Hyponormality of pairs of Toeplitz operators with matrix-valued symbols

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AMS Annual Meeting Special Session on

Analytic Function Spaces and Operators on Them

January 9, 2016

(with In Sung Hwang and Woo Young Lee; two related papers have appeared in Adv. Math., J. Funct. Anal.)

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Abstract

We consider pairs of Toeplitz operators on the Hardy space of the unit circle, whose symbols are matrix-valued.
We obtain a complete characterization of (joint) hyponormality when the symbols are of bounded type.
As in the scalar case (RC, W.Y. Lee, Memoirs AMS(2001)), we reduce joint hyponormality to the hyponormality of single Toeplitz operators.
As a corollary, we obtain a simple description of (joint) hyponormality when the symbols are rational matrix-valued.
We study the self-commutators of Toeplitz pairs with matrix-valued rational symbols, and derive the associated rank formulae.

Notation and Preliminaries

$L^\infty \equiv L^\infty(\mathbb{T})$; $H^\infty \equiv H^\infty(\mathbb{T})$; $L^2 \equiv L^2(\mathbb{T})$; $H^2 \equiv H^2(\mathbb{T})$,
$P : L^2 \to H^2$ orthogonal projection
$T \in \mathcal{L}(\mathcal{H})$: algebra of bounded operators on a Hilbert space $\mathcal{H}$

- normal if $T^*T = TT^*$
- quasinormal if T commutes with T^*T
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- hyponormal if $T^*T \geq TT^*$
- 2-hyponormal if (T, T^2) is (jointly) hyponormal ($k \geq 1$)

$$\begin{pmatrix} [T^*, T] & [T^{*2}, T] \\ [T^*, T^2] & [T^{*2}, T^2] \end{pmatrix} \geq 0$$

quasinormal $\Rightarrow$ subnormal $\Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal

For $\varphi \in L^\infty$, the Toeplitz operator with symbol φ is $T_\varphi : H^2 \to H^2$, given by

$$T_\varphi f := P(\varphi f) \quad (f \in H^2)$$

T_φ is said to be *analytic* if $\varphi \in H^\infty$

Halmos's Problem 5 (1970):

Is every subnormal Toeplitz operator either normal or analytic?

C. Cowen and J. Long (1984): No

- C. Cowen (1988)

  $\varphi \in L^\infty$, $\varphi = \bar{f} + g \ \ (f, g \in H^2)$

$$T_\varphi \textit{ is hyponormal } \Leftrightarrow \ f = c + T_{\bar{h}} g,$$

  for some $c \in \mathbb{C}$, $h \in H^\infty$, $\|h\|_\infty \leq 1$.

- Nakazi-Takahashi (1993)

  For $\varphi \in L^\infty$, let

$$\mathcal{E}(\varphi) := \{k \in H^\infty : \|k\|_\infty \leq 1 \textit{ and } \varphi - k\bar{\varphi} \in H^\infty\}.$$

  Then

$$T_\varphi \textit{ is hyponormal } \Leftrightarrow \ \mathcal{E}(\varphi) \neq \emptyset.$$

Natural Questions:

1) When is T_φ subnormal?

At present, there's no known characterization of subnormality in terms of the symbol φ.

2) Characterize 2-hyponormality for Toeplitz operators

Sample Result:

Theorem

(RC and WY Lee, 2001) Every 2-hyponormal **trigonometric** *Toeplitz operator is subnormal.*

- Cowen-Long (1984): Let $0 < \alpha < 1$, let $\psi : \mathbb{D} \to E$ be conformal, where E is the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$, and let

$$\varphi := \frac{\psi + \alpha\bar{\psi}}{1-\alpha^2}.$$

Question

Let φ be the Cowen & Long symbol. Does it follow that $T_\varphi \cong T_\eta$ for some $\eta \in H^\infty$?

Question

A Reformulation of Halmos's Problem 5

Let T_φ be a non-normal subnormal Toeplitz operator. Does it follow that $T_\varphi \cong T_\eta$ for some $\eta \in H^\infty$?

These two questions remain open.

