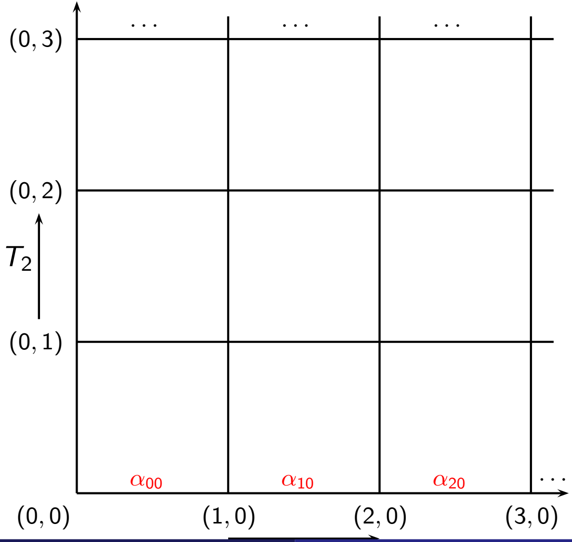
(J. Eschmeier - M. Putinar; 2000) For every $n \geq 3$ there exists a non-normal spherical isometry $\mathbf{T}$ *such that the polynomially convex hull of* $\sigma_T(\mathbf{T})$ *is contained in the unit sphere.*

Theorem

(C. Benhida - RC, 2018) Let $T \equiv (T_1, \cdots, T_n)$ *be a commuting n-tuple of Hilbert space operators, and let* $\widehat{T}$ *be its spherical Aluthge transform. Then*

$$\sigma_T(\widehat{T}) = \sigma_T(T).$$

Construction of spher. isom. 2-var. w.s.



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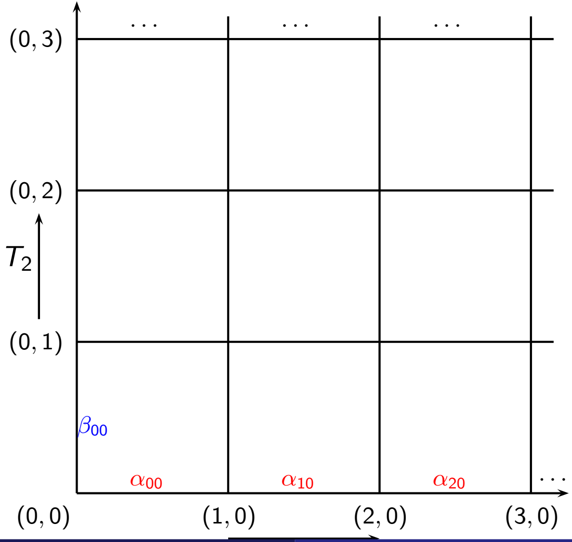

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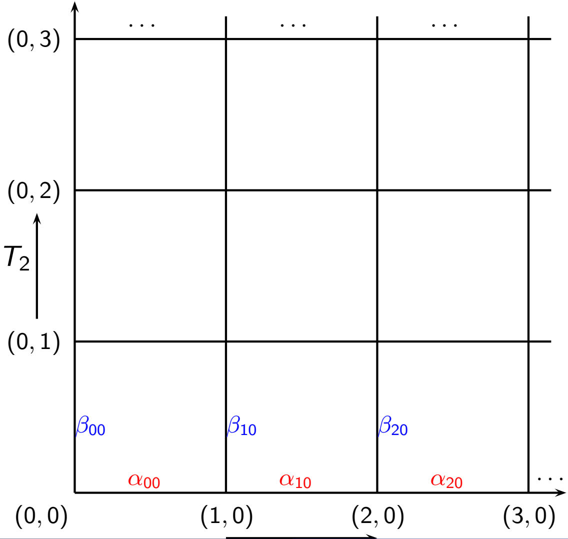

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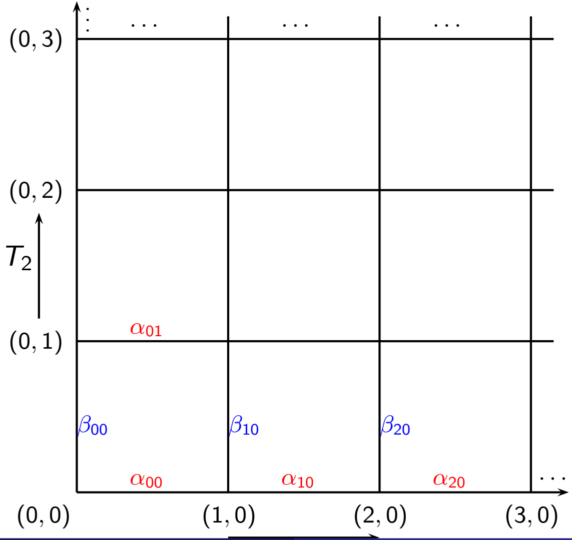

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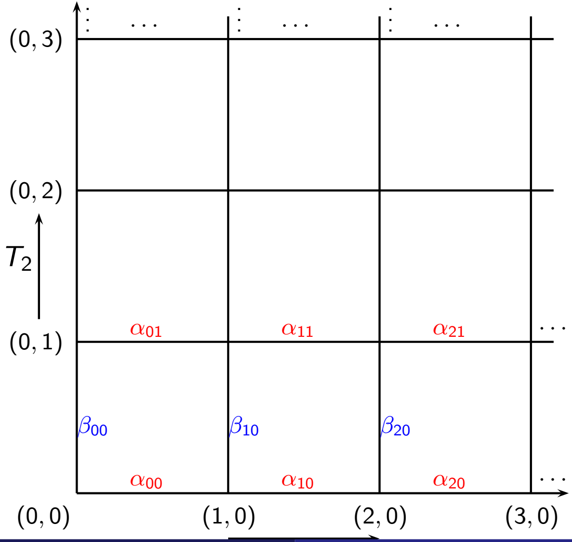
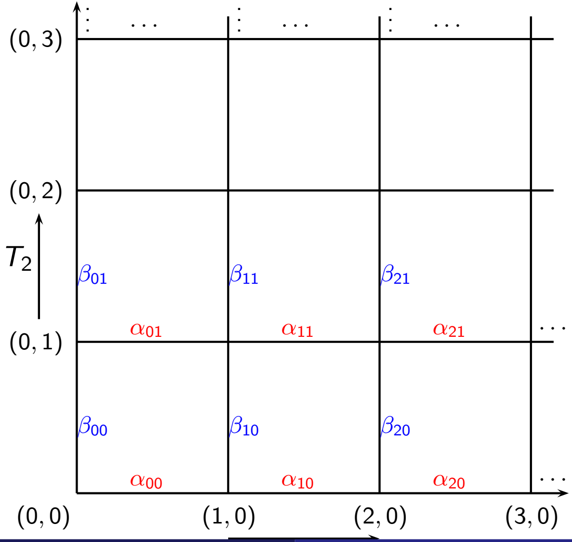


Construction of spher. isom. 2-var. w.s.



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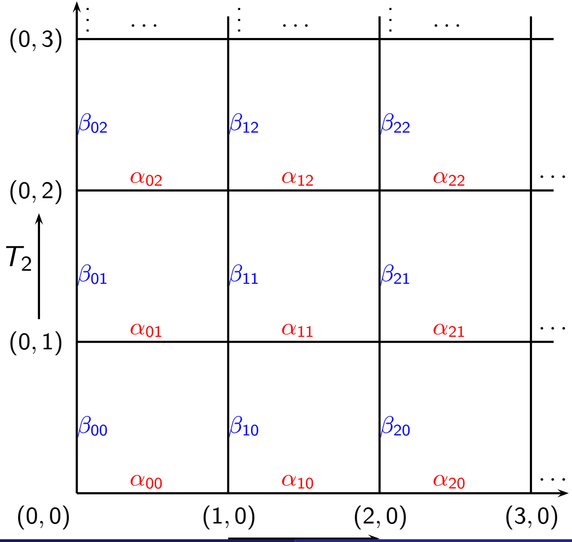


Recursively Generated Weighted Shifts

Recall that a unilateral weighted shift is *recursively generated* if the sequence of moments satisfy a linear relation

$$\gamma_{n+k} = \varphi_0 \gamma_n + \varphi_1 \gamma_{n+1} + \cdots + \varphi_{k-1} \gamma_{n+k-1} \quad (k \geq 1,\ n \geq 0).$$

Theorem

(RC-Fialkow; IEOT, 1993) A subnormal weighted shift W_α is recursively generated if and only if its Berger measure is finitely atomic.