Functions of bounded type and Abrahamse's Theorem

$\varphi \in L^\infty$ is of *bounded type* (or in the Nevanlinna class) if

$$\varphi := \frac{\psi_1}{\psi_2} \quad (\psi_1, \psi_2 \in H^\infty).$$

(Abrahamse, 1976) Assume φ or $\bar{\varphi}$ is of bounded type. If T_φ is hyponormal and $\ker[T_\varphi^*, T_\varphi]$ is invariant for T_φ, then T_φ is normal or analytic.

Thus, the answer to Halmos's Problem 5 is affirmative if φ is of bounded type.

Block Toeplitz Operators

$M_n := M_{n\times n}$ $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$ $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ For $\Phi \in L^\infty_{M_n}$, $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes the *block* Toeplitz operator with symbol Φ defined by

$$T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$.

A *block Hankel* operator with symbol $\Phi \in L^\infty_{M_n}$ is the operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where $J_n(f)(z) := \overline{z} I_n f(\overline{z})$ for $f \in L^2_{\mathbb{C}^n}$.

We easily see that

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \dots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \dots & T_{\varphi_{nn}} \end{bmatrix} \quad \text{and} \quad H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \dots & H_{\varphi_{1n}} \\ & \vdots & \\ H_{\varphi_{n1}} & \dots & H_{\varphi_{nn}} \end{bmatrix},$$

where

$$\Phi = \begin{bmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \dots & \varphi_{nn} \end{bmatrix} \in L^\infty_{M_n}.$$

For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if $\Theta^*\Theta = I_m$ almost everywhere on $\mathbb{T}$. Given $\Phi, \Psi \in L^\infty_{M_n}$,

$$T^*_\Phi = T_{\Phi^*}$$

$$H^*_\Phi = H_{\widetilde{\Phi}} \quad (\text{recall that } \widetilde{\Phi}(z) := \Phi^*(\overline{z}))$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}$$

Block Toeplitz operators have been studied by D.Z. Arov, E. Basor, A. Böttcher, R.G. Douglas, H. Dym, I. Feldman, I. Gohberg, S. Grudsky, C. Gu, W. Helton, J. Hendricks, I.S. Hwang, D.-O. Kang, M.A. Kaashoek, I. Koltracht, W.Y. Lee, A. Rogozhin, D. Rutherford, I. Spitkovsky, H. Woerdeman, D. Zheng, Y. Zucker, and others.

R.G. Douglas, *Banach Algebra Techniques in the Theory of Toeplitz Operators*, Amer. Math. Soc., 1980.

$\Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n}$ is of *bounded type* if each entry φ_{ij} is of bounded type.
Φ is *rational* if each entry φ_{ij} is a rational function.

The *shift* operator S on $H^2_{\mathbb{C}^n}$ is defined by

$$S := T_{zI_n}.$$

The Beurling-Lax-Halmos Theorem. *A nonzero subspace $\mathcal{M}$ of $H^2_{\mathbb{C}^n}$ is invariant for S if and only if $\mathcal{M} = \Theta H^2_{\mathbb{C}^m}$, where Θ is an inner matrix function. Furthermore, Θ is unique up to a unitary constant right factor.*

As a consequence, if $\ker H_\Phi \neq \{0\}$, then

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ.

Hyponormality of Block Toeplitz Operators

(C. Gu, J. Hendricks and D. Rutherford, 2006) For $\Phi \in L^\infty_{M_n}$, let

$$\mathcal{E}(\Phi) := \left\{K \in H^\infty_{M_n} : \ ||K||_\infty \leq 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n}\right\}.$$

Then T_Φ is hyponormal if and only if Φ is normal (i.e. $\Phi^*\Phi = \Phi\Phi^*$) and $\mathcal{E}(\Phi)$ is nonempty.

Theorem

(Gu, Hendricks and Rutherford, 2006) For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:

1. *Φ is of bounded type;*
2. $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ *for some square inner matrix function Θ;*
3. *$\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.*

Definition: Θ and A are right coprime if they do not have a common nontrivial right factor.

Example

Let $\Phi := \left(\begin{smallmatrix} z & z \\ z & z \end{smallmatrix}\right)$ then we can write

$$\Phi = \Theta A^* = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

but $\Theta := \left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]$ and $A := \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ are not right coprime because $\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} z & -z \\ 1 & 1 \end{smallmatrix}\right]$ is a common right inner divisor, i.e.,

$$\Theta = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & -z \\ 1 & 1 \end{bmatrix}$$

$$A = \sqrt{2} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & -z \\ 1 & 1 \end{bmatrix}.$$

Given a matrix-valued symbol Φ, write

$$\Phi \equiv \Phi_-^* + \Phi_+ = \Theta A^*.$$

If Φ and Φ^* are of bounded type, then

$$\Phi_+ = \Theta_1 A^*$$

and

$$\Phi_- = \Theta_2 B^*,$$

where Θ_1 and A are right coprime, and Θ_2 and B are also right coprime.

Necessary Condition for Hyponormality:

Theorem

$\Theta_2 | \Theta_1$.

Thus, WLOG, we can always assume:

$\Phi_+ = \Theta_2 \Theta_0 A^*$

$\Phi_- = \Theta_2 B^*$

If Φ is rational, then each Θ_i is a finite Blaschke product. In general, Θ and A need not be (right) coprime!

(Recall that $\Phi = \Theta A^*$, and that Θ and A are right coprime if they do not have a common nontrivial right factor.)

Proposition

(2010, CHL) Let $B \in H^{\infty}_{M_n}$ be rational and let $\Theta \equiv \theta I_n$, where θ is a finite Blaschke product. The following statements are equivalent:

(a) *$B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$ (the zero set of θ);*

(b) *B and Θ are right coprime;*

(c) *B and Θ are left coprime.*

Corollary

Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ *is a matrix-valued rational function. We may write*

$$\Phi_- = \Theta B^*,$$

where $\Theta := \theta I_n$ *with a finite Blaschke product* θ*. Assume that* $B(\alpha)$ *is invertible for each* $\alpha \in \mathcal{Z}(\theta)$*. If* T_Φ *is subnormal then* T_Φ *is normal or analytic.*

Quasinormal Block Toeplitz Operators

Yakubovich's Theorem (2006). If $T \in \mathcal{B}(\mathcal{H})$ is a pure subnormal operator with finite rank self-commutator and without point masses then it is unitarily equivalent to a Toeplitz operator T_Φ with a matrix-valued analytic rational symbol Φ.

On the other hand, Ito and Wong proved in 1972 that every quasinormal Toeplitz operator is either normal or analytic, i.e., the answer to the Halmos's Problem 5 is affirmative for quasinormal Toeplitz operators.

However, this is not true for the cases of matrix-valued symbols: indeed, if

$$\Phi \equiv \begin{bmatrix} \overline{z} & \overline{z} + 2z \\ \overline{z} + 2z & \overline{z} \end{bmatrix}.$$

then T_Φ is quasinormal, but it is neither normal nor analytic. But since

$$T_\Phi = \begin{bmatrix} U_+^* & U_+^* + 2U_+ \\ U_+^* + 2U_+ & U_+^* \end{bmatrix},$$

one can prove that T_Φ us unitarily equivalent to

$$2 \begin{bmatrix} U_+^* + U_+ & 0 \\ 0 & -U_+ \end{bmatrix},$$

that is, the direct sum of a normal operator, $2(U_+^* + U_+)$, and an analytic Toeplitz operator, $-2U_+$.

Theorem (CHKL, 2013)

Every pure quasinormal operator with finite rank self-commutator is unitarily equivalent to a Toeplitz operator with a matrix-valued analytic rational symbol.

Corollary (CHKL, 2013)

Every pure quasinormal Toeplitz operator with a matrix-valued rational symbol is unitarily equivalent to an analytic Toeplitz operator.

Hyponormal Toeplitz pairs

In 2001, RC and W.Y. Lee, we completely characterized the hyponormality of Toeplitz pairs $\mathbf{T} \equiv (T_\varphi, T_\psi)$, when both symbols φ and ψ are trigonometric polynomials.