Functional Calculus

$$\begin{pmatrix} 1 & T & T^2 & \dots & T^k \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & & \cdots & \gamma_{k+1} \\ \gamma_2 & \cdots & & \cdots & \gamma_{k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_k & \gamma_{k+1} & & \cdots & \gamma_{2k} \end{pmatrix} \geq 0.$$

When W_α is recursively generated we can interpret

$$\gamma_{n+k} = \varphi_0 \gamma_n + \varphi_1 \gamma_{n+1} + \cdots + \varphi_{k-1} \gamma_{n+k-1} \quad (k \geq 1,\ n \geq 0).$$

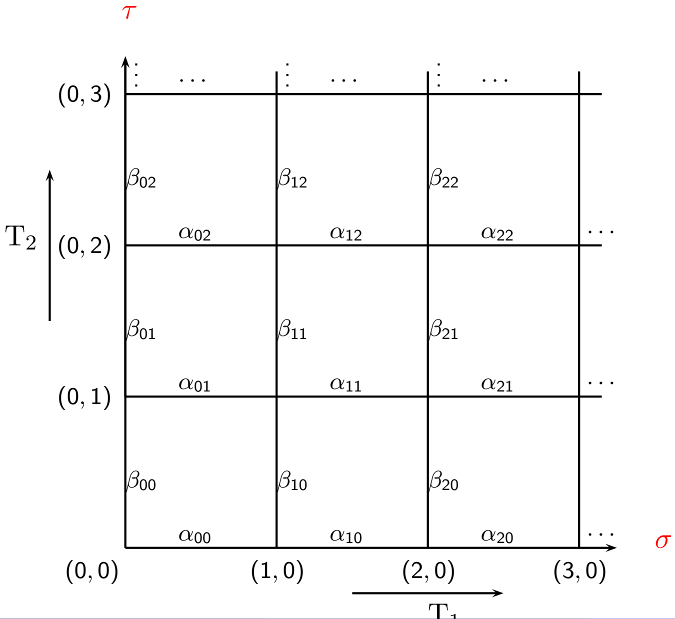
as

$$T^k = \varphi_0 1 + \varphi_1 T + \cdots + \varphi_{k-1} T^{k-1}.$$

(RC-Fialkow; 1993) Let μ be the Berger measure of W_α. Then

$$\text{supp } \mu \subseteq \{t : t^k - (\varphi_0 + \varphi_1 t + \cdots + \varphi_{k-1} t^{k-1}) = 0\}.$$

For 2-variable weighted shifts, a similar construction using the moments of (α, β) works.



τ
σ
T_2
T_1
(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
β_{02} β_{12} β_{22}
α_{02} α_{12} α_{22}
β_{01} β_{11} β_{21}
α_{01} α_{11} α_{21}
β_{00} β_{10} β_{20}
α_{00} α_{10} α_{20}

Theorem

(RC-Yoon; 2016) Let $W_{(\alpha,\beta)}$ be a spherical isometry, and assume that the zero-th row is subnormal with finitely atomic Berger measure σ.

(i) Each horizontal row is recursively generated, and its moments satisfy the same linear relation as the zero-th row.

(ii) Each vertical column is recursively generated, and its moments satisfy the linear relation obtained from (ii) which appropriately reflects the condition $\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1$ ($\mathbf{k} \in \mathbb{Z}_+^2$).

(iii) The Berger measure of $W_{(\alpha,\beta)}$ is finitely atomic, with support contained in the Cartesian product of σ and τ, where τ is the Berger measure of the zero-th column of $W_{(\alpha,\beta)}$.

Theorem (cont.)

(iv) If $\Lambda^{(0)}$ and ${}^{(0)}\Lambda$ are the Riesz functional of the zero-th row and zero-th column of $W_{(\alpha,\beta)}$, resp., then

$$ {}^{(0)}\Lambda(p(t)) = \Lambda^{(0)}(p(1-t)) $$

for every polynomial p. As a result,

$$ \text{supp}\ \tau = 1 - \text{supp}\ \sigma. $$

Proof.

Let W_0 and W_1 denote the 0-th row and first row, respectively; each is a unilateral weighted shift.

Key Identity:

$$\begin{aligned}\beta^2_{(0,0)}\gamma_{k_1}(W_1) &= \beta^2_{(k_1,0)}\gamma_{k_1}(W_0) \\ &= \left(1-\alpha^2_{(k_1,0)}\right)\gamma_{k_1}(W_0) \\ &= \gamma_{k_1}(W_0)+\gamma_{k_1+1}(W_0),\end{aligned}$$

One is naturally led to the following question.