Key building block: For φ and ψ trigonometric polynomials, the hyponormality of $\mathbf{T} \equiv (T_\varphi, T_\psi)$ forces the co-analytic parts of φ and ψ to necessarily coincide up to a constant multiple; equivalently,

$$[ECAP] \qquad \varphi - \beta\psi \in H^2 \text{ for some } \beta \in \mathbb{C}.$$

We show that (ECAP) can be extended to the case of matrix functions of bounded type.

(ECAP) still holds (properly interpreted) for pairs of Toeplitz operators whose symbols are matrix-valued trigonometric polynomials.

We first observe that if $\mathbf{T} = (T_\varphi, T_\psi)$ then the self-commutator of $\mathbf{T}$ can be expressed as:

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_\varphi^*, T_\varphi] & [T_\psi^*, T_\varphi] \\ [T_\varphi^*, T_\psi] & [T_\psi^*, T_\psi] \end{pmatrix}$$

$$= \begin{pmatrix} H^*_{\overline{\varphi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\varphi_-}} & H^*_{\overline{\varphi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\varphi_-}} \\ H^*_{\overline{\psi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\psi_-}} & H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\psi_-}} \end{pmatrix}.$$

The hyponormality of Toeplitz pairs is also related to the kernels of Hankel operators involved with the analytic and co-analytic parts of the symbol.

(**C. Gu**) If neither φ nor ψ is analytic and if (T_φ, T_ψ) is hyponormal, then

$$\ker H_{\overline{\varphi_+}} \subseteq \ker H_{\overline{\psi_-}} \quad \text{and} \quad \ker H_{\overline{\psi_+}} \subseteq \ker H_{\overline{\varphi_-}}.$$

One might be tempted to guess that (ECAP) still holds for Toeplitz pairs whose symbols are matrix-valued trigonometric polynomials. However, this is not the case; take

$$\Phi := \begin{pmatrix} z^{-1} + 2z & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Psi := \begin{pmatrix} 0 & 0 \\ 0 & z^{-1} + 2z \end{pmatrix}.$$

A calculation shows that $\mathbf{T} := (T_\Phi, T_\Psi)$ is hyponormal, but $\Phi_- \neq \Lambda \Psi_-$ for any constant matrix $\Lambda \in M_2$.

On the other hand,

$$\Phi_- = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^*$$

so that

$$\Theta \equiv \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \text{ and } B \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ are not right coprime.}$$

As we might expect, once the condition "Θ and B are right coprime" is assumed then a matrix-valued version of (ECAP) is within reach.

To proceed, we consider some basic facts.

(i) Recall that for each $\Phi \in L^\infty_{M_n}$, if we put

$$\mathcal{E}(\Phi) := \left\{ K \in H^\infty_{M_n} : \ ||K||_\infty \leq 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n} \right\}.$$

then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

(ii) If $\Phi \in L^\infty_{M_n}$, then

$$[T^*_\Phi, T_\Phi] = H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi + T_{\Phi^*\Phi - \Phi\Phi^*} \ .$$

Since the normality of Φ is a necessary condition for the hyponormality of T_Φ, the positivity of $H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi$ is an essential condition for the hyponormality of T_Φ. Thus, we isolate this property as a new notion, weaker than hyponormality.

Definition

For $\Phi, \Psi \in L^\infty_{M_n}$, let

$$[T_\Phi, T_\Psi]_p := H^*_{\Psi^*}H_\Phi - H^*_{\Phi^*}H_\Psi.$$

Then $[T^*_\Phi, T_\Phi]_p$ is called the *pseudo-selfcommutator* of T_Φ. Also T_Φ is said to be *pseudo-hyponormal* if $[T^*_\Phi, T_\Phi]_p$ is positive semidefinite.

Definition

Let $\Phi, \Psi \in L^\infty_{M_n}$. For a Toeplitz pair $\mathbf{T} \equiv (T_\Phi, T_\Psi)$, the *pseudo-commutator* of $\mathbf{T}$ is defined by

$$[\mathbf{T}^*, \mathbf{T}]_p := \begin{pmatrix} [T_\Phi^*, T_\Phi]_p & [T_\Psi^*, T_\Phi]_p \\ [T_\Phi^*, T_\Psi]_p & [T_\Psi^*, T_\Psi]_p \end{pmatrix}$$

Then $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ is said to be *pseudo-(jointly) hyponormal* if $[\mathbf{T}^*, \mathbf{T}]_p \geq 0$.