Question

If W_0 is recursively generated, is it also the case that V_0 is recursively generated?

To study this question, we will take advantage of the theory of truncated moment problems in two real variables.

We will make use of the moment matrix associated with $W_{(\alpha,\beta)}$; that is, the infinite matrix $M(\alpha,\beta)$ whose rows and columns are indexed by $\mathbf{k} \in \mathbb{Z}_+^2$ and whose $(\mathbf{i},\mathbf{j})$-entry is given by $\gamma_{\mathbf{i}+\mathbf{j}}$.

As typically done in the theory of truncated real moment problems, it is natural to label the rows and columns of $M(\alpha,\beta)$ using the homogenous monomials of ascending degree $1, S, T, S^2, ST, T^2, S^2, S^2T, ST^2, T^3, \cdots$. For instance, when we refer to the entry in the position $((1,2),(0,1))$, we mean the entry corresponding to row $(1,2)$ and column $(0,1)$, that is, the row labeled by the monomial ST^2 and the column labeled by the monomial T.

Lemma

Let $W_{(\alpha,\beta)}$ be a 2-variable weighted shift, let $c > 0$ and fix $\mathbf{k} \in \mathbb{Z}_+^2$. The following statements are equivalent.

(i) $\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = c$.

(ii) $\gamma_{\mathbf{k}+\varepsilon_1} + \gamma_{\mathbf{k}+\varepsilon_2} = c\gamma_{\mathbf{k}}$. $\quad (S + T = c\,1)$

Corollary

Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c > 0$. Then the columns of the moment matrix $M(\alpha, \beta)$ satisfy the linear relation $S + T = c\ 1$.

Corollary

Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c > 0$, and let σ and τ be the Berger measures of W_0 and V_0, respectively. Then $\operatorname{supp} \tau = c - \operatorname{supp} \sigma := \{c - s : s \in \operatorname{supp} \sigma\}$.

Proof.

Since the columns of the moment matrix $M(\alpha, \beta)$ satisfy the linear relation $S + T = c\ 1$, the Riesz functionals Λ_{W_0} and Λ_{V_0} for σ and τ (resp.) satisfy the condition

$$\Lambda_\beta(p(t)) = \Lambda_\alpha(p(c - t)),$$

for every polynomial p in one real variable. This immediately leads to the desired result about the supports of the Berger measures. □

We are now ready to prove that for spherically quasinormal 2-variable weighted shifts the property of being recursively generated **transfers** from the 0-th row in the weight diagram to the 0-th column.

Theorem

Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c > 0$, and assume that the unilateral weighted shift W_0 (which corresponds to the 0-th row in the weight diagram of $W_{(\alpha,\beta)}$) is recursively generated. Then the unilateral weighted shift V_0 ((which corresponds to the 0-th column) is also recursively generated.

Sketch of Proof.

In the moment matrix, $S + T = c\,\mathit{1}$. Thus, at the level of polynomials in the indeterminates s and t, one can replace any occurrence of s by $c - t$. As a consequence, the same holds for the columns of M. Thus, the linear relation

$$S^k = \varphi_0 1 + \varphi_1 S + \cdots + \varphi_{k-1} S^{k-1}$$

can be rewritten (in terms of T) as

$$(c\,\mathit{1} - T)^k = \varphi_0 1 + \varphi_1 (c\,\mathit{1} - T) + \cdots + \varphi_{k-1} (c\,\mathit{1} - T)^{k-1}. \qquad (3)$$

Now solve for T:

$$T^k = \sum_{j=0}^{k-1} (-1)^{k-j} \psi_j T^j, \qquad (4)$$

where

$$\psi_j := \sum_{i=j}^{k-1} \binom{i}{j} \varphi_i c^{i-j} - \binom{k}{j} c^{k-j}. \quad \square$$

Corollary

Let $W_{(\alpha,\beta)}$ *be a spherically quasinormal* 2*-variable weighted shift, with constant* $c > 0$*, and assume that the unilateral weighted shift* W_0 *(which corresponds to the* 0*-th row in the weight diagram of* $W_{(\alpha,\beta)}$*) is recursively generated. Let* σ *be the Berger measure of* W_0*, and let* μ *be the Berger measure of* $W_{(\alpha,\beta)}$*. Then*

(i) $\operatorname{supp}\mu \subseteq \operatorname{supp}\sigma \times (c - \operatorname{supp}\sigma)$*; and*

(ii) μ *is finitely atomic.*

The following problem arises naturally.

Problem

Consider a spherically quasinormal 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ and let σ be the Berger measure of W_0. Since $W_{(\alpha,\beta)}$ is subnormal, let μ be the Berger measure of $W_{(\alpha,\beta)}$.

(i) Describe μ in terms of σ.

(ii) Assume that W_0 is recursively generated. We know that μ is finitely atomic, and that $\operatorname{supp}\mu \subseteq \operatorname{supp}\sigma \times (c - \operatorname{supp}\sigma)$. What else can we say? Can we give a concrete formula for the atoms and densities of μ?

We know that W_0 carries all the information about $W_{(\alpha,\beta)}$; therefore we know that the atoms and densities of μ must algorithmically be obtained from those of σ.

Example. Assume that σ is 2-atomic, and write $\sigma \equiv \lambda_0 \delta_{s_0} + \lambda_1 \delta_{s_1}$, with $0 \leq s_0 < s_1 \leq 1$ and $\lambda_0, \lambda_1 > 0$. We know that

$$\operatorname{supp} \mu \subseteq \{(s_0, c - s_0), (s_0, c - s_1), (s_1, c - s_0), (s_1, c - s_1)\}.$$

Moreover, $\operatorname{supp} \mu$ must have at least two atoms, because σ (and τ) are 2-atomic. Thus, we can postulate that

$$\mu = \rho_{00} \delta_{(s_0, c - s_0)} + \rho_{01} \delta_{(s_0, c - s_1)} + \rho_{10} \delta_{(s_1, c - s_0)} + \rho_{11} \delta_{(s_1, c - s_1)},$$

with $\rho_{ij} \geq 0$ $(i, j = 1, 2)$.

We now write the moment equations in Vandermonde form, as follows:

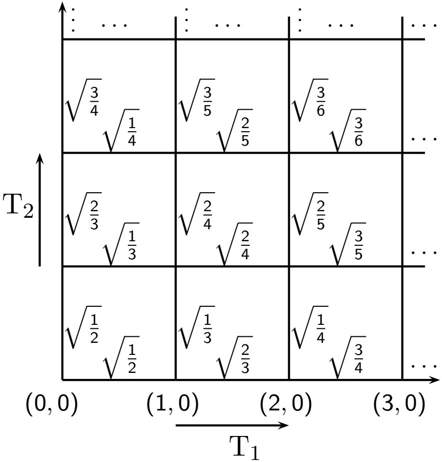
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ s_0 & s_0 & s_1 & s_1 \\ c - s_0 & c - s_1 & c - s_0 & c - s_1 \\ s_0(c - s_0) & s_0(c - s_1) & s_1(c - s_0) & s_1(c - s_1) \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{pmatrix} = \begin{pmatrix} \gamma_{(0,0)} \\ \gamma_{(0,1)} \\ \gamma_{(1,0)} \\ \gamma_{(1,1)} \end{pmatrix}$$

Solving this equation, one easily obtains that μ is always 2-atomic.

We conclude this section with an intriguing question.

Question

Let W_0 be the Bergman shift $\text{shift}(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots)$*, and build a spherically quasinormal 2-variable weighted shift W. For this shift the j-th row is identical to the j-column, for every $j \geq 0$. Note also that W is a close relative of the Drury-Arveson 2-variable weighted shift, in that the j-row of W is the Agler A_{j+2} shift. What is the Berger measure of W?*



$\sqrt{\frac{3}{4}}$ $\sqrt{\frac{1}{4}}$
$\sqrt{\frac{3}{5}}$ $\sqrt{\frac{2}{5}}$
$\sqrt{\frac{3}{6}}$ $\sqrt{\frac{3}{6}}$
$\sqrt{\frac{2}{3}}$ $\sqrt{\frac{1}{3}}$
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$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$
$\sqrt{\frac{1}{3}}$ $\sqrt{\frac{2}{3}}$
$\sqrt{\frac{1}{4}}$ $\sqrt{\frac{3}{4}}$
T_2
$(0,0)$
$(1,0)$
$(2,0)$
$(3,0)$
T_1

xiè xiè nǐmen!