Lemma

Let $\Phi, \Psi \in L^\infty_{M_n}$. *If* $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *is hyponormal then* Φ *and* Ψ *commute.*

Theorem

Let $\Phi, \Psi \in L^\infty_{M_n}$ *and let* $\mathbf{T} := (T_\Phi, T_\Psi)$. *Then the following are equivalent:*

(i) $\mathbf{T}$ *is hyponormal;*

(ii) $\mathbf{T}$ *is pseudo-hyponormal,* Φ *and* Ψ *are normal, and* $\Psi\Phi = \Phi\Psi$.

Theorem

Let $\Phi \in H^\infty_{M_n}$ *and* $\Psi \in L^\infty_{M_n}$. *If*

$$\Phi \equiv \Phi_+ = A^*\Theta \quad \text{(left coprime)}$$

then

$\mathbf{T} = (T_\Phi, T_\Psi)$ *is pseudo-hyponormal* $\Longleftrightarrow$ $T_{\Psi^{1,\Theta}}$ *is pseudo-hyponormal* ,

where $\Psi^{1,\Theta} := \Psi_-^* + P_{H_0^2}(\Psi_+\Theta^*)$.

Lemma

Let $\mathbf{T} := (T_\Phi, T_\Psi)$ *be a pseudo-hyponormal Toeplitz pair with bounded type symbols* $\Phi, \Psi \in L^\infty_{M_n}$. *Suppose the inner parts of the right coprime factorizations of* Φ_+ *and* Ψ_+ *commute. If* Φ *and* Ψ *have a common matrix pole, then this pole has the same order.*

Theorem

Let $\mathbf{T} := (T_\Phi, T_\Psi)$ *be a hyponormal Toeplitz pair with matrix-valued trigonometric polynomial symbols whose outer coefficients are invertible. Then*

$$\deg(\Phi_-) = \deg(\Psi_-).$$

Theorem

(Hyponormality of Polynomial Toeplitz Pairs) *Let* $\Phi, \Psi \in L^\infty_{M_n}$ *be matrix-valued trigonometric polynomials of the form*

$$\Phi(z) := \sum_{j=-m}^{N} A_j z^j \quad \text{and} \quad \Psi(z) := \sum_{j=-\ell}^{M} B_j z^j$$

satisfying

(i) *the outer coefficients* $A_{-m}, A_N, B_{-\ell}$ *and* B_M *are invertible;*

(ii) *the "co-analytic" outer coefficients* A_{-m} *and* $B_{-\ell}$ *are diagonal-constant.*

If $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *is hyponormal then*

$$\Phi - \Lambda\Psi \in H^2_{M_n} \quad \text{for some constant matrix } \Lambda \in M_n.$$

Corollary

Let $\Phi, \Psi \in L^\infty_{M_n}$ *be matrix-valued trigonometric polynomials such that the outer coefficients of* Φ *and* Ψ *are invertible and the co-analytic outer coefficients of* Φ *and* Ψ *are diagonal-constant. If* $\mathbf{T} := (T_\Phi, T_\Psi)$ *is hyponormal then*

$$\deg(\Phi_+) = \deg(\Psi_+).$$

We now obtain the rank formula for self-commutators.

Theorem

Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *be a hyponormal Toeplitz pair with rational symbols* $\Phi, \Psi \in L^\infty_{M_n}$ *of the form*

$$\Phi_+ = \theta_0 \theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2 \theta_1 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime)}.$$

If θ_0 *and* θ_2 *are* **not coprime** *and* $B(\gamma_0)$ *and* $D(\gamma_0)$ *are diagonal-constant for some* $\gamma_0 \in \mathcal{Z}(\theta_0)$*, then*

$$[\mathbf{T}^*, \mathbf{T}] = \left([T_\Phi^*, T_\Phi] \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \bigoplus 0 \qquad (1)$$

In particular,

$$\text{rank}\, [\mathbf{T}^*, \mathbf{T}] = \text{rank}\, [T_\Phi^*, T_\Phi]. \qquad (2)$$